

LOWER BOUNDS OF SIGNED DOMINATION NUMBER OF A GRAPH

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ABSTRACT. Recently, Zhang et al. [6] obtained some lower bounds of the signed domination number of a graph. In this paper, we obtain some new lower bounds of the signed domination number of a graph which are sharper than those of them.

1. Introduction

Throughout this paper, let G be a finite connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *neighborhood* of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of vertices adjacent to v . The *closed neighborhood* of v denoted by $N[v] = N(v) \cup \{v\}$ and the degree of v in G by $d(v)$. If S is a subset of V , we set $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v] = S \cup N(S)$. We use $|X|$ for the cardinality of a set X . Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of G , respectively, and $\lceil x \rceil$ denote the minimum integer not less than x . For a graph $G = (V, E)$, let V_o and V_e be the set of vertices having odd degree and even degree, respectively. And, for a function $f : V \rightarrow \{-1, 1\}$, the *weight* of f is $w(f) = \sum_{v \in V} f(v)$. For a vertex v in V , we define $f[v] = \sum_{u \in N[v]} f(u)$. A *signed dominating function* of G is a function $f : V \rightarrow \{-1, 1\}$ such that $f[v] \geq 1$ for all $v \in V$. The signed domination number $\gamma_s(G)$ of G is the minimum weight of a signed dominating function on G . We call a signed dominating function having weight $\gamma_s(G)$ a γ_s -function of G . Notice that if we change the set $\{-1, 1\}$ of the above definition to $\{0, 1\}$, then f is the dominating function and the corresponding value is the domination number. Dunbar

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et al. [2] gave a lower bound for the signed domination number for an r -regular graph G in Theorem 1. Zhang et al. [6] gave a lower bound for the signed domination number of a graph which is better than that of Theorem 1. In this paper, we obtain some different lower bounds which are sharper than that of Theorem 2.

THEOREM 1. ([2]) *For any r -regular graph G of order n , $\gamma_s(G) \geq \frac{n}{r+1}$.*

THEOREM 2. ([6]) *For any graph G of order n ,*

$$\gamma_s(G) \geq \frac{\delta(G) + 2 - \Delta(G)}{\delta(G) + 2 + \Delta(G)}n.$$

2. Some lower bounds

In this section we give our main results.

THEOREM 3. *For any graph G of order n , we have*

$$\gamma_s(G) \geq \frac{(\delta(G) + 2 - \Delta(G))n + 2n_o}{\delta(G) + 2 + \Delta(G)},$$

where n_o is the number of vertices having odd degree in G . In particular, if the degree of each vertex of G is odd, then

$$\gamma_s(G) \geq \frac{\delta(G) + 4 - \Delta(G)}{\delta(G) + 2 + \Delta(G)}n.$$

Proof. Let f be a γ_s -function of G and let m be the number of edges in G . Let $\mathcal{P}_f = \{v \in V | f(v) = 1\}$ and let $\mathcal{N}_f = \{v \in V | f(v) = -1\}$. It is clear that for any $v \in V_o$, $f[v] = \sum_{u \in N[v]} f(u) \geq 2$ and for any $v \in V_e$, $f[v] = \sum_{u \in N[v]} f(u) \geq 1$. This implies that

$$\sum_{v \in V} f[v] = \sum_{v \in V_o} f[v] + \sum_{v \in V_e} f[v] \geq 2|V_o| + |V_e| = n + n_o.$$

Note that

$$\sum_{v \in \mathcal{P}_f} d(v) - \sum_{v \in \mathcal{N}_f} d(v) = \sum_{v \in V} d(v) - 2 \sum_{v \in \mathcal{N}_f} d(v) = 2 \sum_{v \in \mathcal{P}_f} d(v) - \sum_{v \in V} d(v).$$

On the other hand,

$$\begin{aligned} \sum_{v \in V} f[v] &= \sum_{v \in V} f(v) + \sum_{v \in V} \sum_{u \in N(v)} f(u) \\ &= 2|\mathcal{P}_f| - n + \sum_{v \in V} f(v)d(v) \\ &= 2|\mathcal{P}_f| - n + \sum_{v \in \mathcal{P}_f} d(v) - \sum_{v \in \mathcal{N}_f} d(v). \end{aligned}$$

Since

$$\sum_{v \in V} d(v) - 2 \sum_{v \in \mathcal{N}_f} d(v) \leq 2m - 2(n - |\mathcal{P}_f|)\delta(G),$$

we have

$$(1) \quad n + n_0 \leq \sum_{v \in V} f[v] \leq 2|\mathcal{P}_f| - n + 2m - 2(n - |\mathcal{P}_f|)\delta(G).$$

Therefore

$$(2) \quad 2(\delta(G) + 1)(|\mathcal{P}_f| - n) \geq n_0 - 2m.$$

Since

$$2 \sum_{v \in \mathcal{P}_f} d(v) - \sum_{v \in V} d(v) \leq 2|\mathcal{P}_f|\Delta(G) - 2m,$$

we have

$$(3) \quad n + n_0 \leq \sum_{v \in V} f[v] \leq 2|\mathcal{P}_f| - n + 2|\mathcal{P}_f|\Delta(G) - 2m.$$

Therefore

$$(4) \quad 2(\Delta(G) + 1)|\mathcal{P}_f| \geq 2n + n_0 + 2m.$$

From (2) and (4), we have

$$|\mathcal{P}_f| \geq \frac{(2 + \delta(G))n + n_0}{\delta(G) + 2 + \Delta(G)}.$$

Since $\gamma_s(G) = 2|\mathcal{P}_f| - n$, it completes the proof. □

Notice that the lower bound in Theorem 3 is better than that of Theorem 2.

For a regular graph G , the Theorem 3 gives the following.

COROLLARY 1. ([2]) For any r -regular graph of order n , we have

$$\gamma_s(G) \geq \begin{cases} \frac{2n}{r+1} & \text{if } r \text{ is odd,} \\ \frac{n}{r+1} & \text{if } r \text{ is even.} \end{cases}$$

Theorem 3 gives an alternative proof for Theorem 1 in ([5]).

COROLLARY 2. ([5]) For any graph G of order n , we have

$$\gamma_s(G) \geq 2 \left[\max \left\{ n + \frac{n_o - 2m}{2(\delta(G) + 1)}, \frac{2n + n_o + 2m}{2(\Delta(G) + 1)} \right\} \right] - n,$$

where m is the number of edges in G and n_o is the number of vertices having odd degree in G .

Proof. Let f be a γ_s -function of G . From (2), (4) in the proof of Theorem 3, we get

$$|\mathcal{P}_f| \geq n + \frac{n_o - 2m}{2(\delta(G) + 1)}, \quad |\mathcal{P}_f| \geq \frac{2n + n_o + 2m}{2(\Delta(G) + 1)}.$$

The corollary comes from the fact that $\gamma_s(G) = 2|\mathcal{P}_f| - n$. □

Let $\Delta(G) = d_1 \geq d_2 \geq \dots \geq d_{n-1} \geq d_n = \delta(G)$ be the degree sequence of G . Then, we can obtain the following lower bound of $\gamma_s(G)$ which is related to the degree sequence of G .

THEOREM 4. For any graph G of order n , we have

$$\gamma_s(G) \geq 2 \left[\frac{2(n + m - s_t + td_{t+1}) + n_o}{2(d_{t+1} + 1)} \right] - n,$$

where m is the number of edges in G , n_o is the number of vertices having odd degree in G , $s_t = \sum_{i=1}^t d_i$ and $t = \left\lceil \max \left\{ n + \frac{n_o - 2m}{2(\delta(G) + 1)}, \frac{2n + n_o + 2m}{2(\Delta(G) + 1)} \right\} \right\rceil$.

Proof. We use the notations introduced in the proof of Theorem 3. In the proof of Corollary 2, we showed that $|\mathcal{P}_f| \geq t$. We note that

$$\begin{aligned} \sum_{v \in V} f[v] &= 2|\mathcal{P}_f| - n + \left(\sum_{v \in \mathcal{P}_f} d(v) - \sum_{v \in \mathcal{N}_f} d(v) \right) \\ &= 2|\mathcal{P}_f| - n + \left(2 \sum_{v \in \mathcal{P}_f} d(v) - \sum_{v \in V} d(v) \right). \end{aligned}$$

Since

$$2 \sum_{v \in \mathcal{P}_f} d(v) - \sum_{v \in V} d(v) \leq 2 \left(\sum_{i=1}^t d_i + (|\mathcal{P}_f| - t)d_{t+1} \right) - 2m,$$

we have

$$\sum_{v \in V} f[v] \leq 2|\mathcal{P}_f| - n + 2(s_t + (|\mathcal{P}_f| - t)d_{t+1}) - 2m.$$

By using the fact that $n + n_o \leq \sum_{v \in V} f[v]$ and this inequality, we have

$$|\mathcal{P}_f| \geq \frac{2(n + m - s_t + td_{t+1}) + n_o}{2(d_{t+1} + 1)}.$$

Now, the theorem results from this inequality and the fact that $\gamma_s(G) = 2|\mathcal{P}_f| - n$. □

If the lower bound of $|\mathcal{P}_f|$ in the proof of Theorem 4, say $\ell(t)$, is greater than t , we replace t in the Theorem 4 by $\ell(t)$. We can repeat this process until t and $\ell(t)$ become equal.

3. More on cubic graphs

In this section, we study the signed domination number for the cubic graphs. In the case of cubic graphs, the notion of signed domination can be interpreted in terms of 2-packing set. In this section, let G be a cubic graph of order n . A subset S of $V(G)$ is said to be a *2-packing set* of G if for any two distinct vertices s, s' in S , the distance $d(s, s')$ between s and s' is at least 3. Let $P_2(G) = \max\{|S| : S \text{ is 2-packing set of } G\}$ and call it the *2-packing number* of G . Since $|N[v]| = 4$ for all vertices v of V , $f[v] \geq 1$ if and only if the set $\mathcal{N}_f = \{u | f(u) = -1\}$ intersects $N[v]$ in at most one vertex for every signed function $f : V \rightarrow \{-1, 1\}$. Therefore a signed function $f : V \rightarrow \{-1, 1\}$ is signed dominating function if and only if \mathcal{N}_f is 2-packing set of G . Since $\gamma_s(G) = n - 2P_2(G)$, $\gamma_s(G) = n - 2P_2(G) \geq \frac{n}{2}$ by Corollary 1. O. Favaron [3] gave an upper bound for the signed domination number of cubic graph as follows.

THEOREM 5. ([3]) *Every connected cubic graph different from the Petersen graph satisfies*

$$\gamma_s(G) \leq \frac{3n}{4}.$$

The cubic graphs having diameter at most 2 can be characterized as follows.

COROLLARY 3. *Let G be a cubic graph. Then the diameter of G is at most 2 if and only if*

$$\gamma_s(G) = n - 2.$$

THEOREM 6. *Let G be cubic graph of order n . Then the following statements are equivalent.*

- (1) *The signed domination number $\gamma_s(G)$ is $\frac{n}{2}$.*
- (2) *The perfect domination number is $\frac{n}{4}$.*

Proof. If the signed domination number $\gamma_s(G)$ is $\frac{n}{2}$ and f is γ_s -function of G , \mathcal{N}_f is 2-packing set of G such that $|\mathcal{N}_f| = \frac{n}{4}$. Since $\sum_{v \in \mathcal{N}_f} |N[v]| = 4|\mathcal{N}_f| = n$ and $N[v] \cap N[v'] = \emptyset$ for v, v' in \mathcal{N}_f , each vertex of G except \mathcal{N}_f is adjacent to exactly one vertex in \mathcal{N}_f . Therefore \mathcal{N}_f is perfect dominating set of G and there is no perfect dominating set of G having fewer vertices. Hence, the perfect domination number of G is $\frac{n}{4}$. Conversely, let S be a perfect dominating set with $\frac{n}{4}$ vertices. Then S is also 2-packing set of G . It is clear that there is no 2-packing set with $\frac{n}{4} + 1$ vertices. Hence, the 2-packing number of G is $\frac{n}{4}$. Therefore the signed domination number $\gamma_s(G)$ is $\frac{n}{2}$. \square

Let G be a cubic graph of order n . If the perfect domination number of G is $\frac{n}{4}$, then the perfect domination number of every ℓ -fold covering graph \tilde{G} of G is $\ell \frac{n}{4}$. Hence, there are infinitely many cubic graphs whose signed domination number attain our lower bound. The following Theorem gives an upper bound for the signed domination number of some cubic graphs.

THEOREM 7. *Every connected cubic graph whose every vertex is contained in at least one triangle satisfies*

$$\gamma_s(G) \leq \frac{2n}{3}.$$

Proof. For $v \in V$, let $N_i[v]$ ($N_i[S]$ resp.) be the set of vertices at distances at most i from v (from some vertex of S resp.) We construct 2-packing set M of G as follows:

$M := u$ where u is arbitrary vertex; $V := V - N_2[M]$;
 while $V \neq \emptyset$ do
 chose $v \in N_3[M] \cap V$; $M := M \cup \{v\}$; $V := V - N_2[M]$.

When the process stops, M is 2-packing set of G . After choice of first vertex u , we eliminate at most 8 vertices since there is at least one triangle containing u . After each of the following choice v , we eliminate at most 6 vertices since at least one neighbor of v and one vertex at distance 2 from v have already been eliminated and there is at least one triangle containing v . Therefore $n \leq 8 + 6(|M| - 1)$ and thus $|M| \geq \frac{(n-2)}{6}$. If there exists some vertex u being contained to at least two triangles, we initialize the algorithm with the above vertex u . Then $|N_2(u)| \leq 6$ and get $n \leq 6 + 6(|M| - 1)$, thus $|M| \geq \frac{n}{6}$. If every vertex of G is contained to exactly one triangle, for any $u \in V$ there exists x such that $x \in N_2(u)$ and degree of x in induced subgraph $\langle N_2(u) \rangle$ is 2. Let v be the only neighbor of x which is not in $N_2(u)$. Then $|N_2(v) - N_2(u)|$ is at most 5. If we initialize the algorithm u and choose v as the second vertex of M , we get $n \leq 8 + 5 + 6(|M| - 2)$, thus $|M| \geq \frac{(n-1)}{6}$. Since G is a cubic graph, n is even number. Therefore, $|M| \geq \frac{n}{6}$. So in case of a cubic graph whose every vertex is contained to at least one triangle, $\gamma_s(G) = n - P_2(G) \leq \frac{2n}{3}$. \square

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