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CONDITIONAL INTEGRALS ON ABSTRACT WIENER AND HILBERT SPACES WITH APPLICATION TO FEYNMAN INTEGRALS

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Abstract. In this paper, we define conditional integrals on abstract Wiener and Hilbert spaces and then obtain a formula for evaluating the integrals. We use this formula to establish the existence of conditional Feynman integrals for the classes $\Lambda^q(B)$ and $\Lambda^q(H)$ of functions on abstract Wiener and Hilbert spaces and then specialize this result to provide the fundamental solution to the Schrödinger equation with the forced harmonic oscillator.

1. Introduction

We are concerned with the fundamental solution to the Schrödinger equation for a quantum mechanical particle of mass m in $\mathbb{R}^n$

$$(1)\qquad i\hbar\frac{\partial}{\partial t}\Gamma(t,\vec{\eta}) = -\frac{\hbar^2}{2m}\ \triangle\,\Gamma(t,\vec{\eta}) + V(\vec{\eta})\Gamma(t,\vec{\eta}),$$
$$\Gamma(0,\vec{\eta}) = \psi(\vec{\eta}),\quad \vec{\eta}\in\mathbb{R}^n$$

where $\triangle$ is the Laplacian on $\mathbb{R}^n$, $\hbar$ is Planck's constant and V is a suitable potential. According to Feynman [12], the fundamental solution $K(t,\vec{\eta},0,\vec{\xi})$ to the Schrödinger equation (1) such that

$$\Gamma(t,\vec{\eta}) = \int_{\mathbb{R}^n} K(t,\vec{\eta},0,\vec{\xi})\,\psi(\vec{\xi})d\vec{\xi}$$

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can be expressed as the formal path integral:

$$K(t,\vec{\eta},0,\vec{\xi}) = \int_{C_{\vec{\xi},\vec{\eta}}[0,t]} \exp\left\{\frac{i}{\hbar}S(x)\right\}\mathcal{D}(x), \tag{2}$$

where $C_{\vec{\xi},\vec{\eta}}$ is the space of paths x such that $x(0)=\vec{\xi}$ and $x(t)=\vec{\eta}$, $\mathcal{D}(x)$ is a uniform "measure" which does not exist and $S(x)$ is the action integral associated with the path x; i.e.,

$$S(x) = \int_0^t \left[\frac{m}{2}\left(\frac{dx}{ds}\right)^2 - V(x(s))\right] ds.$$

The basic problem of quantum mechanics is to find the solution $\Gamma(t,\vec{\eta})$ or the fundamental solution $K(t,\vec{\eta},0,\vec{\xi})$ to equation (1).

In [14], Gelfand and Yaglom made an attempt to give sense to the formal integral in equation (2) by introducing a Wiener measure with complex variance parameter. Unfortunately their attempt was failed as pointed out by Cameron in [2].

There has been several rigorous approaches to equation (2) to provide the fundamental solution to equation (1). See for examples [16, 21, 25].

In [7] and [10], the concept of the conditional Feynman integral has been introduced and is used to provide an method of getting the fundamental solution to the Schrödinger equation and to obtain the kernel of operator-valued Feynman integrals of various functions.

The main purpose of this paper is to establish the existence of conditional analytic Feynman integral on abstract Wiener and Hilbert spaces for a wider classes of functions than the Fresnel class on abstract Wiener spaces considered in [6, 7]. The latter result establishes a formula for the conditional analytic Feynman integral of functions involving unbounded potentials and then use it to obtain the fundamental solution to the Schrödinger equation with the forced harmonic oscillator.

This paper is organized as follows. In Section 1, we give an introduction for this paper. In Section 2, we recall some preliminary materials which will be needed in this paper. In Section 3, we define conditional integrals on abstract Wiener and Hilbert spaces and then obtain a formula for evaluating the integrals. In Section 4, we establish the existence of conditional analytic Feynman integrals for the classes $\Lambda^q(B)$ and $\Lambda^q(H)$ of functions on abstract Wiener and Hilbert spaces and use this result to provide the fundamental solution to the Schrödinger equation. In Section 5, we specialize the result of Section 4 to Feynman path integrals to obtain the fundamental solution to the Schrödinger equation with the potential $V(t,\xi) = \frac{m\omega^2}{2}\xi^2 + f(t)\xi$, the forced harmonic oscillator with external force $f(t)$.

2. Preliminaries

Let H be a real separable infinite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. Let $\mathcal{P}$ be the set of all orthogonal projections on H with finite dimensional range. For P_1, $P_2 \in \mathcal{P}$, we define $P_1 < P_2$ if $P_1(H) \subseteq P_2(H)$. For $P \in \mathcal{P}$, let

$$\mathcal{C}_P = \{P^{-1}B : B \text{ a Borel set in range of } P\}$$

and

$$\mathcal{C} = \bigcup_P \mathcal{C}_P.$$

A cylinder measure is a finitely additive nonnegative measure on $(H,\ \mathcal{C})$ such that its restriction to $\mathcal{C}_P$ is countably additive for all $P \in \mathcal{P}$. The canonical Gauss measure m on H is the cylinder measure on $(H,\mathcal{C})$ characterized by

$$\int_H e^{i\langle h,h_1\rangle}\,dm(h) = e^{-|h_1|^2/2}.$$

Let $\|\cdot\|$ be a measurable norm on H. It is well known that H is not complete with respect to $\|\cdot\|$ (see [22]). Let B denote the completion of H under $\|\cdot\|$ and let i denote the natural injection. The adjoint operator i^* maps the strong dual B^* continuously, one-to-one, onto a dense subspace of $H^* \approx H$. Gross proved that the induced measure mi^{-1} on the cylinder sets in B is indeed countably additive and hence extends to a countably additive measure ν on $\mathcal{B}$-the Borel σ-field on B. The triple (H,B,ν) is called an *abstract Wiener space* and ν is called the *abstract Wiener measure*. For more details, see [22].

Let $\{e_j | j \geq 1\}$ be a complete orthonormal set in H such that the e_j's are in B^*. For each $h \in H$ and $x \in B$, let

$$(h,x)^\sim = \lim_{n\to\infty} \sum_{j=1}^{n} \langle h,e_j\rangle (e_j,x)$$

if the limit exists and is equal to 0 otherwise. It is shown that for each $h(\neq 0)$ in H, $(h,\cdot)^\sim$ is a Gaussian random variable on B with mean zero, variance $|h|^2$, and that $(h,\lambda x)^\sim = \lambda(h,x)^\sim$ for all $\lambda > 0$. It is easy to see that if $\{h_1,h_2,\ldots,h_n\}$ is a orthogonal set in H, then the random variable $(h_i,x)^\sim$s are independent.

A function f defined on H of the form

$$f(h) = \phi(\langle h_1, h\rangle, \ldots, \langle h_k, h\rangle)$$

is called a cylinder function, where $h_i \in H$ and ϕ is a complex valued Borel function on $\mathbb{R}^n$. We denote by $R(f)$ the random variable $\phi(\widetilde{(h_1, x)}, \widetilde{(h_2, x)}, \ldots, \widetilde{(h_k, x)})$ on $(B, \mathcal{B}, \nu)$. This mapping is extended as follows. For more details, see [15, 21].

DEFINITION. Let $\mathcal{L}(H, \mathcal{C}, m)$ be the set of all complex valued continuous functions f on H such that the net $\{R(f \circ P) : P \in \mathcal{P}\}$ is Cauchy net in ν-probability. Furthermore, for $f \in \mathcal{L}(H, \mathcal{C}, m)$, let

$$R(f) = \lim_{P \in \mathcal{P}} \nu - \text{probability } R(f \circ P).$$

The mapping R will be called *m-lifting.*

DEFINITION. Let

$$\mathcal{L}^1(H, \mathcal{C}, m) = \left\{ f \in \mathcal{L}(H, \mathcal{C}, m) : \int_B |R(f)|\, d\nu < \infty \right\}$$

and for $f \in \mathcal{L}^1(H, \mathcal{C}, m)$, define

$$\int_H f dm = \int_B R(f)\, d\nu.$$

Let A be a self-adjoint, trace class operator with eigenvalues $\{\alpha_k\}$ and corresponding eigenfunctions $\{e_k\}$. Let

$$\widetilde{(x, Ax)} = \lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j \widetilde{(e_j, x)}^2$$

if the limit exists and is equal to 0 otherwise.

Let X be an $\mathbb{R}^n$-valued measurable function and Y a $\mathbb{C}$-valued integrable function of $(B, \mathcal{B}(B), \nu)$. Let P_X be the probability distribution of X, i.e., for all $A \in \mathcal{B}(\mathbb{R}^n)$, $P_X(A) = \nu(X^{-1}(A))$. Then by Radon-Nikodym Theorem, there exists a Borel measurable and P_X-integrable function ψ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P_X)$ such that

$$\int_{X^{-1}(A)} Y d\nu = \int_A \psi(\vec{\xi})\, dP_X,$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$. The function $\psi(\vec{\xi}), \xi \in \mathbb{R}^n$ is unique up to Borel null sets in $\mathbb{R}^n$. The function $\psi(\vec{\xi})$, written $E[Y|X = \vec{\xi}]$, is called the *conditional integral of* Y *on abstract Wiener space given vector-valued conditioning function* X. Thus we have

$$\int_{X^{-1}(A)} Y\,d\nu = \int_A E[Y|X=\vec{\xi}]\,dP_X, \tag{3}$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$.

3. A formula for conditional integral on abstract Wiener and Hilbert spaces

In this section, we give a formula for evaluating conditional integrals on abstract Wiener and Hilbert spaces that include the results of Park and Skoug given in [23, 24] as special cases.

3.1. Conditional integrals on abstract Wiener spaces

Let $\{g_1, g_2, \ldots, g_n\}$ be an orthonormal set in H. Let B_1 be the n-dimensional subspace of H generated by $\{g_1, g_2, \ldots, g_n\}$. The mapping

$$Q(x) = \sum_{j=1}^{n} (g_j, x)^{\sim} g_j \tag{4}$$

defines a continuous operator on B. Since $(g_j, x)^{\sim} = \langle g_j, x\rangle$ for $x \in H$, Q is the lifting to B of the orthogonal projection of H onto B_1. Hence Q is a projection with the range B_1 and $B = B_0 \oplus B_1$ where B_0 is the null space of Q. Let $X : B \to \mathbb{R}^n$ be defined by

$$X(x) = ((g_1, x)^{\sim}, (g_2, x)^{\sim}, \ldots, (g_n, x)^{\sim}). \tag{5}$$

Define $[\cdot] : \mathbb{R}^n \to H$ by $[\vec{\xi}] = \sum_{j=1}^{n} \xi_j g_j$ for $\vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ and we write $[x] = [X(x)] = \sum_{j=1}^{n} (g_j, x)^{\sim} g_j$ for $x \in B$.

LEMMA 3.1.1. *Let* $F \in \mathcal{L}^1(B, \mathcal{B}(B), \nu)$ *and* X *be as in* (5). *Then*

$$E[F] = \int_B F(x)d\nu(x) = \int_{\mathbb{R}^n} E[F(x - [x] + [\vec{\xi}])]\,d\nu \circ X^{-1}(\vec{\xi}).$$

Proof. Let Y and Z be $\mathcal{B}(B)$-measurable functions on B defined by

$$\begin{aligned} Y(x) &= (I - Q)(x) = x - [X(x)], \\ Z(x) &= Q(x) = [X(x)]. \end{aligned} \tag{6}$$

Let $B_1 = Q(B) = \{[X(x)] | x \in B\}$ and $B_0 = Y(B)$. Since Y and Z are independent, we have

$$\begin{aligned}\int_B F(x)d\nu(x) &= \int_B F(x - [X(x)] + [X(x)])d\nu(x) \\ &= \int_{B_0 \times B_1} F(y+z)d(\nu \circ Y^{-1} \times \nu \circ Z^{-1})(y,z) \\ &= \int_{B_0}\int_{B_1} F(y+z)d(\nu \circ Y^{-1})(y)d(\nu \circ Z^{-1})(z).\end{aligned}$$

Since $Z(x) = Q(x) = [X(x)]$, by the change of variables formula and (6), we have

$$\begin{aligned}&\int_{B_0}\int_{B_1} F(y+z)d(\nu \circ Y^{-1})(y)d(\nu \circ Z^{-1})(z) \\ &= \int_{\mathbb{R}^n}\int_{B_0} F(y + [\vec{\xi}])d(\nu \circ Y^{-1})(y)d(\nu \circ X^{-1})(\vec{\xi}) \\ &= \int_{\mathbb{R}^n}\int_{B} F(x - [x] + [\vec{\xi}])d\nu(x)d(\nu \circ X^{-1})(\vec{\xi}) \\ &= \int_{\mathbb{R}^n} E[F(x - [x] + [\vec{\xi}])]d(\nu \circ X^{-1})(\vec{\xi}).\end{aligned}$$

Hence we complete the proof. □

LEMMA 3.1.2. *Let $F \in \mathcal{L}^1(B, \mathcal{B}(B), \nu)$. Then*

$$\int_{X^{-1}(A)} F(x)d\nu(x) = \int_A E[F(x - [x] + [\vec{\xi}])]d(\nu \circ X^{-1})(\vec{\xi})$$

for every $A \in \mathcal{B}(\mathbb{R}^n)$.

Proof. Using Lemma 3.1.1, for every $A \in \mathcal{B}(\mathbb{R}^n)$ and its indicator function I_A, we have

$$\begin{aligned}&\int_{X^{-1}(A)} F(x)d\nu(x) \\ &= \int_B I_{X^{-1}(A)}(x)F(x)d\nu(x) \\ &= \int_B \{(I_A \circ X) \cdot F\}(x)d\nu(x) \\ &= \int_{\mathbb{R}^n} E\big[\{(I_A \circ X) \cdot F\}(x - [x] + [\vec{\xi}])\big]d\nu \circ X^{-1}(\vec{\xi}) \\ &= \int_{\mathbb{R}^n} E\big[I_A(X(x - [x] + [\vec{\xi}]))F(x - [x] + [\vec{\xi}])\big]d\nu \circ X^{-1}(\vec{\xi})\end{aligned}$$

$$= \int_{\mathbb{R}^n} E\big[I_A(\vec{\xi})F(x-[x]+[\vec{\xi}])\big]d\nu \circ X^{-1}(\vec{\xi})$$
$$= \int_{\mathbb{R}^n} I_A(\vec{\xi})E\big[F(x-[x]+[\vec{\xi}])\big]d\nu \circ X^{-1}(\vec{\xi})$$
$$= \int_A E\big[F(x-[x]+[\vec{\xi}])\big]d\nu \circ X^{-1}(\vec{\xi}).$$

Hence the proof is completed. □

THEOREM 3.1.3. *Let $F \in \mathcal{L}^1(B, \mathcal{B}(B), \nu)$. Then*

$$E\big[F(x)|X(x) = \vec{\xi}\big] = E\big[F(x-[x]+[\vec{\xi}])\big]$$

for a.e. $\vec{\xi} \in \mathbb{R}^n$.

Proof. By equation (3) and Lemma 3.1.2 , we have

$$\int_A E\big[F(x)|X(x) = \vec{\xi}\big]\, d(\nu \circ X^{-1})(\vec{\xi})$$
$$= \int_{X^{-1}(A)} F(x)d\nu(x)$$
$$= \int_A E\big[F(x-[x]+[\vec{\xi}])\big]d(\nu \circ X^{-1})(\vec{\xi})$$

for any $A \in \mathcal{B}(\mathbb{R}^n)$. Hence the desired result is obtained. □

COROLLARY 3.1.4. *Let $F \in \mathcal{L}^1(B, \mathcal{B}(B), \nu)$. Then*

$$E\big[F(x)|X(x) = \vec{\xi}\big] = \int_{B_0} F(y+[\vec{\xi}])d\nu_0(y)$$

where $\nu_0 = \nu \circ Y^{-1}$ is the abstract Wiener measure on B_0 induced by Y.

3.2. Conditional integrals on Hilbert spaces

Let $\{g_1, g_2, \ldots, g_n\}$ be an orthonormal set in H. Define an $\mathbb{R}^n$-valued function y on H by

(7) $$y(h) = ((g_1, h), \ldots, (g_n, h)), \quad h \in H.$$

For any $f \in \mathcal{L}^1(H, \mathcal{C}, m)$ and a function y on H, we define

$$E_H\left[f \mid y(\cdot) = \overrightarrow{\xi}\right] = E\left[R(f) \mid R(y)(\cdot) = \overrightarrow{\xi}\right].$$

Such $E_H\left[f \mid y = \overrightarrow{\xi}\right]$ is called a *conditional integral of f on H given $y = \overrightarrow{\xi}$.*

REMARK. Let E be a topological vector space. For $\lambda > 0$, define $f^\lambda(x) = f(\lambda^{-\frac{1}{2}}x)$, $x \in E$. Then for any $\mathbb{C}$(or $\mathbb{R}^n$)-valued function f on H with $f^\lambda \in \mathcal{L}^1(H, \mathcal{C}, m)$,

$$E_H\left[f^\lambda \mid y^\lambda = \overrightarrow{\xi}\right] = E\left[R(f^\lambda) \mid R(y^\lambda) = \overrightarrow{\xi}\right],$$

for all $\lambda > 0$.

THEOREM 3.2.1. *Let $f \in \mathcal{L}^1(H, \mathcal{C}, m)$. Then there exists a $\mathcal{B}(\mathbb{R}^n)$-measurable function $E_H[f \mid y = \overrightarrow{\xi}]$ such that*

$$E_H\left[f \mid y(\cdot) = \overrightarrow{\xi}\right] = E_H\left[f(\cdot - [y(\cdot)] + [\overrightarrow{\xi}])\right]$$

for all $\overrightarrow{\xi} \in \mathbb{R}^n$.

Proof. Since $R(f) \in \mathcal{L}^1(B, \mathcal{B}(B), \nu)$ for any $f \in \mathcal{L}^1(H, \mathcal{C}, m)$, there exists a $\mathcal{B}(\mathbb{R}^n)$-measurable function $E\left[R(f) \mid R(y)(\cdot) = \overrightarrow{\xi}\right]$. Note that

$$R(y)(x) = ((g_1, x)\tilde{}, (g_2, x)\tilde{}, \ldots, (g_n, x)\tilde{}) = X(x), \quad x \in B.$$

Hence by the definition of $E_H\left[f \mid y(\cdot) = \overrightarrow{\xi}\right]$ and Theorem 3.1.3 , we get

$$\begin{aligned} E_H\left[f \mid y(\cdot) = \overrightarrow{\xi}\right] &= E\left[R(f) \mid R(y)(\cdot) = \overrightarrow{\xi}\right] \\ &= E\left[R(f) \mid X(\cdot) = \overrightarrow{\xi}\right] \\ &= E\left[R(f)(\cdot - [X(\cdot)] + [\overrightarrow{\xi}])\right] \\ &= E_H\left[f(\cdot - [y(\cdot)] + [\overrightarrow{\xi}])\right]. \end{aligned}$$

Hence the proof follows. □

4. Conditional analytic Wiener and Feynman integrals

In this section, we give an evaluation of the analytic Wiener and Feynman integrals of functions in the classes of $\Lambda^q(B)$ and $\Lambda^q(H)$ and show that the conditional analytic Feynman integral is used to provide the fundamental solution to the Schrödinger equation.

4.1. Conditional analytic Wiener and Feynman integrals on abstract Wiener spaces

We begin with the definition of the conditional analytic Wiener and Feynman integral of a function F on B given a function X.

DEFINITION. Let X be an $\mathbb{R}^n$-valued measurable function on B and let F be a $\mathbb{C}$-valued measurable function on B such that the integral

$$E[F^\lambda] = \int_B F(\lambda^{-\frac{1}{2}}x)d\nu(x)$$

exists as a finite number for all $\lambda > 0$. If for a.e. $\vec{\eta} \in \mathbb{R}^n$, $J_\lambda(\vec{\eta}) = E\big[F^\lambda | X^\lambda = \vec{\eta}\big]$ exists for all $\lambda > 0$ and has an analytic continuation to $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Re} z > 0\}$, denoted by $J^*_\lambda(\vec{\eta})$, then J^*_λ is defined to be the conditional analytic Wiener integral of F on B given X with parameter $\lambda \in \mathbb{C}^+$ and we write

$$E^{\mathrm{anw}_\lambda}[\,F\,|\,X = \vec{\eta}\,] = J^*_\lambda(\vec{\eta}).$$

If for fixed real $q \neq 0$, the limit

$$\lim_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} E^{\mathrm{anw}_\lambda}[\,F\,|\,X = \vec{\eta}\,]$$

exists for a.e. $\vec{\eta} \in \mathbb{R}^n$, then we denote the value of this limit by $E^{\mathrm{anf}_q}[F|X = \vec{\eta}]$ and call it the *conditional analytic Feynman integral of F on B given X with parameter q.*

Let $\mathcal{M}(H)$ be the class of all $\mathbb{C}$-valued Borel measures on H with bounded variation. Let $\mathcal{F}(H)$ be the class of all functions f on H of the form

$$f(h_1) = \int_H e^{i\langle h,h_1\rangle} d\sigma(h) \tag{8}$$

for some $\sigma \in \mathcal{M}(H)$. $\mathcal{F}(H)$ is the Fresnel class of Albeverio and Hoegh-Krohn [1]. It is known [21] that each function of the form (8) can be extended to B uniquely by

$$F(x) = \int_H e^{i(h,x)^\sim} d\sigma(h).$$

Given two $\mathbb{C}$-valued measurable function F and G on B, F is said to be equal to G s-almost surely(s-a.s.) if for each $\alpha > 0$, $\nu\{x \in B | F(\alpha x) \neq G(\alpha x)\} = 0$ (for more detail, see [5, 19]). For a measurable function F

on B, let $[F]$ denote the equivalence class of functionals which are equal to F s-a.s. The class of equivalence classes defined by

$$\mathcal{F}(B) = \{[F] | F(x) = \int_H e^{i\widetilde{(h,x)}} d\sigma(h), \quad \sigma \in \mathcal{M}(H)\}$$

is called the *Fresnel class of functions on B*.

It is known that $\mathcal{F}(B)$ forms a Banach algebra over the complex field and that $\mathcal{F}(H)$ and $\mathcal{F}(B)$ are isometrically isomorphic (see [1, 17, 21]).

As customary, we will identify a function with its s-equivalence class and think of $\mathcal{F}(B)$ as a class of functions on B rather than as a class of equivalence classes.

For any $q \in \mathbb{R}$, $q \neq 0$, we denote by $\Lambda^q(H)$ and $\Lambda^q(B)$, respectively, the class of all functions g on H of the form given by

$$g(h) = \exp\{\frac{i}{2}(h, Ah)\} \int_H e^{i\langle h, h_1\rangle} d\sigma(h_1) \tag{9}$$

and G on B of the form given by

$$G(x) = \exp\{\frac{i}{2}\widetilde{(x, Ax)}\} \int_H e^{i\widetilde{(h,x)}} d\sigma(h) \tag{10}$$

for some $\sigma \in \mathcal{M}(H)$ and some self adjoint trace class operator A on H such that $(I + \frac{1}{q}D)$is invertible where $D = (I - Q)A(I - Q)$ (see [11, 21]). It is known [21] that G is the m-lifting of g.

For a self-adjoint trace class operator A with eigenvalues $\{\alpha_j\}$, the Fredholm determinant of $(I + A)$, denoted by $\det(I + A)$, defined by

$$\det(I + A) = \prod_{j=1}^{\infty} (1 + \alpha_j)$$

and the Maslov index of $(I + A)$, denote by ind(I+A), is the number of negative eigenvalues of $(I + A)$.

Let F be a $\mathbb{C}$-valued measurable function on B such that

$$J(\lambda) = \int_B F(\lambda^{-\frac{1}{2}}x) d\nu(x)$$

exists for all real $\lambda > 0$. If $J(\lambda)$ has an analytic continuation to $\mathbb{C}^+$, denoted by $J^*(\lambda)$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F on B with parameter λ, and for $\lambda \in \mathbb{C}^+$ we write $E^{\text{anw}_\lambda}[F] = J^*(\lambda)$. If for $q \neq 0$, $\lim\limits_{\substack{\lambda \to -iq \\ \lambda \in \mathbb{C}^+}} E^{\text{anw}_\lambda}[F]$ exists, we call the limit analytic Feynman integral of F with parameter q and we denote it by $E^{\text{anf}_q}[F]$.

It is known [21] that $E^{\text{anf}_q}[G]$ exists for all $G \in \Lambda^q(B)$.

THEOREM 4.1.1. *Let $G \in \Lambda^q(B)$ be given by (10) and let X be as in (5). Then the conditional analytic Feynman integral $E^{\mathrm{anf}_q}[G \,|\, X = \vec{\xi}\,]$ exists and we have*

$$
\begin{aligned}
&E^{\mathrm{anf}_q}[G \,|\, X = \vec{\xi}\,] \\
&= \left|\det(I + \frac{1}{q}D)\right|^{-\frac{1}{2}} e^{-\frac{\pi i}{2}\mathrm{Ind}(I+\frac{1}{q}D)} \cdot e^{\frac{i}{2}\langle[\vec{\xi}],A[\vec{\xi}]\rangle} \cdot \int_H \exp\left\{ i\langle h, [\vec{\xi}]\rangle - \frac{i}{2q} \right. \\
&\left. \cdot \left\langle \left(I + \frac{1}{q}D\right)^{-1} (I-Q)(A[\vec{\xi}] + h), (I-Q)(A[\vec{\xi}] + h) \right\rangle \right\} d\sigma(h).
\end{aligned}
$$

Proof. For any $\lambda > 0$ and $\vec{\xi} \in \mathbb{R}^n$, we have

$$
\begin{aligned}
(11)\quad & E[G^\lambda(x - [x] + [\sqrt{\lambda}\vec{\xi}])] \\
&= E\left[\exp\left\{\frac{i}{2\lambda}(x - [x] + [\sqrt{\lambda}\vec{\xi}], A(x - [x] + [\sqrt{\lambda}\vec{\xi}])\tilde{)}\right\}\right. \\
&\quad \left.\cdot \int_H e^{\frac{i}{\sqrt{\lambda}}(h, x - [x] + [\sqrt{\lambda}\vec{\xi}])\tilde{}} d\sigma(h)\right] \\
&= E\left[\exp\left\{\frac{i}{2\lambda}((I-Q)x + [\sqrt{\lambda}\vec{\xi}], A(I-Q)x + A[\sqrt{\lambda}\vec{\xi}])\tilde{}\right\}\right. \\
&\quad \left.\cdot \int_H e^{\frac{i}{\sqrt{\lambda}}(h, (I-Q)x + [\sqrt{\lambda}\vec{\xi}])\tilde{}} d\sigma(h)\right] \\
&= E\left[\exp\left\{\frac{i}{2\lambda}(x, Dx)\tilde{} + 2((I-Q)A[\sqrt{\lambda}\vec{\xi}], x)\tilde{} + \langle[\sqrt{\lambda}\vec{\xi}], A[\sqrt{\lambda}\vec{\xi}]\rangle\right\}\right. \\
&\quad \left.\cdot \int_H e^{\left\{\frac{i}{\sqrt{\lambda}}((I-Q)h, x)\tilde{} + i\langle h, [\vec{\xi}]\rangle\right\}} d\sigma(h)\right] \\
&= \exp\left\{\frac{i}{2}\langle[\vec{\xi}], A[\vec{\xi}]\rangle\right\} \int_H e^{i\langle h, [\vec{\xi}]\rangle} E\left[\exp\left\{\frac{i}{2\lambda}(x, Dx)\tilde{}\right.\right. \\
&\quad \left.\left. + \frac{i}{\sqrt{\lambda}}(k, x)\tilde{}\right\}\right] d\sigma(h) \\
&= \exp\left\{\frac{i}{2}\langle[\vec{\xi}], A[\vec{\xi}]\rangle\right\} \cdot \int_H e^{i\langle h, [\vec{\xi}]\rangle} \left\{ \int_B \exp\left\{\frac{i}{2\lambda}\sum_{j=1}^{\infty} \alpha_j (e_j, x)\tilde{}^2\right.\right. \\
&\quad \left.\left. + \frac{i}{\sqrt{\lambda}}\sum_{j=1}^{\infty}\langle k, e_j\rangle (e_j, x)\tilde{}\right\} d\nu(x)\right\} d\sigma(h)
\end{aligned}
$$

$$= \exp\left\{\frac{i}{2}\langle[\vec{\xi}], A[\vec{\xi}]\rangle\right\} \cdot \int_H e^{i\langle h,[\vec{\xi}]\rangle} \prod_{j=1}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(1-\frac{i\alpha_j}{\lambda})y^2 + \frac{i\langle k, e_j\rangle}{\sqrt{\lambda}} y dy\right] d\sigma(h)$$

$$= \left\{\prod_{j=1}^{\infty}(1-\frac{i\alpha_j}{\lambda})\right\}^{-\frac{1}{2}} \cdot e^{\frac{i}{2}\langle[\vec{\xi}],A[\vec{\xi}]\rangle} \int_H e^{i\langle h,[\vec{\xi}]\rangle} \cdot \exp\left\{-\frac{1}{2\lambda}\sum_{j=1}^{\infty}\frac{\langle k, e_j\rangle^2}{1-\frac{i}{\lambda}\alpha_j}\right\} d\sigma(h)$$

where $k = (I-Q)(A[\vec{\xi}]+h)$ and $\{\alpha_j\}$ and $\{e_j\}$ are eigenvalues and eigenvectors of D, respectively. Since $\sum_{j=1}^{\infty}|\alpha_j| < \infty$, the infinite product and series appeared in (11) converges absolutely. Hence by Theorem 3.1.3, there exists a version $J_\lambda(\vec{\xi})$ such that

$$J_\lambda(\vec{\xi}) = E\left[G^\lambda(x - [x] + [\sqrt{\lambda}\vec{\xi}])\right]$$

for all $\lambda > 0$ and $\vec{\xi} \in \mathbb{R}^n$.

We now show that $E^{\text{anw}_\lambda}[G|X = \vec{\xi}]$ exists for all $\vec{\xi} \in \mathbb{R}^n$. We may assume that $1 + \frac{\alpha_j}{q} < 0$ for $j = 1, 2, \ldots, m$, $\quad m = \text{Ind}(I + \frac{1}{q}D)$ and $1 + \frac{\alpha_j}{q} > 0$ for $j \geq m+1$. Since D is a trace class operator and $I + \frac{1}{q}D$ is invertible, $|\alpha_j| \to 0$, as $j \to \infty$ and $\alpha_j \neq -q$ for all j. Hence we can choose $\delta > 0$ such that α_j is not in $[-q-\delta, -q+\delta]$ for all j. Let $\Omega = \mathbb{C}^+ \cup \{z \in \mathbb{C} \,|\, \text{Re}z = 0, |q + \text{Im}z| \leq \delta\}$. For $z \in \Omega$, let

$$A_1(z) = \prod_{j=1}^{m} z^{\frac{1}{2}}(z - i\alpha_j)^{-\frac{1}{2}},$$

$$A_2(z) = \prod_{j=m+1}^{\infty}(1 - \frac{i\alpha_j}{z})^{-\frac{1}{2}}$$

and

$$A_3(z) = \int_H \exp\left\{i\langle h, [\vec{\xi}]\rangle - \frac{1}{2}\sum_{j=1}^{\infty}\frac{\langle k, e_j\rangle^2}{z - i\alpha_j}\right\} d\sigma(h).$$

By the similar arguments as in the proof of Theorem 3.2 in [21], we can show that A_1, A_2 and A_3 are continuous function on Ω and analytic in $\mathbb{C}^+$. Thus

$$J_z^*(\vec{\xi}) = \exp\left\{-\frac{i}{2}\langle[\vec{\xi}], A[\vec{\xi}]\rangle\right\} A_1(z)A_2(z)A_3(z)$$

is continuous on Ω and analytic in $\mathbb{C}^+$. It is easy to see that $J_\lambda^*(\vec{\xi}) = J_\lambda(\vec{\xi})$ for real $\lambda > 0$ and hence $E^{\mathrm{anw}_\lambda}[G|X = \vec{\xi}]$ exists for all $\vec{\xi} \in \mathbb{R}^n$.

We finally show that $E^{\mathrm{anf}_q}[G|X = \vec{\xi}]$ exists. We note that

$$\begin{aligned} E^{\mathrm{anf}_q}[\,G\,|\,X = \vec{\xi}\,] &= \lim_{z \to -iq} J_z^*(\vec{\xi}) \\ &= \exp\left\{-\frac{i}{2}\langle[\vec{\xi}], A[\vec{\xi}]\rangle\right\} A_1(-iq)A_2(-iq)A_3(-iq). \end{aligned}$$

But we get

$$A_1(-iq) = \left(\prod_{j=1}^{m}\left|1 + \frac{\alpha_j}{q}\right|^{-\frac{1}{2}}\right) \mathrm{e}^{-\frac{\pi}{2}i\mathrm{Ind}(I+\frac{1}{q}D)},$$

$$A_2(-iq) = \prod_{j=m+1}^{\infty}\left(1 + \frac{\alpha_j}{q}\right)^{-\frac{1}{2}} \qquad \text{as} \quad 1 + \frac{\alpha_j}{q} \geq 0$$

and

$$\begin{aligned} A_3(-iq) &= \int_H \exp\left\{ i\langle h, [\vec{\xi}]\rangle - \frac{i}{2}\sum_{j=1}^{\infty}\frac{\langle k, e_j\rangle^2}{q + \alpha_j}\right\} d\sigma(h) \\ &= \int_H \exp\left\{ i\langle h, [\vec{\xi}]\rangle - \frac{i}{2q}\left\langle k, (I + \frac{1}{q}D)^{-1}k\right\rangle\right\} d\sigma(h). \end{aligned}$$

Hence the desired result is obtained. □

COROLLARY 4.1.2. *Let σ be the measure concentrated at $0 \in H$ in Theorem 4.1.1. Then we have*

$$\begin{aligned} &E^{\mathrm{anf}_q}\left[\exp\left\{\frac{i}{2}(x, A\tilde{x})\right\} |X(x) = \vec{\xi}\right] \\ &= \left|\det(I + \frac{1}{q}D)\right|^{-\frac{1}{2}} e^{-\frac{\pi i}{2}\mathrm{Ind}(I+\frac{1}{q}D)} \cdot e^{\frac{i}{2}\langle[\vec{\xi}],A[\vec{\xi}]\rangle} \\ &\quad \cdot \exp\left\{-\frac{i}{2q}\left\langle (I - Q)A[\vec{\xi}], \left(I + \frac{1}{q}D\right)^{-1}(I - Q)A[\vec{\xi}]\right\rangle\right\}. \end{aligned}$$

COROLLARY 4.1.3. [7] *Let $A = 0$ in Theorem 4.1.1. Then we have*

$$E^{\mathrm{anf}_q}\left[\int_H \exp\left\{i\widetilde{(h,x)}\right\} d\sigma(h) | X(x) = \vec{\xi}\right]$$
$$= \int_H \exp\{i\langle h, [\vec{\xi}]\rangle\} \cdot \exp\left\{-\frac{i}{2q}\left(|h|^2 - \sum_{j=1}^{n}\langle g_j, h\rangle^2\right)\right\} d\sigma(h).$$

In our next theorem, we need the following summation procedure (see [20, p.340]):

$$\overline{\int_{\mathbb{R}^n}} f(\vec{\xi})\, d\vec{\xi} = \lim_{A\to\infty} \int_{\mathbb{R}^n} f(\vec{\xi}) \exp\left\{-\frac{|\vec{\xi}|^2}{2A}\, d\vec{\xi}\right\} \tag{12}$$

whenever the expression on the right exists. Of course if $f \in L^1(\mathbb{R}^n)$, it is clear using the dominated convergence theorem that

$$\overline{\int_{\mathbb{R}^n}} f(\vec{\xi})\, d\vec{\xi} = \int_{\mathbb{R}^n} f(\vec{\xi})\, d\vec{\xi}.$$

THEOREM 4.1.4. *Let G and X be as in Theorem 4.1.1. Then for all $\lambda \in \mathbb{C}^+$,*

$$\int_{\mathbb{R}^n} \left(\frac{\lambda}{2\pi}\right)^{n/2} \exp\left\{-\frac{\lambda}{2}|\vec{\xi}|^2\right\} E^{\mathrm{anw}_\lambda}[G|X = \vec{\xi}]\, d\vec{\xi} = E^{\mathrm{anw}_\lambda}[G] \tag{13}$$

and for all real $q \neq 0$,

$$\overline{\int_{\mathbb{R}^n}} \left(\frac{q}{2\pi i}\right)^{n/2} \exp\left\{\frac{iq}{2}|\vec{\xi}|^2\right\} E^{\mathrm{anf}_q}[G|X = \vec{\xi}]\, d\vec{\xi} = E^{\mathrm{anf}_q}[G]. \tag{14}$$

Proof. From Lemma 3.1.1, we have, for any real $\lambda > 0$,

$$E[G^\lambda] = \int_{\mathbb{R}^n} E\left[G^\lambda(x - [x] + [\vec{\xi}])\right] d\nu \circ (X^\lambda)^{-1}(\vec{\xi}).$$

Since $d\nu \circ (X^\lambda)^{-1}(\vec{\xi}) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \exp\left\{-\frac{\lambda}{2}|\vec{\xi}|^2\right\} d\vec{\xi}$, we have

$$E[G^\lambda] = \int_{\mathbb{R}^n} \left(\frac{\lambda}{2\pi}\right)^{n/2} \exp\left\{-\frac{\lambda}{2}|\vec{\xi}|^2\right\} E[G^\lambda | X^\lambda = \vec{\xi}]\, d\vec{\xi}.$$

Let $U(\vec{\xi}, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \exp\left\{-\frac{\lambda}{2}|\vec{\xi}|^2\right\} E^{\mathrm{anw}_\lambda}[G|X]$. As shown in the proof of Theorem 4.1.1, $E^{\mathrm{anw}_\lambda}[G|X]$ is analytic in $\mathbb{C}^+$ and continuous on Ω.

Thus $U(\vec{\xi},\lambda)$ is analytic in $\mathbb{C}^+$ and continuous on Ω. A simple application of Morera's Theorem gives the proof of equation (13). To prove equation (14), it suffices in view of equation (13) to show that

$$\lim_{\lambda\to -iq}\int_{\mathbb{R}^n} U(\vec{\xi},\lambda)\,d\vec{\xi} = \overline{\int_{\mathbb{R}^n}} U(\vec{\xi},-iq)\,d\vec{\xi}.$$

But this follows from the use of the dominate convergence theorem that

$$\begin{aligned}\overline{\int_{\mathbb{R}^n}} U(\vec{\xi},-iq)\,d\vec{\xi} &= \lim_{A\to\infty}\int_{\mathbb{R}^n} U(\vec{\xi},-iq)\,\exp\left\{-\frac{|\vec{\xi}|^2}{2A}\right\}d\vec{\xi}\\ &= \lim_{A\to\infty}\int_{\mathbb{R}^n}\lim_{\lambda\to -iq} U(\vec{\xi},\lambda)\,\exp\left\{-\frac{|\vec{\xi}|^2}{2A}\right\}d\vec{\xi}\\ &= \lim_{A\to\infty}\lim_{\lambda\to -iq}\int_{\mathbb{R}^n} U(\vec{\xi},\lambda)\,\exp\left\{-\frac{|\vec{\xi}|^2}{2A}\right\}d\vec{\xi}\\ &= \lim_{\lambda\to -iq}\lim_{A\to\infty}\int_{\mathbb{R}^n} U(\vec{\xi},\lambda)\,\exp\left\{-\frac{|\vec{\xi}|^2}{2A}\right\}d\vec{\xi}\\ &= \lim_{\lambda\to -iq}\int_{\mathbb{R}^n} U(\vec{\xi},\lambda)\,d\vec{\xi}.\end{aligned}$$

□

COROLLARY 4.1.5. *Let G and X be in Theorem 4.1.1. Let ψ be given by*

$$\psi(\vec{\eta}) = \int_{\mathbb{R}^n}\exp\{i\langle\vec{u},\vec{\eta}\rangle\}d\mu(\vec{u}) \tag{15}$$

where μ is a $\mathbb{C}$-valued Borel measure on $\mathbb{R}^n$ with bounded variation. For $\vec{\eta}\in\mathbb{R}^n$, let $K_{\vec{\eta}}$ be the function on B given by

$$K_{\vec{\eta}}(\cdot) = G(\cdot)\,\psi\Big(X(\cdot)+\vec{\eta}\Big).$$

Then for all $q\neq 0$, we have that

$$\begin{aligned}\Gamma(\vec{\eta},q) &\equiv E^{\text{anf}_q}[K_{\vec{\eta}}]\\ &= \overline{\int_{\mathbb{R}^n}} E^{\text{anf}_q}\Big[G\big|X=\vec{\xi}-\vec{\eta}\Big]\left(\frac{q}{2\pi i}\right)^{n/2}\exp\left\{\frac{iq|\vec{\xi}-\vec{\eta}|^2}{2}\right\}\psi(\vec{\xi})d\vec{\xi}.\end{aligned}$$

Proof. We note that for $\vec{\eta} \in \mathbb{R}^n$, $K_{\vec{\eta}}$ belongs to $\Lambda^q(B)$ and

$$E\left[K_{\vec{\eta}}|X^\lambda + \vec{\eta} = \vec{\xi}\right] = E\left[G^\lambda \,|\, X^\lambda = \vec{\xi} - \vec{\eta}\right] \psi(\vec{\xi})$$

for all $\vec{\xi} \in \mathbb{R}^n$ and $\lambda > 0$. The proof follows from this note and Theorem 4.1.4 □

REMARK. Let

$$H(\vec{\eta}, \vec{\xi}, q) = E^{\mathrm{anf}_q}\left[G|X = \vec{\xi} - \vec{\eta}\right] \left(\frac{q}{2\pi i}\right)^{n/2} \exp\left\{\frac{iq|\vec{\xi} - \vec{\eta}|^2}{2}\right\}.$$

Then Corollary 4.1.5 gives that $H(\vec{\eta}, \vec{\xi}, q)$ with $q = \frac{m}{\hbar}$ is the fundamental solution to the Schrödinger equation (1) with $\psi \in \mathcal{F}(\mathbb{R}^n)$.

4.2. Conditional analytic Feynman integrals on Hilbert spaces

In this section, we define conditional analytic Feynman integrals on H and give the evaluation of the conditional analytic Feynman integrals involving quadratic functions on H.

DEFINITION. Let f be a $\mathbb{C}$-valued function on H such that for all real $\lambda > 0$, $f^\lambda \in \mathcal{L}^1(H, \mathcal{C}, m)$ and y be as in (7). Suppose that a.e $\vec{\xi} \in \mathbb{R}^n$, $K_\lambda(\vec{\xi}) = E_H[\, f^\lambda \,|\, y^\lambda = \vec{\xi}]$ exists for all $\lambda > 0$ and has an analytic continuation to $\mathbb{C}^+$, denoted by $K_z^*(\vec{\xi})$. Then $K_z^*(\vec{\xi}) = E^{\mathrm{ang}_z}[\, f \,|\, y = \vec{\xi}]$ is called the *conditional analytic Gauss integral of f on H with parameter z*. If for $q \neq 0$, the limit

$$\lim_{\substack{z \to -iq \\ \lambda \in \mathbb{C}^+}} E^{\mathrm{ang}_z}[f|y = \vec{\xi}] = E^{\mathrm{anf}_q}[f|y = \vec{\xi}]$$

exists for a.e. $\vec{\xi} \in \mathbb{R}^n$, we define $E^{\mathrm{anf}_q}[f|y = \vec{\xi}]$ to be the *conditional analytic Feynman integral of f on H* with parameter q.

REMARK. Suppose f satisfies that there exists a $\mathbb{C}$-valued function F on B with the condition

$$R(f^\lambda) = F^\lambda,$$

for all real $\lambda > 0$. Then for all real $\lambda > 0$,

$$E^{\mathrm{anw}_\lambda}[F|X = \vec{\xi}] = E^{\mathrm{ang}_\lambda}[f|y = \vec{\xi}]$$

and hence

$$E^{\mathrm{anf}_q}[F|X = \vec{\xi}] = E^{\mathrm{anf}_q}[f|y = \vec{\xi}].$$

We now define analytic Feynman integrals for functions on H. Let f be a $\mathbb{C}$-valued function on H such that for all real $\lambda > 0$, $f^\lambda \in \mathcal{L}^1(H, \mathcal{C}, m)$. Suppose that $K(\lambda) = \int_H f^\lambda dm$ exists for all $\lambda > 0$. If $K(\lambda)$ has an analytic continuation to $\mathbb{C}^+$, denoted by $K^*(z)$, then $K^*(z)$ is defined to be the analytic Gauss integral of f on H with parameter z, and for $z \in \mathbb{C}^+$ we write $E^{\mathrm{ang}_z}[F] = K^*(z)$. If for $q \neq 0$, $\lim_{\substack{z \to -iq \\ z \in \mathbb{C}^+}} E^{\mathrm{ang}_z}[f]$ exists, we call the limit analytic Feynman integral of f with parameter q and we denote it by $E^{\mathrm{anf}_q}[f]$. It is known [21] that $E^{\mathrm{anf}_q}[g]$ exists for all $g \in \Lambda^q(H)$.

THEOREM 4.2.1. *Let $g \in \Lambda^q(H)$ be given by (9) and let y be as in (7). Then the conditional analytic Feynman integral $E^{\mathrm{anf}_q}[\,g\,|\,y = \vec{\xi}\,]$ exists and we have*

$$
\begin{aligned}
&E^{\mathrm{anf}_q}[\,g\,|\,y = \vec{\xi}\,] \\
&= \left|\det(I + \frac{1}{q}D)\right|^{-\frac{1}{2}} e^{-\frac{\pi i}{2}\mathrm{Ind}(I + \frac{1}{q}D)} \cdot e^{\frac{i}{2}\langle[\vec{\xi}], A[\vec{\xi}]\rangle} \cdot \int_H \exp\Bigg\{ i\langle h, [\vec{\xi}]\rangle \\
&\quad - \frac{i}{2q}\left\langle \left(I + \frac{1}{q}D\right)^{-1} (I - Q)(A[\vec{\xi}] + h), (I - Q)(A[\vec{\xi}] + h)\right\rangle \Bigg\} d\sigma(h).
\end{aligned}
$$

THEOREM 4.2.2. *Let $g \in \Lambda^q(H)$ be given by (9) and let y be as in (7). Then for all $\lambda \in \mathbb{C}^+$,*

$$
\int_{\mathbb{R}^n} \left(\frac{\lambda}{2\pi}\right)^{n/2} \exp\left\{-\frac{\lambda}{2}|\xi|^2\right\} E^{\mathrm{ang}_\lambda}[g|y = \vec{\xi}]\, d\vec{\xi} = E^{\mathrm{ang}_\lambda}[g] \tag{16}
$$

and for all real $q \neq 0$,

$$
\overline{\int_{\mathbb{R}^n}} \left(\frac{q}{2\pi i}\right)^{n/2} \exp\left\{\frac{iq}{2}|\xi|^2\right\} E^{\mathrm{anf}_q}[g|y = \vec{\xi}]\, d\vec{\xi} = E^{\mathrm{anf}_q}[g]. \tag{17}
$$

COROLLARY 4.2.3. *Let $g \in \Lambda^q(H)$ be given by (9) and let y be as in (7). Let ψ be given by*

$$
\psi(\vec{\eta}) = \int_{\mathbb{R}^n} \exp\{i\langle \vec{u}, \vec{\eta}\rangle\} d\mu(\vec{u}) \tag{18}
$$

where μ is a $\mathbb{C}$-valued Borel measure on $\mathbb{R}^n$ with bounded variation. For $\vec{\eta} \in \mathbb{R}^n$, let $L_{\vec{\eta}}$ be the function on H given by

$$
L_{\vec{\eta}}(\cdot) = g(\cdot)\psi(y(\cdot) + \vec{\eta}).
$$

Then for all $q \neq 0$, *we have that*

$$\begin{aligned}\Gamma(\vec{\eta}, q) &\equiv E^{\mathrm{anf}_q}(L_{\vec{\eta}}) \\ &= \overline{\int_{\mathbb{R}^n}} E^{\mathrm{anf}_q}[g|y = \vec{\xi} - \vec{\eta}] \left(\frac{q}{2\pi i}\right)^{n/2} \exp\left\{\frac{iq|\vec{\xi} - \vec{\eta}|^2}{2}\right\} \psi(\vec{\xi})d\vec{\xi}.\end{aligned}$$

5. Applications to Feynman path integrals

We consider the case where $B = C_1[0,T], H = C_1'[0,T]$ and ν is Wiener measure m_w. Let S be the operator on $C_1'[0,T]$ defined by

$$Sf(\tau) = \int_0^\tau f(u)du. \tag{19}$$

Then S is a bounded linear operator and the adjoint operator S^* of S is given by

$$S^*f(\tau) = f(T)\tau - \int_0^\tau f(u)du. \tag{20}$$

The operator $A = S^*S$ is a self-adjoint trace class operator on $C_1'[0,T]$ given by

$$\begin{aligned}Af(\tau) &= S^*\left(\int_0^\tau f(u)du\right) \\ &= \tau \int_0^T f(u)du - \int_0^\tau \int_0^s f(u)duds \\ &= \int_0^T \min(\tau, s)f(s)ds. \end{aligned} \tag{21}$$

Furthermore $\langle f, Ag\rangle = \langle Sf, Sg\rangle = \int_0^T f(s)g(s)ds$ for all $f, g \in C_1'[0,T]$ and so A is positive definite, i.e.$\langle f, Af\rangle \geq 0$ for all $f \in C_1[0,T]$.

The real-valued function Z on $[0,T] \times C_1[0,T]$ defined by

$$Z(t,x) \equiv z(t) = x(t) - \frac{t}{T}x(T)$$

is called a *pinned Wiener process* on $(C_1[0,T], \mathcal{B}(C_1[0,T]), m_w)$ and $z(0) = 0$ and $z(T) = 0$. This process $\{z(t) : t \in [0,T]\}$ is uniquely determined by the mean function $E[z(t)] = 0$ for every $t \in [0,T]$ and the covariance function $E[z(s), z(t)] = k(s,t) = \min\{s,t\} - \frac{st}{T}$. Let A be the integral operator defined as above. Then it can be shown that the operator $D = (I-Q)A(I-Q)$ on $C_1'[0,T]$ is expressed by

$$Df(s) = \int_0^T k(s,t)\big(f(t) - \frac{t}{T}f(T)\big)\,dt, \qquad s \in [0,T], \qquad f \in C_1'[0,T].$$

and that the eigenvectors $\{e_n\}$ and the eigenvalues $\{\alpha_n\}$ of the operator D are given by

$$\alpha_n = \frac{T^2}{n^2\pi^2} \quad \text{and} \quad e_n(s) = \frac{\sqrt{2T}}{n\pi}\sin\frac{n\pi}{T}s. \tag{22}$$

LEMMA 5.0.1. *For any real α, $t \in [0,T]$,*

$$\sum_{n=1}^{\infty} \frac{T}{n^2\pi^2 + \alpha T^2}\cos\left(\frac{n\pi}{T}t\right) = \frac{\cosh\sqrt{\alpha}(T-t)}{2\sqrt{\alpha}\sinh\sqrt{\alpha}T} - \frac{1}{2\alpha T}. \tag{23}$$

Proof. To prove this lemma, we use a known result that

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{n^2-a^2} = \frac{1}{2a^2} - \frac{\pi\cos(ax)}{2a\sin(a\pi)}, \quad -\pi \le x \le \pi,$$

where a is not an integer. If we let $a = \frac{i\sqrt{\alpha}T}{\pi}$ and $x = \frac{\pi(T-t)}{T}$, then

$$\sum_{n=1}^{\infty}(-1)^n\frac{\pi^2}{n^2\pi^2+\alpha T^2}\cos\left(n\pi - \frac{n\pi}{T}t\right) = \frac{\pi^2\cosh\sqrt{\alpha}(T-t)}{2\sqrt{\alpha}T\sinh\sqrt{\alpha}T} - \frac{\pi^2}{2\alpha T^2}.$$

Hence we obtain

$$\sum_{n=1}^{\infty}\frac{T}{n^2\pi^2+\alpha T^2}\cos\left(\frac{n\pi}{T}t\right) = \frac{\cosh(\sqrt{\alpha}(T-t))}{2\sqrt{\alpha}\sinh\sqrt{\alpha}T} - \frac{1}{2\alpha T}.$$

□

LEMMA 5.0.2. *For a real number α, let*

$$R(s,t,\alpha) = \sum_{n=1}^{\infty}\frac{1}{1+\alpha\alpha_n}e_n(s)e_n(t), \quad s,t \in [0,T]$$

where α_n and e_n are as above. Then for each $t \in [0,T]$

$$R(s,t,\alpha) = \begin{cases} \dfrac{\sinh\sqrt{\alpha}(T-t)\sinh\sqrt{\alpha}s}{\sqrt{\alpha}\sinh\sqrt{\alpha}\,T}, & 0 \le s \le t; \\[2ex] \dfrac{\sinh\sqrt{\alpha}(T-s)\sinh\sqrt{\alpha}t}{\sqrt{\alpha}\sinh\sqrt{\alpha}\,T}, & t \le s \le T. \end{cases}$$

Proof. Using (22) and Lemma 5.0.1 we have

$$\begin{aligned}
&R(s,t,\alpha)\\
&= \sum_{n=1}^{\infty} \frac{n^2\pi^2}{n^2\pi^2+\alpha T^2}\,\frac{2T}{n^2\pi^2}\sin\left(\frac{n\pi}{T}s\right)\sin\left(\frac{n\pi}{T}t\right)\\
&= \sum_{n=1}^{\infty} \frac{T}{n^2\pi^2+\alpha T^2}\left[\cos\frac{n\pi}{T}(s-t)-\cos\frac{n\pi}{T}(s+t)\right]\\
&= \frac{1}{2\sqrt{\alpha}\sinh\sqrt{\alpha}\,T}\left[\cosh\sqrt{\alpha}(T-|s-t|)-\cosh\sqrt{\alpha}(T-|s+t|)\right]\\
&= \begin{cases} \dfrac{\sinh\sqrt{\alpha}\,(T-t)\,\sinh\sqrt{\alpha}\,s}{\sqrt{\alpha}\sinh\sqrt{\alpha}\,T}, & s\le t;\\[2mm] \dfrac{\sinh\sqrt{\alpha}\,(T-s)\,\sinh\sqrt{\alpha}\,t}{\sqrt{\alpha}\sinh\sqrt{\alpha}\,T}, & s\ge t. \end{cases}
\end{aligned}$$

□

For the partition $0=t_0<t_1<\cdots<t_n=T$, let $g_i\in C_1'[0,T]$ be defined by

$$g_i(\tau)=\frac{1}{\sqrt{t_i-t_{i-1}}}\int_0^{\tau} 1_{[t_{i-1},t_i)}(u)du,\quad i=1,2,\dots,n. \tag{24}$$

Then $\{g_1,g_2,\dots,g_n\}$ is an orthonormal set in $C_1'[0,T]$ and

$$(g_i,x)\tilde{}=\frac{1}{\sqrt{t_i-t_{i-1}}}(x(t_i)-x(t_{i-1})).$$

We note that for $x\in C_1[0,T]$, $(x(t_1),x(t_2),\dots,x(t_n))=(\xi_1,\xi_2,\dots,\xi_n)$ if and only if $(g_j,x)\tilde{}=(t_j-t_{j-1})^{-\frac{1}{2}}(\xi_j-\xi_{j-1})$ for all $j=1,2,\dots,n$, where $\xi_0=0$.

THEOREM 5.0.3. *Let F be measurable function on $C_1[0,T]$ defined by*

$$F(x)=\exp\left\{-\frac{a}{2}i\int_{t_1}^{T}x^2(s)\,ds\right\}\cdot\int_{C_1'[0,T]}\exp\left\{ib\int_{t_1}^{T}h(s)\,x(s)\,ds\right\}d\sigma(h).$$

Then for all $(\xi_1,\xi_2)\in\mathbb{R}^2$, $E^{\mathrm{anf}_q}[F\,|\,X=(\xi_1,\xi_2)]$ exists and is given by the formula

$$\begin{aligned}
&E^{\mathrm{anf}_q}[F\,|X=(\xi_1,\xi_2)]\\
&=\left|\frac{\sin\sqrt{\frac{a}{q}}(T-t_1)}{\sqrt{\frac{a}{q}}(T-t_1)}\right|^{-\frac{1}{2}}\exp\left\{\frac{\pi i}{2}\left[\frac{\sqrt{\frac{a}{q}}(T-t_1)}{\pi}\right]\right\}
\end{aligned}$$

$$\times \exp\left\{ -\frac{ia}{2\sqrt{-\frac{a}{q}}}(\xi_2^2+\xi_1^2)\coth\sqrt{-\frac{a}{q}}(T-t_1) \right.$$

$$\left. +\frac{ia}{\sqrt{-\frac{a}{q}}}\cdot\frac{\xi_2\xi_1}{\sinh\sqrt{-\frac{a}{q}}(T-t_1)} - \frac{iq}{2}\frac{(\xi_2-\xi_1)^2}{T-t_1} \right\}$$

$$\times \int_{C_1'[0,T]} \exp\left\{ ib\int_{t_1}^{T} \frac{\xi_2\sinh\sqrt{-\frac{a}{q}}(t-t_1)+\xi_1\sinh\sqrt{-\frac{a}{q}}(T-t)}{\sinh\sqrt{-\frac{a}{q}}(T-t)} h(t)dt \right.$$

$$-\frac{ib^2}{q}\int_{t_1}^{T}\int_{t_1}^{t} \frac{\sinh\sqrt{-\frac{a}{q}}(T-t)\sqrt{-\frac{a}{q}}\sinh\sqrt{-\frac{a}{q}}(s-t_1)}{\sqrt{-\frac{a}{q}}\sinh\sqrt{-\frac{a}{q}}(T-t_1)}\cdot$$

$$\left. h(s)\,h(t)\,ds\,dt \right\} d\sigma(h).$$

Proof. By using the integral operator A in (21), we can express F on classical Wiener space $C_1[0,T]$ as a function G on abstract Wiener space B which is given by

$$G(x) = \exp\left\{ -\frac{a}{2} i (x, A\tilde{x)} \right\} \int_H e^{i(bAh,\tilde{x)}} d\sigma(h).$$

If g_1 and g_2 are taken as

$$g_i(\tau) = \frac{1}{\sqrt{t_i - t_{i-1}}}\int_0^{\tau} 1_{[t_{i-1},t_i)}(u)du, \quad i=1,2$$

where $t_0 = 0$, $t_2 = T$. Then we have

$$E^{\mathrm{anf}_q}\Big[F\,\Big|\,X=(\xi_1,\xi_2)\Big] = E^{\mathrm{anf}_q}\left[G\,\middle|\,(g_1,\tilde{x)} = \frac{\xi_1}{\sqrt{t_1}},\ (g_2,\tilde{x)} = \frac{\xi_2-\xi_1}{\sqrt{T-t_1}}\right].$$

By Theorem 4.1.1, we have

$$E^{\mathrm{anf}_q}\left[G\,\middle|\,(g_1,\tilde{x)} = \frac{\xi_1}{\sqrt{t_1}},\ (g_2,\tilde{x)} = \frac{\xi_2-\xi_1}{\sqrt{T-t_1}}\right]$$

$$= \left|\det\left(I-\frac{a}{q}D\right)\right|^{-\frac{1}{2}} \exp\left\{-\frac{\pi i}{2}\mathrm{Ind}\left(I-\frac{a}{q}D\right)\right\}\exp\left\{-\frac{a}{2}i\langle[\vec{\xi}],A[\vec{\xi}]\rangle\right\}$$

$$\times \int_H \exp\left\{ i\langle bAh,[\vec{\xi}]\rangle - \frac{i}{2q}\left\langle\left(I-\frac{a}{q}D\right)^{-1}(I-Q)(aA[\vec{\xi}]-bAh),\right.\right.$$

$$(I-Q)(aA[\vec{\xi}]-bAh)\Big\rangle\Big\}\, d\sigma(h)$$

where $[\vec{\xi}](\tau)=\frac{\xi_2-\xi_1}{T-t_1}\tau+\xi_1,\quad 0\le\tau\le T-t_1$ and $D=(I-Q)A(I-Q)$ is the operator on $C_1'[0,T-t_1]$. First we observe that

$$\begin{aligned}\Big|\det\Big(I-\frac{a}{q}D\Big)\Big|^{-\frac12} &= \Big|\prod_{n=1}^{\infty}1-\frac{a}{q}\frac{(T-t_1)^2}{(n\pi)^2}\Big|^{-\frac12}\\ &= \Big|\prod_{n=1}^{\infty}1-\Big(\sqrt{\frac{a}{q}}\Big)^2\cdot\frac{(T-t_1)^2}{(n\pi)^2}\Big|^{-\frac12}\\ &= \left|\frac{\sin\sqrt{\frac{a}{q}}(T-t_1)}{\sqrt{\frac{a}{q}}(T-t_1)}\right|^{-\frac12}\end{aligned}$$

and

$$\exp\Big\{-\frac{\pi i}{2}\mathrm{Ind}\Big(I-\frac{a}{q}D\Big)\Big\}=\exp\left\{-\frac{\pi i}{2}\left[\frac{\sqrt{\frac{a}{q}}(T-t_1)}{\pi}\right]\right\}.$$

By the simple computation, the followings are obtained

$$\begin{aligned}-\frac{a}{2}\,i\langle[\vec{\xi}],A[\vec{\xi}]\rangle &= -\frac{a}{2}\,i\int_0^{T-t_1}[\vec{\xi}]^2(s)ds\\ &= \frac{a}{6}i(T-t_1)(\xi_1^2+\xi_2\xi_1+\xi_2^2)\end{aligned}\tag{25}$$

and

$$i\langle bAh,[\vec{\xi}]\rangle=ib\int_0^{T-t_1}h(s+t_1)[\vec{\xi}](s)ds.\tag{26}$$

Moreover we can get

$$\begin{aligned}&-\frac{i}{2q}\Big\langle\Big(I-\frac{a}{q}D\Big)^{-1}(I-Q)(aA[\vec{\xi}]-bAh),(I-Q)(aA[\vec{\xi}]-bAh)\Big\rangle\\ &= -\frac{i}{2q}\sum_{j=1}^{\infty}\frac{\langle k,e_j\rangle^2}{1-\frac{a}{q}\alpha_j}\\ &= -\frac{i}{2q}\int_0^{T-t_1}\int_0^{T-t_1}R\Big(s,t,-\frac{a}{q}\Big)\Big(a[\vec{\xi}](s)-bh(s+t_1)\Big)\\ &\quad\Big(a[\vec{\xi}](t)-bh(t+t_1)\Big)\,dsdt\end{aligned}$$

$$= -\frac{ia^2}{2q}\int_0^{T-t_1}\int_0^{T-t_1} R\left(s,t,-\frac{a}{q}\right)[\vec{\xi}](s)[\vec{\xi}](t)dsdt$$
$$+\frac{iab}{q}\int_0^{T-t_1}\int_0^{T-t_1} R\left(s,t,-\frac{a}{q}\right)[\vec{\xi}](s)h(t+t_1)dsdt$$
$$-\frac{ib^2}{2q}\int_0^{T-t_1}\int_0^{T-t_1} R\left(s,t,-\frac{a}{q}\right)h(s+t_1)h(t+t_1)dsdt$$

where $k = (I-Q)(aA[\vec{\xi}] - bAh)$. But equation (25) and simple computation, we have

$$-\frac{ia^2}{2q}\int_0^{T-t_1}\int_0^{T-t_1} R\left(s,t,-\frac{a}{q}\right)[\vec{\xi}](s)[\vec{\xi}](t)dsdt - \frac{ia}{2}\langle[\vec{\xi}], A[\vec{\xi}]\rangle$$
$$= -\frac{ia^2}{2q}\int_0^{T-t_1}\int_0^{T-t_1} R\left(s,t,-\frac{a}{q}\right)[\vec{\xi}](s)[\vec{\xi}](t)dsdt$$
$$-\frac{ia}{2}\int_0^{T-t_1}[\vec{\xi}]^2(s)ds$$
$$= -\frac{ia}{2\sqrt{-\frac{a}{q}}}(\xi_2^2+\xi_1^2)\coth\left(\sqrt{-\frac{a}{q}}(T-t_1)\right)$$
$$+\frac{ia}{\sqrt{-\frac{a}{q}}}\cdot\frac{\xi_2\xi_1}{\sinh\sqrt{-\frac{a}{q}}(T-t_1)} - \frac{iq}{2}\frac{(\xi_2-\xi_1)^2}{T-t_1}.$$

Also, by equation (26) and simple computation, we can get

$$\frac{iab}{q}\int_0^{T-t_1}\int_0^{T-t_1} R\left(s,t,-\frac{a}{q}\right)h(s+t_1)[\vec{\xi}](t)dsdt + i\langle bAh, [\vec{\xi}]\rangle$$
$$= \frac{iab}{q}\int_0^{T-t_1}\int_0^{T-t_1} R\left(s,t,-\frac{a}{q}\right)h(s+t_1)[\vec{\xi}](t)dsdt$$
$$+ ib\int_0^{T-t_1}[\vec{\xi}](t)h(t+t_1)dt$$
$$= ib\int_{t_1}^{T}\frac{\xi_2\sinh\sqrt{-\frac{a}{q}}(t-t_1)+\xi_1\sinh\sqrt{-\frac{a}{q}}(T-t)}{\sinh\sqrt{-\frac{a}{q}}(T-t_1)}h(t)dt$$

and

$$-\frac{ib^2}{2q}\int_0^{T-t_1}\int_0^{T-t_1} R\left(s,t,-\frac{a}{q}\right)h(s+t_1)h(t+t_1)dtds$$

$$= -\frac{ib^2}{q}\int_{t_1}^{T}\int_{t_1}^{t}\frac{\sinh\sqrt{-\frac{a}{q}}(T-t)\sinh\sqrt{-\frac{a}{q}}(s-t_1)}{\sqrt{-\frac{a}{q}}\sinh\sqrt{-\frac{a}{q}}(T-t_1)}h(s)\,h(t)\,ds\,dt.$$

By combining all the computation as above, we obtain the desired results. □

COROLLARY 5.0.4. [13, p.64] *Let $T-t_1<\frac{\pi}{\omega}$. The fundamental solution to the Schrödinger equation with the potential $V(t,\xi)=\frac{m\omega^2}{2}\xi^2+f(t)\xi$ where $f\in C_1'[0,T]$:*

$$\frac{\partial U}{\partial t}=\frac{i\hbar}{2m}\frac{\partial^2 U}{\partial \xi^2}+\frac{i}{\hbar}\left(-\frac{m\omega^2}{2}\xi^2+f(t)\xi\right)U \tag{27}$$

with the initial state $U(0,\xi)=\psi(\xi)$, is given by

$$\begin{aligned}
&K(\xi_2,T;\xi_1,t_1)\\
&=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega(T-t_1)}}\exp\left\{\frac{im\omega}{2\hbar\sin\omega(T-t_1)}\right.\\
&\times\left[\cos\omega(T-t_1)(\xi_2^2+\xi_1^2)-2\xi_2\xi_1+\frac{2\xi_2}{m\omega}\int_{t_1}^{T}f(t)\sin\omega(t-t_1)dt\right.\\
&+\frac{2\xi_1}{m\omega}\int_{t_1}^{T}f(t)\sin\omega(T-t)dt\\
&\left.\left.-\frac{2}{m^2\omega^2}\int_{t_1}^{T}\int_{t_1}^{t}f(s)f(t)\sin\omega(T-t)\sin\omega(s-t_1)dsdt\right]\right\}.
\end{aligned}\tag{28}$$

Proof. We first note that if we let $q=\frac{m}{\hbar}$, $a=\frac{m\omega^2}{\hbar}$, $b=\frac{1}{\hbar}$, the function F on $C_1[0,T]$ given in Theorem 5.0.3 is expressed as

$$F(x)=\exp\left\{-\frac{im\omega^2}{2\hbar}\int_{t_1}^{T}x^2(s)\,ds+\frac{i}{\hbar}\int_{t_1}^{T}f(s)\,x(s)\,ds\right\}.$$

We also note that since $T-t_1<\frac{\pi}{\omega}$, we see that ind $\big(\det\big(I-\omega^2 D\big)\big)=1$. With these notes and Theorem 5.0.3, we can see that for $(\xi_2,T,\xi_1,t_1,q)\in\mathbb{R}\times[0,T]\times\mathbb{R}\times[0,T]\times(\mathbb{R}-\{0\})$ with $t_1<T$,

$$\begin{aligned}
&\Gamma(\xi_2,T;\xi_1,t_1;-iq)\\
&=\sqrt{\frac{q}{2\pi i(T-t_1)}}\exp\left\{\frac{iq(\xi_2-\xi_1)^2}{2(T-t_1)}\right\}E^{\mathrm{anf}_q}[F|X=(\xi_1,\xi_2)]
\end{aligned}\tag{29}$$

is equal to (28). But in view of Corollary 4.1.5, (29) is the fundamental solution to the Schrödinger equation (27). Hence we complete the proof. □

REMARK. The Maslov index does not appear in the expression (27) which is given in Feynman and Hibbs' book [13]. This is because they consider values of $T - t_1 < \frac{\pi}{\omega}$.

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