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THE RADON-NIKODÝM THEOREM FOR A NONABSOLUTE INTEGRAL ON MEASURE SPACES

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ABSTRACT. We prove the Radon-Nikodým theorem for a nonabsolute integral on measure spaces endowed with metric topologies and hence provide a descriptive definition of the integral.

The Radon-Nikodým theorem is a well-known result in measure theory which has found applications in many problems in abstract analysis. It also provides a notion of differentiation in abstract measure spaces.

We showed in [10] that a Henstock-type nonabsolute integral, which we call the H-integral, can be defined on measure spaces endowed with metric topologies, and in [9] we proved that for measurable functions, the absolute H-integrability is equivalent to, among other integrabilities, the well-known Lebesgue integrability.

In this paper we shall prove the Radon-Nikodým theorem for the H-integral. What we shall present here is a significant improvement on the results we have proved in [6] which are for the Euclidean space. As a consequence of the Radon-Nikodým theorem, the primitive of a H-integrable function will be completely characterized, thus giving rise to a descriptive definition of the H-integral.

1. Preliminaries

Let (X, d) be a metric space with topology $\mathcal{T}$ induced by the metric d on X and let (X, Ω, ι) be a measure space such that $\mathcal{T} \subset \Omega$. The measure ι is assumed to be non-negative and countably additive. Furthermore, the following condition will be assumed throughout this paper.

(*) For every measurable set $W \in \Omega$ and every $\varepsilon > 0$, there exist an open set U and a closed set Y such that $Y \subset W \subset U$ and $\iota(U \setminus Y) < \varepsilon$.

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Let $\mathcal{T}_1$ be the set of all *open balls*, i.e. sets of the form $\{y \in X : d(x,y) < r\}$, where $x \in X$ and $r > 0$, and denoted by $B(x,r)$. We shall also call its closure a *closed ball.* Throughout this paper we shall assume that $\iota(U) > 0$ and $\iota(U) = \iota(\overline{U})$ for all $U \in \mathcal{T}_1$, where $\overline{U}$ denotes, as usual, the closure of U.

Consider the following sets:

$$\begin{aligned}\mathcal{H}_1 &= \left\{\overline{B_1} \setminus \overline{B_2} : B_1, B_2 \in \mathcal{T}_1 \text{ where } B_1 \not\subseteq B_2 \text{ and } B_2 \not\subseteq B_1\right\}, \\ \mathcal{H}_2 &= \left\{\bigcap_{i\in\Lambda} X_i \neq \emptyset : X_i \in \mathcal{H}_1 \text{ and } \Lambda \text{ is a finite index set}\right\}.\end{aligned}$$

More precisely, members of $\mathcal{H}_1$ are either closed balls or scalloped balls. A typical member of $\mathcal{H}_2$ is a finite intersection of a combination of closed balls and scalloped balls.

We shall call members of $\mathcal{H}_2$ *generalised intervals* or, where there is no ambiguity, *intervals*. Note that intervals are relatively compact, though not necessarily closed or compact. Also note that $\iota(I) = \iota(\overline{I})$ for each interval I.

EXAMPLE 1. Let X be the two-dimensional Euclidean space $\Re^2$. The metrics d_1 and d_2 in X are given by

$$d_1(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

$$d_2(x,y) = \left[(x_1 - y_1)^2 + (x_2 - y_2)^2\right]^{\frac{1}{2}},$$

for each $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X. It is well-known that the d_1-open balls are squares without the boundaries, and the d_2-open balls are open circular discs. It is easy to see that when the metric d_1 is used, a generalised interval looks like a polygon with edges each being vertical or horizontal, and each edge is not necessarily included. When the metric d_2 is used instead, a generalised interval is a simply connected domain in the plane with piecewise circular edges, and each arc may or may not be included.

We next present the necessary and standard terminology in defining a Henstock-type integral.

Let E be a finite union of (possibly just one) mutually disjoint intervals and call it an *elementary set.* Throughout this paper, we shall let an elementary set E with finite measure, that is $\iota(E) < +\infty$, be fixed and define integrability on E.

A set $\{(I_i, x_i) : i = 1, 2, \ldots, n\}$ of interval-point pairs is called a *partial division* of E if $I_1, I_2, \ldots, I_n$ are mutually disjoint subintervals of E such that $E \setminus \bigcup_{i=1}^{n} I_i$ is either empty or an elementary subset of E, and for each i, we have $x_i \in \overline{I_i}$. We call x_i the *associated point* of I_i. A *division* of E is a partial division $\{(I_i, x_i) : i = 1, 2, \ldots, n\}$ such that the union of I_i is E.

Let $\delta : \overline{E} \to \Re^+$ be a positive function. We call δ a *gauge* on E. Note that we need to consider gauges defined on $\overline{E}$ and not just E, because for each interval-point pair (I, x) in a partial division, the associated point x comes from $\overline{I}$ and not just I.

Let a gauge δ on E be given. An interval-point pair (I, x) is *δ-fine* if $I \subset B(x, \delta(x))$. A partial division $\{(I_i, x_i) : i = 1, 2, \ldots, n\}$ of E is *δ-fine* if (I_i, x_i) is δ-fine for each $i = 1, 2, \ldots, n$. Since divisions themselves are partial divisions, the δ-fine divisions of E are similarly defined. A gauge δ_1 is said to be *finer* than a gauge δ_2 on E if for every $x \in \overline{E}$ we have $\delta_1(x) \leq \delta_2(x)$. The existence of δ-fine divisions has been proved in [10].

We shall now define the H-integral and the M-integral where the latter is a McShane-type integral. For brevity and where there is no ambiguity, $D = \{(I, x)\}$ shall denote a finite collection of interval-point pairs (I, x), and the corresponding Riemann sum shall be denoted by $(D) \sum f(x)\iota(I)$. All functions f considered in this paper are real-valued point functions defined on $\overline{E}$.

DEFINITION 1. A function f is said to be H-integrable on E to a real number A if for every $\varepsilon > 0$, there exists a gauge δ on E such that for every δ-fine division $D = \{(I, x)\}$ of E, we have

$$(1) \qquad \left|(D) \sum f(x)\iota(I) - A\right| < \varepsilon.$$

We write $(H) \int_E f = A$. The H-integrability of f on any elementary subset of E is similarly defined.

It has been proved that the H-integral is uniquely determined, and closed under addition, scalar multiplication, monotone convergence and controlled convergence. Furthermore, the Cauchy criterion of integrability and Henstock's lemma also hold [9]. We have also proved in [10] that the H-integral is a nonabsolute one. If both f and $|f|$ are H-integrable, we say that f is *absolutely H-integrable.*

Given a gauge δ on E, a set $D = \{(I_i, x_i) : i = 1, 2, \ldots, n\}$ is called a *δ-fine McShane partial division* of E if I_i are mutually disjoint subintervals of E such that $I_i \subset B(x_i, \delta(x_i))$ for each $i = 1, 2, \ldots, n$ and where

x_i is in $\overline{E}$, though not necessarily in $\overline{I_i}$. Again, if further the union of I_i is E, we call D a δ-fine McShane division of E. Obviously, a δ-fine partial division of E is a δ-fine McShane partial division of E but not conversely.

If (1) in Definition 1 holds for every δ-fine McShane division D of E, we say that f is *M-integrable* on E to the value A. The M-integrability of f on any elementary subset of E is similarly defined. Obviously, a function which is M-integrable on E is H-integrable on E.

Given a function f which is H-integrable on E, the *primitive* F of f on E is given by

$$F(E_0) = (H) \int_{E_0} f$$

for each elementary subset E_0 of E. Note that F is an elementary-set function which is finitely additive over elementary subsets of E (see [9]). The finite additivity here is defined in the standard manner. The primitive of a M-integrable function is similarly defined.

Let us extend the domains of H-integrability and M-integrability to measurable sets. Given a function f on $\overline{E}$ and for every measurable subset W of $\overline{E}$, the function f_W is given by $f_W(x) = f(x)$ if $x \in W$ and 0 otherwise. This leads to the following definition.

DEFINITION 2. Let W be a measurable subset of $\overline{E}$. A function f is said to be H-integrable on W to a real number A if f_W is H-integrable on E to the number A and we write $(H) \int_W f = A$. If f_W is absolutely H-integrable on E, then we say that f is absolutely H-integrable on W.

If f is H-integrable on W to the value A and F is the primitive of f on E, we shall write $F(W) = A$. The M-integrability on a measurable subset of $\overline{E}$ is similarly defined.

We have proved in [9] that for measurable functions, the absolute H-integral, the M-integral and the Lebesgue integral are all equivalent. The Lebesgue integral of f on a measurable subset W of $\overline{E}$, if exists, shall be denoted by $(L) \int_W f$.

Note that if f is H-integrable on E, for every $\varepsilon > 0$, there is a gauge G_1 such that for any G_1-fine division $D = \{(I, x)\}$ of E, we have

$$(D) \sum |f(x)\iota(I) - F(I)| < \varepsilon$$

where F is the primitive of f. For every gauge G on E, we then put

$$A_G = \sup_{D_G} (D_G) \sum |F(I)|$$

where the supremum is taken over all G-fine divisions $D_G = \{(I, x)\}$ of E and let

$$A = \inf_G A_G$$

where the infimum is taken over all gauges G on E. The value A thus defined is called the *Henstock variation* of F on E.

The following result, whose proof requires the notion of Henstock variation, will be used in proving the main theorem later. It essentially says that as far as H-integrability is concerned, we can ignore sets of measure zero.

THEOREM 3. *If a function f is H-integrable on E and*

$$(H) \int_I f = 0$$

for all subintervals I of E, then $f(x) = 0$ almost everywhere in $\overline{E}$.

Proof. Let $W = \{x \in \overline{E} : f(x) > 0\}$. It suffices to prove that W is of measure zero. To this end, we define, for each positive integer n,

$$W_n = \{x \in W : 0 < f(x) \leq n\}$$

and $f_n = f_{W_n}$. For a fixed n, the function f_n is absolutely H-integrable on E and so is Lebesgue integrable on E with integral value given by the Henstock variation A. In view of the hypothesis and by applying Henstock's lemma, we see that $A = 0$. Therefore

$$(L) \int_{W_n} f = (L) \int_E f_n = 0.$$

Since $f(x) > 0$ for all $x \in W_n$, it follows that W_n is of measure zero. Consequently, the set $W = \bigcup_{n=1}^{\infty} W_n$ is also of measure zero. □

In what follows, we shall provide a necessary and sufficient condition for a function to be H-integrable in terms of the (LG)-condition [7] which we shall define later. We will need this result in proving the main theorem. The notion of generalised absolute continuity [3] will also be introduced.

The following result for the case of the real-line is due to Lu and Lee [8]. Its extension to our setting is straightforward.

THEOREM 4. *If a function f is H-integrable on E, then there is a sequence $\{X_i\}$ of closed subsets of $\overline{E}$ such that $X_i \subset X_{i+1}$ for all i*

where $\overline{E} \setminus \bigcup_{i=1}^{\infty} X_i$ *is of measure zero,* f *is Lebesgue integrable on each* X_i*, and*

$$\lim_{i\to\infty} (L) \int_{X_i} f = (H) \int_E f.$$

COROLLARY 5. *If a function* f *is H-integrable on* E*, then* f *is measurable.*

Proof. Let the sequence of closed sets $\{X_i\}$ be as in Theorem 4. For each i, the function f_{X_i} is Lebesgue integrable on E and so is measurable. It is easy to see that

$$\lim_{i\to\infty} f_{X_i}(x) = f(x)$$

almost everywhere in $\overline{E}$. Hence f is measurable. □

Theorem 4 has shed some light on how a necessary and sufficient condition for H-integrability can be formulated. We shall also need the following definitions.

DEFINITION 6. Let f be a function on $\overline{E}$. A sequence $\{X_i\}$ of measurable sets with union $\overline{E}$ such that f is H-integrable on each X_i is said to be a basic sequence of f on E. The sequence is monotone if $X_i \subset X_{i+1}$ for each i.

The next definition is due to Liu [7] in giving necessary conditions for Henstock integrability on the real line.

DEFINITION 7. Let $\{X_i\}$ be a sequence of measurable sets with union $\overline{E}$. A function f is said to satisfy the (LG)-condition on $\{X_i\}$ if for every $\varepsilon > 0$, there is a positive integer N such that for each $i \geq N$, there exists a gauge δ_i on E satisfying the condition that for every δ_i-fine division $D = \{(I, x)\}$ of E, we have

$$\left| (D) \sum_{x \notin X_i} f(x)\iota(I) \right| < \varepsilon. \tag{2}$$

Here $(D) \sum_{x \notin X_i}$ sums over all interval-point pairs (I, x) in D with $x \notin X_i$.

The following result is proved by Lee [4] for a Henstock-type integral in the Euclidean space. It basically follows from the fact that

$$(D) \sum_{x \notin X_i} = (D) \sum - (D) \sum_{x \in X_i}$$

for each division $D = \{(I, x)\}$ of E.

THEOREM 8. *Let f be a function on $\overline{E}$ and let $\{X_i\}$ be a basic sequence of f on E. Suppose that $\lim_{i\to\infty} (H) \int_{X_i} f = A$. Then f is H-integrable on E with integral value A if and only if f satisfies the (LG)-condition on $\{X_i\}$.*

With Theorems 4 and 8, we can now provide a necessary and sufficient condition for H-integrability.

THEOREM 9. *A function f is H-integrable on E if and only if f has a basic sequence $\{X_i\}$ on E such that $\lim_{i\to\infty} (H) \int_{X_i} f < +\infty$ and f satisfies the (LG)-condition on $\{X_i\}$.*

Proof. Suppose f is H-integrable on E and let $\{X_i\}$ be a sequence of closed sets satisfying the conditions in Theorem 4. Let $Y = \overline{E} \setminus \bigcup_{i=1}^{\infty} X_i$ be the set of measure zero. Then f is Lebesgue integrable, and hence is H-integrable, on the measurable set $X_1 \cup Y$. So $\{X_1 \cup Y, X_2, X_3, \dots\}$ is a basic sequence of f on E. Since f is H-integrable on E, by Theorem 8, f satisfies the (LG)-condition on $\{X_1 \cup Y, X_2, X_3, \dots\}$. The converse follows immediately from Theorem 8. □

The main theorem shall be proved with the aid of the above result.

2. The main theorem

We first state without proof the well-known Radon-Nikodým theorem for the Lebesgue integral (see [1, Theorem 19.23]). We shall not state the original statement but a form we will use later.

THEOREM 10. *Let F be a non-negative function defined on the set of all measurable subsets Y of $\overline{E}$ which is finitely additive over measurable sets and which is ι-absolutely continuous on E. Then there exists a non-negative function f which is Lebesgue integrable on E such that for any measurable subset Y of $\overline{E}$, we have*

$$F(Y) = (L) \int_Y f. \tag{3}$$

Moreover, the function f is unique in the sense that if g is a non-negative Lebesgue integrable function on E for which (3) holds, then $f = g$ almost everywhere in $\overline{E}$.

In the above theorem, the function F is *ι-absolutely continuous* on E in the sense that for every $\varepsilon > 0$, there exists $\eta > 0$ such that for every measurable subset Y of $\overline{E}$ satisfying the condition $\iota(Y) < \eta$, we have $|F(Y)| < \varepsilon$, where $|F(Y)|$ denotes the measure of $F(Y)$ on the real line.

Throughout the remainder of this chapter, we shall write F to denote a real-valued elementary-set function which is finitely additive over subintervals of E. Our objective is to define a sequence of non-negative finitely additive measures on E in terms of F so that we can apply Theorem 10 to obtain a sequence of Lebesgue integrable functions, and in turn a H-integrable function. We begin with a few definitions.

DEFINITION 11. Let Y be a measurable subset of $\overline{E}$. A function F is said to be $AC_{\Delta}(Y)$ if for every $\varepsilon > 0$, there exist a gauge δ on E and $\eta > 0$ such that for every δ-fine partial division $D = \{(I, x)\}$ of E with $x \in Y$ satisfying the condition that $(D)\sum \iota(I) < \eta$, we have

$$\left|(D)\sum F(I)\right| < \varepsilon. \tag{4}$$

It is well-known that the primitive of a Lebesgue integrable function is absolutely continuous [1, Theorem 12.34]. On the real line, it has been proved that the primitive of a Kurzweil-Henstock integrable function is generalised absolutely continuous in some restricted sense [2]. The above definition is a generalisation of the concept of absolute continuity to our abstract setting.

DEFINITION 12. A function F is said to be strongly ACG_{Δ} on E if there exist measurable sets $X_1, X_2, \cdots$ with union $\overline{E}$ such that F is $AC_{\Delta}(X_i)$ for each i, and if the following (L)-condition on E holds: For every subinterval I_0 of E and every $\varepsilon > 0$, there is a positive integer N such that for each $i \geq N$, there exists a gauge δ_i on E satisfying the condition that for every δ_i-fine division $D = \{(I, x)\}$ of I_0, we have

$$\left|(D)\sum_{x \notin X_i} F(I)\right| < \varepsilon.$$

In the above definition, we may assume that δ_i is also a candidate for the gauge δ in the definition of $AC_{\Delta}(Y)$ with $Y = X_i$ for each i.

Let us begin with an elementary-set function F which is strongly ACG_{Δ} on E. Then there exist measurable sets $X_1, X_2, \cdots$ with union E such that for each i and every $\varepsilon > 0$, the condition in Definition 11 holds with Y, δ and η replaced by X_i, δ_i and η_i respectively. For each i

and for each subinterval J of E, we define

$$K_i(J) = \inf_{\delta} \sup_{D_\delta}(D_\delta) \sum_{x \in X_i} F(I)$$

and

$$|K_i|(J) = \inf_{\delta} \sup_{D_\delta}(D_\delta) \sum_{x \in X_i} |F(I)|,$$

where in each case above the infimum is over all gauges δ and the supremum is over all δ-fine divisions $D_\delta = \{(I, x)\}$ of J. We proceed to define, for each subinterval J of E,

$$F_i(J) = \inf_P(P) \sum K_i(I) \tag{5}$$

and

$$|F_i|(J) = \inf_P(P) \sum |K_i|(I),$$

where in each case the infimum is over all partitions $P = \{I\}$ of J.

It can be proved that F_i and $|F_i|$ are finitely additive over subintervals of E (see [6]). We shall call the sequence $\{F_i\}$ of elementary-set functions, where each F_i is as defined in (5), the *derived sequence* of F on $\{X_i\}$.

For each subinterval I of E, we further define

$$F_i^+(I) = \frac{|F_i|(I) + F_i(I)}{2} \quad \text{and} \quad F_i^-(I) = \frac{|F_i|(I) - F_i(I)}{2}.$$

It is easy to see that F_i^+ and F_i^- are well-defined and non-negative elementary-set functions such that for each subinterval I of E, we have

$$F_i(I) = F_i^+(I) - F_i^-(I).$$

Let us extend the domains of F_i^+ and F_i^- to all measurable subsets of $\overline{E}$. Firstly, for each measurable subset Y of E, we define

$$F_i^{++}(Y) = \inf \left\{ \sum_{j=1}^{\infty} F_i^+(I_j) : Y \subset \bigcup_{j=1}^{\infty} I_j \right\}$$

and

$$F_i^{--}(Y) = \inf \left\{ \sum_{j=1}^{\infty} F_i^-(I_j) : Y \subset \bigcup_{j=1}^{\infty} I_j \right\},$$

where the I_j, $j = 1, 2, \ldots$, are subintervals of E. Note that F_i^{++} and F_i^{--} agree with F_i^+ and F_i^- respectively on each subinterval of E and are finitely additive over measurable subsets of E (see [6]). Since $F_i^{++}(Z) = 0 = F_i^{--}(Z)$ if Z is of measure zero, we define

$$F_i^{++}(\overline{E}) = F_i^{++}(E) \quad \text{and} \quad F_i^{--}(\overline{E}) = F_i^{--}(E).$$

The following two lemmas can be proved using the same techniques we employ in [6].

LEMMA 13. *The functions F_i^{++} and F_i^{--} are ι-absolutely continuous on E.*

LEMMA 14. *Suppose F is strongly ACG_Δ on E and $\{F_i\}$ is the derived sequence of F on $\{X_i\}$. Then $F_i(I) \to F(I)$ as $i \to \infty$ for each subinterval I of E.*

In order to apply Theorem 9 in proving the main theorem, we need to prove that the (LG)-condition is satisfied. Note that the (L)-condition involves the elementary-set function F whereas the (LG)-condition involves the point function f. The two conditions are equivalent when f is H-integrable on E with primitive F in view of Henstock's lemma. However, when we do not know if f is H-integrable, we cannot prove that the (L)-condition implies the (LG)-condition. We therefore need to impose, in addition, the following (WL)-condition.

DEFINITION 15. Let $\{X_k\}$ be a sequence of measurable sets with union $\overline{E}$. A sequence $\{F_k\}$ of elementary-set functions is said to satisfy the (WL)-condition on $\{X_k\}$ if for every $\varepsilon > 0$, there is a positive integer N such that for each $i \geq N$, there exists a gauge δ_i on E satisfying the condition that for every δ_i-fine division $D = \{(I, x)\}$ of E, we have

$$\left| \sum_{k=i+1}^{\infty} (D_k) \sum F_k(I) \right| < \varepsilon,$$

where $D_k = \left\{ (I, x) \in D : x \in X_k \setminus \bigcup_{j=1}^{k-1} X_j \right\}$ for $k \geq i + 1$.

The following result is pivotal in proving the main theorem.

THEOREM 16. *Let f be a function on $\overline{E}$ and $\{X_i\}$ be a basic sequence of f on E. For each k, let F_k be the primitive of f_{X_k}. Then f satisfies the (LG)-condition on $\{X_i\}$ if and only if $\{F_k\}$ satisfies the (WL)-condition on $\{X_k\}$.*

Proof. Suppose f satisfies the (LG)-condition on $\{X_i\}$ and let $\varepsilon > 0$ be given. Choose a positive integer N such that for each $i \geq N$, there exists a gauge δ_i such that for every δ_i-fine division D of E, we have

$$\left|(D) \sum_{x \notin X_i} f(x)\iota(I)\right| < \frac{\varepsilon}{2}$$

and that $\sum_{k=N}^{\infty} 2^{-k} < \frac{\varepsilon}{2}$. For each $k \geq N$, since f_{X_k} is H-integrable on E, we may assume that for every δ_k-fine division D of E, we have

$$(D) \sum |f_{X_k}(x)\iota(I) - F_k(I)| < \varepsilon 2^{-k}.$$

Now let $i \geq N$ be fixed. We choose a gauge δ_i^* on E such that $\delta_i^*(x) \leq \delta_k(x)$ if $x \in X_k \setminus \bigcup_{j=1}^{k-1} X_j$ for $k \geq i+1$. Then for every δ_i^*-fine division D of E we let $D_k = \left\{ (I,x) \in D : x \in X_k \setminus \bigcup_{j=1}^{k-1} X_j \right\}$ for $k \geq i+1$. Consequently, we obtain

$$\begin{aligned} \left| \sum_{k=i+1}^{\infty} (D_k) \sum F_k(I) \right| &\leq \left| \sum_{k=i+1}^{\infty} (D_k) \sum \{ f(x)\iota(I) - F_k(I) \} \right| \\ &+ \left| (D) \sum_{x \notin X_i} f(x)\iota(I) \right| \\ &< \varepsilon. \end{aligned}$$

The converse follows in a similar manner just that this time round we consider instead the inequality

$$\begin{aligned} \left| (D) \sum_{x \notin X_i} f(x)\iota(I) \right| &= \left| \sum_{k=i+1}^{\infty} (D_k) \sum f(x)\iota(I) \right| \\ &\leq \left| \sum_{k=i+1}^{\infty} (D_k) \sum \{ f(x)\iota(I) - F_k(I) \} \right| \\ &+ \left| \sum_{k=i+1}^{\infty} (D_k) \sum F_k(I) \right|. \end{aligned}$$

This completes the proof. □

The Radon-Nikodým theorem for the H-integral as stated below can now be proved as in [6, Theorem 10] by applying Lemmas 13 and 14 and Theorems 9, 10 and 16. The uniqueness of the function f in the theorem follows readily from Theorem 3.

THEOREM 17. *Let F be an elementary-set function which is finitely additive and strongly ACG_Δ on E such that its derived sequence $\{F_i\}$ on $\{X_i\}$ satisfies the (WL)-condition on $\{X_i\}$. Then there exists a function f which is H-integrable on E such that for any elementary subset E_0 of E, we have*

$$F(E_0) = (H)\int_{E_0} f. \tag{6}$$

Moreover, the function f is unique in the sense that if g is a H-integrable function on E for which (6) holds, then $f = g$ almost everywhere in $\overline{E}$.

The function f obtained in Theorem 17 is in a way a derivative of F. If f exists, we say that F is *Radon-Nikodým differentiable* on E and we write $D_{RN}F = f$. We call $D_{RN}F$ the *Radon-Nikodým derivative* of F.

3. Descriptive definition of the H-integral

The descriptive definition of the Kurzweil-Henstock integral on the real line is well-known. More precisely, a function f is Kurzweil-Henstock integrable on a closed interval $[a,b]$ if and only if there exists a function F which is generalised absolutely continuous in the restricted sense on $[a,b]$ such that $F'(x) = f(x)$ almost everywhere (see [2, Theorem 6.22] or [11]).

In [6], we have given a complete characterization of the primitive of a Henstock-type integral in the Euclidean space. In this section, we shall characterize the primitive of a H-integrable function and hence give a descriptive definition of the H-integral.

We first prove that the primitive of a function which is H-integrable on E is strongly ACG_Δ on E. A similar result for a Henstock-type integral in the Euclidean space has been proved in [5]. Our proof here is an independent one.

THEOREM 18. *If a function f is H-integrable on E, then its primitive F is strongly ACG_Δ on E.*

Proof. Since f is H-integrable on E, by Theorem 4 and in view of Theorem 8, the function f has a monotone basic sequence $\{X_i\}$ on

E such that f is absolutely H-integrable on each X_i and f satisfies the (LG)-condition on $\{X_i\}$. For convenience, we write $X_i = Y$. By Henstock's lemma, for every $\varepsilon > 0$ there exists a gauge δ on E such that for every δ-fine partial division $D = \{(I,x)\}$ of E, we have

$$(D)\sum |F(I) - f(x)\iota(I)| < \varepsilon,$$

and for every δ-fine McShane partial division $D = \{(I,x)\}$ of E, we have

$$(D)\sum |F_Y(I) - f_Y(x)\iota(I)| < \varepsilon.$$

It follows that for every δ-fine partial division $D = \{(I,x)\}$ of E with $x \in Y$, we have

$$\begin{aligned}(D)\sum |F(I) - F_Y(I)| &\leq (D)\sum |F(I) - f(x)\iota(I)| \\ &+ (D)\sum |F_Y(I) - f_Y(x)\iota(I)| \\ &< 2\varepsilon.\end{aligned}$$

Now, as f is absolutely H-integrable on Y, we can choose $\eta > 0$ and modify δ if necessary such that for every δ-fine partial division $D = \{(I,x)\}$ with $x \in Y$ and $(D)\sum \iota(I) < \eta$, we have

$$\left|(D)\sum F(I)\right| \leq (D)\sum |F(I) - F_Y(I)| + (D)\sum |F_Y(I)| < 3\varepsilon.$$

Hence F is $AC_\Delta(Y)$. Finally, we note that for every subinterval I_0 of E and for every δ-fine division D of I_0, we have

$$\begin{aligned}\left|(D)\sum_{x\notin X_i} F(I)\right| &\leq \left|(D)\sum_{x\notin X_i} f(x)\iota(I)\right| \\ &+ \left|(D)\sum_{x\notin X_i} \{f(x)\iota(I) - F(I)\}\right|.\end{aligned}$$

Since f satisfies the (LG)-condition on $\{X_i\}$, it follows that F satisfies the (L)-condition. Therefore F is strongly ACG_Δ on E. □

Note that in the above theorem, we also have $f_{X_i} \to f$ almost everywhere in $\overline{E}$ as $i \to \infty$ and for each i, the function F_i in the derived sequence $\{F_i\}$ of F on $\{X_i\}$ is actually the primitive of f_{X_i}. In view of Theorems 8 and 16, the primitive F of a H-integrable function has its derived sequence $\{F_i\}$ on $\{X_i\}$ satisfying the (WL)-condition on $\{X_i\}$. Conclusively, the primitive of a H-integrable function satisfies all the conditions on F in Theorem 17. Hence we can now state, as desired a descriptive definition of the H-integral.

THEOREM 19. *An elementary-set function F is the primitive of a function f which is H-integrable on E if and only if F is finitely additive and strongly ACG_Δ on E such that its derived sequence $\{F_i\}$ on $\{X_i\}$ satisfies the (WL)-condition on $\{X_i\}$. Furthermore, $D_{RN}F = f$ almost everywhere in $\overline{E}$.*

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