Bull. Korean Math. Soc. **41** (2004), No. 1, pp. 147–151

ON SOME FINITE p-GROUPS

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Abstract. The purpose of this paper is to investigate the order of a finite p-group and determine the structure of such group.

1. Introduction

Wiegold proved that if G is a group with central factor group $G/Z(G)$ of order p^m, then G' is a p-group of order at most $p^{\frac{m(m-1)}{2}}$ (cf. [2]). In this paper we determine the structure of a p-group G such that $|G/Z(G)| = p^m$ and $|G'| = p^{\frac{m(m-1)}{2}-1}$.

The notation in the paper is standard. The center of a group G is denoted by $Z(G)$, and the subgroups $Z_2(G)$ and $Z_3(G)$ of G are given by $Z(G/Z(G)) = Z_2(G)/Z(G)$ and $Z(G/Z_2(G)) = Z_3(G)/Z_2(G)$, respectively. And the commutator subgroup of a group G is denoted by G'. Thus

$$G' = \langle [x, y] | x, y \in G \rangle,$$

where $[x, y] = x^{-1}y^{-1}xy$.

We begin with a lemma.

Lemma 1. *Let p be a prime and let G be an arbitrary group such that $G/Z(G)$ is a finite p-group of order p^n. Then G' is a finite p-group and*

$$|G'| \leq p^{\frac{n(n-1)}{2}}.$$

Proof. The proof can be found in [2, Theorem 2.1]. □

By the above lemma, we have the following(see [1], Lemma 5).

Lemma 2. *Let G be a finite p-group with $|G/Z(G)| = p^m$. Then there exists an integer $s \geq 0$ such that*

$$|G'| = p^{\frac{m(m-1)}{2}-s}$$

Received July 11, 2002.
2000 Mathematics Subject Classification: 20C25.
Key words and phrases: p-group.

and

$$|(G/Z(G))'| \le p^{1+s}.$$

Moreover, if $|(G/Z(G))'| = p^{1+s}$, *then* $Z_2(G)/Z(G)$ *has exponent* p.

DEFINITION 1. A finite abelian p-group is called an elementary abelian p-group if

$$G \cong C_p \times C_p \times \cdots \times C_p,$$

where C_p denotes the cyclic group of order p.

DEFINITION 2. Let G be a finite p-group. Then G is said to be extra-special if the following three conditions hold.

(1) $G' = Z(G)$,
(2) $|G'| = p$, and
(3) G/G' is an elementary abelian p-group.

LEMMA 3. *Let* G *be a finite p-group. Then* G *is extra-special if and only if* $G' = Z(G)$ *and* $|G'| = p$.

Proof. If G is extra-special, then $G' = Z(G)$ and $|G'| = p$ by Definition 2.

Suppose that G be a finite p-group such that $G' = Z(G)$ and $|G'| = p$. Since $G' = Z(G)$, we have $[g, xy] = [g, x][g, y]$ for all $g, x, y \in G$. And it follows from $|G'| = p$ that

$$[g, x^p] = [g, x]^p = 1$$

for all $g, x \in G$. Hence $x^p \in Z(G) = G'$ for all $x \in G$, and so G/G' is an elementary abelian p-group. Thus G is extra-special □

THEOREM 1. *Let* G *be a finite p-group with* $|G/Z(G)| = p^m$. *If*

$$|G'| = p^{\frac{m(m-1)}{2}}$$

then either $G/Z(G)$ *is elementary abelian or* $G/Z(G)$ *is extra-special.*

Proof. The proof can be found in [1, Theorem 6]. □

We can prove the following lemma by easy calculations.

LEMMA 4. *Let* G *be a finite group. Then the following holds.*

1. *If* x, y, z *are elements of* G, *then*
   (a) $[xy, z] = [x, z]^y[y, z] = [x, z][[x, z], y][y, z]$.
   (b) $[x, yz] = [x, z][x, y]^z = [x, z][x, y][[x, y], z]$.
2. *Suppose that* $G' \subseteq Z(G)$. *Then, for any elements* x, y, z *of* G, *we have*
   (a) $[xy, z] = [x, z][y, z]$.

(b) $[x, yz] = [x, z][x, y]$.
(c) $[x^i, y^j] = [x, y]^{ij}$ *for all* $i, j \geq 0$.
(d) $(yx)^i = [x, y]^k y^i x^i$ *for all* $i \geq 1$, *where* $k = \frac{i(i-1)}{2}$.

2. Main theorem

In this section we prove our main theorem.

THEOREM 2. *Let p be a prime and let G be a finite p-group with* $|G/Z(G)| = p^m$. *If* $|G'| = p^{\frac{m(m-1)}{2}-1}$, where $m \geq 3$, *then one of the following holds.*

1. $G = Z_2(G)$, *and* $G/Z(G)$ *is an elementary abelian p-group.*
2. $G = Z_3(G)$, $G/Z(G)$ *is an extra special group and* $|Z_2(G)/Z(G)| = p$.
3. $G = Z_3(G)$, $Z(G/Z(G))$ *is elementary abelian and* $|Z_2(G)/Z(G)| = p^2$.
4. $G/Z_2(G)$ *is of order at most* p^{m-3}, *and* $Z_2(G)/Z(G)$ *has exponent* p.

Proof. By the assumption and Lemma 2, we have $|G'| = p^{\frac{m(m-1)}{2}-1}$ and

$$|(G/Z(G))'| \leq p^2.$$

First,we consider the case when

$$|(G/Z(G))'| = 1.$$

Then $G/Z(G)$ is an abelian group and so $G = Z_2(G)$ and $G' \subseteq Z(G)$.

Suppose that $G/Z(G)$ is not elementary abelian. Then there exists an element $z_0 \in G - Z(G)$ such that $z_0^p \notin Z(G)$. Since $G' \subseteq Z(G)$, it follows from Lemma 4 that the map

$$\varphi : G \longrightarrow [G, z_0], \ \varphi(x) = [x, z_0]$$

is an epimorphism with $\ker(\varphi) = C_G(z_0)$, and so $G/C_G(z_0) \cong [G, z_0]$.

Since $Z(G) \subset \langle z_0, Z(G)\rangle \subseteq C_G(z_0)$, we have $|G/C_G(z_0)| < |G/Z(G)| = p^m$ and so $|G/C_G(z_0)| \leq p^{m-1}$. And $[G, z_0] \subseteq G' \subseteq Z(G)$ and so $[G, z_0]$ is a normal subgroup of G. Put $|G/[G, z_0] : Z(G/[G, z_0])| = p^b$. Since $z_0^p \notin Z(G)$, we have $|[G, z_0]| \leq m - 2$ and $b \leq m - 2$. Because $[G, z_0] \subseteq G'$, we get that $(G/[G, z_0])' = G'/[G, z_0]$ and so $|G'| =$

$|G'/[G,z_0]||[G,z_0]|$. It follows from Lemma 1 that

$$\begin{aligned}\log_p|G'| &\le \frac{b(b-1)}{2}+(m-2)\\ &\le \frac{(m-2)(m-3)}{2}+(m-2)\end{aligned}$$

which forces $m \le 2$. But this is not the case.

Therefore $G/Z(G)$ is an elementary abelian p-group with $G = Z_2(G)$ and (1) holds.

Now, we consider the case when

$$|(G/Z(G))'| = p.$$

Then $(G/Z(G))'$ is a normal subgroup of a finite p-group $G/Z(G)$. Hence, by the property of a finite p-group, we have

$$(G/Z(G))' \cap Z(G/Z(G))$$

is not trivial. Since $|(G/Z(G))'| = p$, it follows that $(G/Z(G))' \subseteq Z((G/Z(G))$ and $|Z(G/Z(G))| \ge p$.

If $|Z(G/Z(G))| = p$, then we obtain $(G/Z(G))' = Z(G/Z(G)) = Z_2(G)/Z(G)$, because $(G/Z(G))' \subseteq Z((G/Z(G))$. It implies that $G/Z(G)$ is an extra special group by Lemma 3 and (2) holds.

If $|Z(G/Z(G))| = p^2$, then $Z(G/Z(G))$ is either cyclic or elementary abelian. Suppose that $Z(G/Z(G))$ is a cyclic of order p^2 and let $Z(G/Z(G)) = \langle aZ(G)\rangle$. Let a be a fixed element of $Z_2(G) - Z(G)$ and let $\Phi(x) = [a,x]$ for all $x \in G$. Because $[a,x]Z(G) = [aZ(G), xZ(G)] = Z(G)$ for all $xZ(G) \in G/Z(G)$, we see that $[a,x] \in Z(G)$. It follows that $\Phi(xy) = [a,xy] = [a,y][a,x]^x = [a,y][a,x] = [a,x][a,y] = \Phi(x)\Phi(y)$ and so Φ is a homomorphism from G into $[a,G]$. We obtain $\ker(\Phi) = C_G(a) \supset Z(G)$. Put $M = \mathrm{im}\Phi$ and $|M| = p^t$. Then $M \cong G/C_G(a)$. Since $\langle a^p Z(G)\rangle$ is cyclic of order p and $a^p \notin Z(G)$, we get that $t \le m-2$. Put $|G/M : Z(G/M)| = p^b$. Since $a \notin Z(G)$ and $aM \in Z(G/M)$, we see that $\langle Z(G)/M, aM\rangle$ is a subgroup of $Z(G/M)$. Because $a^p \notin Z(G)$, we have that $b \le m-2$. Note that $M = \mathrm{im}\Phi \subseteq [a,G] \subseteq G'$ and so $(G/M)' = G'/M$. By Lemma 1, we have that

$$\begin{aligned}|G'| &\le p^{\frac{b(b-1)}{2}+(m-2)}\\ &\le p^{\frac{(m-2)(m-3)}{2}+(m-2)}\end{aligned}$$

and $p^m \le p^2$. This is impossible. Therefore $Z(G/Z(G))$ is an elementary abelian p-group and (3) holds.

If $|Z(G/Z(G))| > p^2$, then we will show that $Z_2(G)/Z(G)$ has exponent p. Suppose that the exponent of $Z_2(G)/Z(G)$ is not equal p. Then there exists an element $b_0 \in Z_2(G) - Z(G)$ such that $b_0^p \notin Z(G)$. For $x \in G$ let $\Phi(x) = [b_0, x] = b_0^{-1}x^{-1}b_0x$. Since $b_0 \in Z_2(G) - Z(G)$ and $[b_0, x] \in Z(G)$, we have that Φ is a homomorphism with $\ker(\Phi) = C_G(b_0)$. Put $M = \mathrm{im}\Phi$ and $|M| = p^t$. Since $b_0^p \notin Z(G)$, we have that $t \leq m - 2$. Put $|G/M : Z(G/M)| = p^b$. It follows from $b_0M \in Z(G/M)$ and $b_0^p \notin Z(G)$ that $b \leq m-2$. Since $M = [b_0, G] \subseteq G'$ and $(G/M)' = G'/M$, by Lemma 1 we get that $m \leq 2$ and this is a contradiction. Therefore $Z_2(G)/Z(G)$ has exponent p.

It is clear that $G/Z_2(G)$ is of order at most p^{m-3}. Since $(G/Z(G))'$ is abelian, we have $(G/Z(G))'' = \{1\}$. It is that $G/Z(G)$ is a finite p-group and so $G/Z(G)$ is a nilpotent group. Therefore $Z_2(G/Z(G)) = G/Z(G)$ and we have $G = Z_3(G)$ and (4) holds.

Finally, we consider the case when

$$|(G/Z(G))'| = p^2.$$

Since $(G/Z(G))'$ is abelian and $G/Z(G)$ is a nilpotent group, we have $Z_2(G/Z(G)) = G/Z(G)$ and $(G/Z(G))' \subseteq Z(G/Z(G))$. We get that $G = Z_3(G)$ and $|Z(G/Z(G))| \geq p^2$.

If $|Z(G/Z(G))| = p^2$, then $Z(G/Z(G))$ is an abelian group of order p^2. Since $Z(G/Z(G))$ has exponent p by Lemma 2, $Z(G/Z(G))$ is an elementary abelian group and (3) holds.

If $|Z(G/Z(G))| > p^2$, then $Z(G/Z(G))$ has exponent p by Lemma 2 and $G/Z_2(G)$ is of order at most p^{m-3} and (4) holds. Hence the assertion of Theorem 2 holds. □

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