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FIXED-POINT THEORY FOR k-SET CONTRACTION

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ABSTRACT. In this note, we get some generalized KKM theorems for the k-set contraction mapping on the nearly-convex sets.

1. Introduction and preliminaries

Recently there are appeared some results on fixed point so-called Kakutani factorizable multifunctions defined on convex sets. In this chapter, we invoke non-convexity of constraint regions in place of convexity.

We digress briefly to list some notations and review some definitions. Suppose that X is a subset of a Hausdorff topological vector space E, we introduce a new class of non-convex sets. A nonempty subset X of a Hausdorff topological vector space E is said to be nearly-convex(Chu and Wu [4]) if for every compact subset A of X and every neighborhood V of the origin 0 of E, there is a continuous mapping $h : A \to X$ such that $x - h(x) \in V$ for all $x \in A$ and $h(A)$ is contained in some convex subset of X. It is clear, every convex set is nearly-convex, but the converse is not true in general.

Throughout this paper, E will denote a Hausdorff locally convex linear topological space and 2^E will denote the family of nonempty subsets of E, while $B(E)$ is the family of nonempty bounded subsets. $T : X \to 2^E$ is said to be closed if the graph $\mathcal{G}_T = \{(x, y) \in X \times E | y \in Tx, \forall x \in X\}$ is a closed subset of $X \times E$.

Let $\wp = \{P | P$ is a family of seminorms which determines the topology on $E\}$. Let $\mathcal{R}^+$ be the set of all nonnegative real numbers. A mapping $\Phi : B(E) \to \mathcal{R}^+$ is called a measure of noncompactness (see, [2]) provided the following conditions hold:

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(i) $\Phi(\overline{co}(\Omega)) = \Phi(\Omega)$ for each $\Omega \in B(E)$, where $\overline{co}(\Omega)$ denotes the closure of the convex hull of Ω,
(ii) $\Phi(\Omega) = 0$ if and only if Ω is precompact,
(iii) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$, for each $A, B \in B(E)$, and
(iv) $\Phi(\lambda\Omega) = \lambda\Phi(\Omega)$, for each $\lambda \geq 0$, $\Omega \in B(E)$.

The above notion is a generalization of the set measure of noncompactness; if $\{p : p \in P\}$, $P \in \wp$ is a family of seminorms which determines the topology on E, then for each $p \in P$ and $\Omega \subset E$, we define the set-measure of noncompactness $\alpha_p : B(E) \to \mathcal{R}^+$ by $\alpha_p(\Omega) = \inf\{\varepsilon > 0 : \Omega$ can be covered by a finite number of sets and each p-diameter of the sets is less than $\varepsilon\}$, where the p-diameter of $A = \sup\{p(x-y) : x, y \in A\}$ for A is a subset of Ω.

A mapping $T : X \to 2^E$ is said to be k-set contraction if there exists $P \in \wp$ such that for each $p \in P$, $\alpha_p(T(\Omega)) \leq k\alpha_p(\Omega)$ with $k \in (0,1)$ for each bounded subset Ω of X and $T(X)$ is bounded.

We generalized the KKM property to the following form for a nearly-convex set X. Assume that X is a nearly-convex subset of a linear space and Y is a topological space. If $T, S : X \to 2^Y$ are two set-valued mappings such that $T(coA \cap X) \subset S(A)$ for each finite subset A of X, then we call S a generalized KKM mapping with respect to T, where $co(A)$ denotes the convex hull of A . Let $T : X \to 2^Y$ be a set-valued mapping such that if $S : X \to 2^Y$ is a generalized KKM mapping with respect to T then the family $\{\overline{Sx} : x \in X\}$ has the finite intersection property (where $\overline{Sx}$ denotes the closure of Sx), then we say that T has the KKM property. Denote

$$\mathrm{KKM}(X,Y) = \{T : X \to 2^Y \mid T \text{ has the KKM property}\}.$$

REMARK. Generalized KKM mappings were first introduced by Park [3], and followed by some others.

We conclude the differences between the convex sets and the nearly-convex sets as follows:

PROPOSITION 1. *Let X be an nearly-convex subset of a Hausdorff topological vector space. Then $\overline{X}$ is convex.*

Proof. Assume that $\overline{X}$ is not convex, then there exist $x_1, x_2, \ldots, x_n \in \overline{X}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in (0,1)$, $\sum_{i=1}^n \lambda_i = 1$ such that $x_\lambda = \sum_{i=1}^n \lambda_i x_i \notin \overline{X}$. Since $\overline{X}^c$ is a neighborhood of x_λ, hence there exists a symmetric convex neighborhood V of the origin 0 such that $x_\lambda + V \subset \overline{X}^c$. Let V' be a symmetric convex neighborhood of the origin 0 such that $V' + V' \subset V$, let $y_1, y_2, \ldots, y_n \in X$ such that $y_i \in x_i + V'$, $i = 1, 2, \ldots, n$, and let $y_\lambda = \sum_{i=1}^n \lambda_i y_i$ and $A = \{y_1, y_2, \ldots, y_n\}$. Then, by the nearly-convexity of X, there is a continuous mapping $h : A \to X$ such that $z - h(z) \in V'$ for all $z \in A$ and $h(A)$ is contained in some convex subset of X. let $z_\lambda = \sum_{i=1}^n \lambda_i h(y_i) \in X$. Then $\sum_{i=1}^n \lambda_i h(y_i) \in \sum_{i=1}^n \lambda_i (y_i + V') = (\sum_{i=1}^n \lambda_i y_i) + V' = y_\lambda + V'$, $y_\lambda + V' \subset [\sum_{i=1}^n \lambda_i (x_i + V')] + V' = (\sum_{i=1}^n \lambda_i x_i) + V' + V' = x_\lambda + V$, and hence $z_\lambda \in x_\lambda + V$. This implies $z_\lambda \in \overline{X}^c$. We get a contraction, and hence $\overline{X}$ is convex. □

REMARK. If X is a closed nearly-convex subset of a Hausdorff topological vector space, then X is convex. But, if X is an nearly-convex subset of a Hausdorff topological vector space, then the conclusion "X is convex" does not hold.

PROPOSITION 2. *Let X and Y be two nonempty subsets of a Hausdorff topological vector space E. If X is nearly-convex and Y is open convex, then $X \cap Y$ is nearly-convex.*

Proof. Suppose K is a compact subset of $X \cap Y$, then by the fact that $K \subset Y$ and Y is open and convex, there is an open neighthood U of the origin 0 of E such that $K + U \subset Y$. For any neighthood V of the origin 0 of E with $V \subset U$, since $K \subset X$ and X is nearly-convex, there is a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in some convex subset of X. Since $h(K) \subset co(h(K)) \subset X$ and $h(K) \subset K+V$, we get $h(K) \subset (K+V) \cap X \subset (K+U) \cap X$, and $h(K) \subset co(h(K)) \subset Y$. So $X \cap Y$ is nearly-convex. □

2. Main results

We now concern some fixed point theorems with domain as a nearly-convex subset of a Hausdorff locally convex space E for a k-set contraction map T, which may not be a compact map.

The following Lemma will play important role.

LEMMA 3. *Let X be a nonempty subset of a locally convex space, Y and Z two topological spaces. Then,*

(i) *$fT \in KKM(X, Z)$, whenever $T \in KKM(X, Y)$ and $f \in C(Y, Z)$*
(ii) *$T|_D \in KKM(D, Y)$, whenever $T \in KKM(X, Y)$ and D is a nonempty subset of X.*

Proof. (i) Let $F : X \to 2^Z$ be a generalized KKM mapping with respect to fT such that Fx is closed for each $x \in X$. Then for any $\{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$, $fT(co(\cup_{i=1}^n x_i) \cap X) \subset \cup_{i=1}^n F x_i$. So $T(co(\cup_{x=1}^n x_i)$

$\cap X) \subset \cup_{i=1}^{n} f^{-1}Fx_i$, which says that $f^{-1}F$ is a generalized KKM mapping with respect to T. Since $T \in KKM(X,Y)$, the family $\{f^{-1}Fx : x \in X\}$ has the finite intersection property, and so does the family $\{Fx : x \in X\}$. This shows that $fT \in KKM(X,Z)$.

(ii) Let $F : D \to 2^Y$ be a generalized KKM mapping with respect to $T|_D$. Then $T|_D(co(A) \cap D) \subset F(A)$, for any finite subset A of D. Define $F' : X \to 2^Y$ by

$$F'(x) = \begin{cases} F(x) & x \in D, \\ Y & x \in X \setminus D. \end{cases}$$

It is clear that for any finite subset B of X, we have $T(co(B) \cap X) \subset F'(B)$. Indeed, we have the followings.

(1) If $B \not\subseteq D$, then there exists some $x \in B \setminus D$, and hence $F'(x) = Y$, so the result is obvious.
(2) If $B \subset D$, since F be a generalized KKM mapping with respect to $T|_D$, the inclusion is true.

Thus F' is a generalized KKM mapping with respect to T. Since $T \in KKM(X,Y)$, hence the family $\{\overline{F'x} : x \in X\}$ has finite intersection property, and hence the family $\{\overline{Fx} : x \in D\}$. So $T|_D \in KKM(D,Y)$. □

The main result of this paper is the following fixed point theorem for the k-set contraction maps.

THEOREM 4. *Let X be a bounded nearly-convex subset of a locally convex space E. If $T \in KKM(X,X)$ is k-set contraction, $0 \leqslant k < 1$ and closed with $\overline{TX} \subset X$, then T has a fixed point in X.*

Proof. Let $\mathcal{N} = \{U_\beta | \beta \in \Lambda\}$ be a local base of E such that U_β is symmetric, open, and convex for each $\beta \in \Lambda$, and let $V \in \mathcal{N}$. Since T is k-set contraction, there exists $P \in \wp$ such that for each $p \in P$, we have $\alpha_p(T(\Omega)) \leq k\alpha_p(\Omega)$ for each subset Ω of X. Take $y \in X$. Let $X_0 = X$, $X_1 = co(T(X_0) \cup \{y\}) \cap X$, and $X_{n+1} = co(T(X_n) \cup \{y\}) \cap X$, for each $n \in N$. Then

(1) $X_{n+1} \subset X_n$, for each $n \in N$,
(2) $T(X_n) \subset X_{n+1}$, for each $n \in N$, and
(3) $\alpha_p(X_{n+1}) \leq \alpha_p(T(X_n)) \leq k\alpha_p(X_n)) \leq \cdots \leq k^n\alpha_p(X)$, for each $n \in N$.

Let $Y = \cap_{i=1}^{\infty} X_i$. Then $T(Y) \subset Y \subset X$ and $\overline{Y}$ is a nonempty compact set, since $y \in Y$ and $\alpha_p(Y) = 0$. Since $\overline{TX} \subset X$ and $T(Y) \subset Y \subset X$, we have $\overline{TY} \subset X$ and $\overline{TY}$ is a compact subset of X. Since $\overline{TY}$ is a compact subset of the nearly-convex set X, there is a continuous mapping h :

$\overline{TY} \to X$ such that $x - h(x) \in V$ for all $x \in \overline{TY}$ and $h(\overline{TY})$ is contained in some convex subset of X.Let $Z = co(h(\overline{TY})$, then $h(\overline{TY}) \subset Z \subset X$. Since $T \in KKM(X,X)$ and Y is a nonempty subset of X, by Lemma 2, we have $T \in KKM(Y,X)$. Next we put $F = h \circ T|_Y$, we have $F \in KKM(Y,Z)$, and F is closed since T is closed, h is continuous, and Y is compact. We now claim that for each $\beta \in \Lambda$, there is an $x_\beta \in Y$ such that $(x_\beta + U_\beta + V) \cap Fx_\beta \neq \phi$. If the above statement is not true, then there is an $U \in \mathcal{N}$ such that $(x+U+V) \cap Fx = \phi$, for all $x \in Y$. Let $K = \overline{F(Y)} = \overline{h(T(Y))} \subset Z$. Then $K = \overline{h(T(Y))} \subset \overline{h(\overline{T(Y)})} = h(\overline{T(Y)})$ and K is a compact subset of Z. Define $G : Y \to 2^Z$ by

$$G(x) = K \backslash (x + U + V) \qquad \text{for each } x \in Y$$

Then

(1) Gx is compact, for each $x \in Y$, and
(2) G is a generalized KKM mapping with respect to F.

To prove (2), we use the contradiction. Assume that there is $\{x_1, x_2, \ldots, x_n\} \in \langle Y \rangle$ such that $F(co\{x_1, x_2, \ldots, x_n\} \cap Y) \not\subseteq \cup_{i=1}^n Gx_i$. Then there exists $\mu \in co\{x_1, x_2, \ldots, x_n\} \cap Y$ and $\nu \in F(\mu) \subset \overline{F(Y)} = K$ such that $\nu \notin \cup_{i=1}^n Gx_i$. Hence $\nu \in x_i + U + V$, for each $i \in \{1, 2, \ldots, n\}$, and hence $\nu \in z + U + V$, for any $z \in co\{x_1, x_2, \ldots, x_n\}$. In particular, $\nu \in \mu + U + V$. Noting that $(\mu + U + V) \cap F(\mu) = \phi$, we conclude that $\nu \notin F(\mu)$. It is a contradiction. Hence G is a generalized KKM mapping with respect to F.

Since $F \in KKM(Y,Z)$, the family $\{Gx : x \in Y\}$ has finite intersection property, and so we conclude that $\cap_{x \in Y} Gx \neq \phi$. Choose $\eta \in \cap_{x \in Y} G(x) \subset K \subset h(\overline{T(Y)})$, then $\eta \in K \backslash (x + U + V)$, for all $x \in Y$. Since $\eta \in h(\overline{T(Y)}) \subset \overline{T(Y)} + V \subset \overline{Y} + V$, hence there is an $x_0 \in Y$ such that $\eta \in x_0 + U + V$. But $\eta \in K \backslash (x_0 + U + V)$, we have reached a contradiction. Therefore, we have proved that for each $\beta \in \Lambda$, there is $x_\beta \in Y$ such that $(x_\beta + U_\beta + V) \cap Fx_\beta \neq \phi$. Let $y_\beta \in (x_\beta + U_\beta + V) \cap Fx_\beta$. Since $\{y_\beta\} \subset K$ and K is compact, we may assume that $\{y_\beta\}$ converges to some $y_\nu \in K$, and since $\{x_\beta\} \subset \overline{Y}$, we assume $\{x_\beta\}$ converges to x_ν. The closeness of F implies that $(x_\nu, y_\nu) \in \mathcal{G}_F$, so we have $y_\nu \in x_\nu + \overline{V}$, and $y_\nu \in F(x_\nu) = h(T(x_\nu))$. Choose $z_\nu \in T(x_\nu)$ such that $y_\nu = h(z_\nu)$. Noting that $z_\nu - h(z_\nu) \in V$, we obtain that $z_\nu \in h(z_\nu) + V = y_\nu + V \in x_\nu + V + \overline{V} \subset x_\nu + V + V + V$, and $T(x_\nu) \cap (x_\nu + V + V + V) \neq \phi$, for any $V \in \mathcal{N}$, which just as before, implies T has a fixed point $x \in X$. □

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