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CANTOR DIMENSION AND ITS APPLICATION

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ABSTRACT. We defined Cantor dimensions of a perturbed Cantor set, and investigated a relation between these dimensions and Hausdorff and packing dimensions of a perturbed Cantor set. In this paper, we introduce another expressions of the Cantor dimensions. Using these, we study some informations which can be derived from power equations induced from contraction ratios of a perturbed Cantor set to give its Hausdorff or packing dimension. This application to a deranged Cantor set gives us an estimation of its Hausdorff and packing dimensions, which is a generalization of the Cantor dimension theorem.

1. Introduction

We ([1]) investigated the Hausdorff dimension and the packing dimension of a perturbed Cantor set whose contraction ratios are uniformly bounded (cf. [4]), that is, the Hausdorff dimension of the perturbed Cantor set is equal to its lower Cantor dimension and its packing dimension is equal to its upper Cantor dimension. Using a weak local dimension based on Cantor dimension, we ([3]) developed some estimation of Hausdorff and packing dimensions of a deranged Cantor set which is the most generalized form of a Cantor set. But for some complexity of the definition, we did not find that the estimation applied to a perturbed Cantor set gives its Cantor dimension theorems. But from an improved form of the definition of Cantor dimension in this paper, we reconstruct nicer theorems, which give generalized estimations of Hausdorff and packing dimensions, than those ([3]) of the weak local dimension theory.

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2. Preliminaries

We recall the definition of a deranged Cantor set ([2], [3]). Let $I_\phi = [0,1]$. Then we obtain the left sub-interval $I_{\tau,1}$ and the right sub-interval $I_{\tau,2}$ of I_τ deleting middle open sub-interval of I_τ inductively for each $\tau \in \{1,2\}^n$, where $n = 0,1,2,\cdots$. Consider $E_n = \cup_{\tau\in\{1,2\}^n} I_\tau$. Then (E_n) is a decreasing sequence of closed sets. For each n, we put $\mid I_{\tau,1} \mid / \mid I_\tau \mid = c_{\tau,1}$ and $\mid I_{\tau,2} \mid / \mid I_\tau \mid = c_{\tau,2}$ for all $\tau \in \{1,2\}^n$,where $\mid I \mid$ denotes the diameter of I. We call $F = \bigcap_{n=0}^\infty E_n$ a deranged Cantor set.

We note that if $y \in F$, then there is $\sigma \in \{1,2\}^N$ such that $\bigcap_{k=0}^\infty I_{\sigma|k} = \{y\}$ (Here $\sigma|k = i_1, i_2, \cdots, i_k$ where $\sigma = i_1, i_2, \cdots, i_k, i_{k+1}, \cdots$). Hereafter, we use $\sigma \in \{1,2\}^N$ and $y \in F$ as the same identity freely.

We ([3]) recall the local Hausdorff dimension $f(\sigma)$ of σ in F

$$f(\sigma) = \inf\{s > 0 : h^s(\sigma) = 0\} = \sup\{s > 0 : h^s(\sigma) = \infty\}$$

where the s-dimensional local Hausdorff measure

$$h^s(\sigma) = \liminf_{k\to\infty}(c_1^s + c_2^s)(c_{\sigma|1,1}^s + c_{\sigma|1,2}^s)(c_{\sigma|2,1}^s + c_{\sigma|2,2}^s)\cdots(c_{\sigma|k,1}^s + c_{\sigma|k,2}^s),$$

and dually the local packing dimension $g(\sigma)$ of σ in F

$$g(\sigma) = \inf\{s > 0 : q^s(\sigma) = 0\} = \sup\{s > 0 : q^s(\sigma) = \infty\}$$

where the s-dimensional local packing measure

$$q^s(\sigma) = \limsup_{k\to\infty}(c_1^s + c_2^s)(c_{\sigma|1,1}^s + c_{\sigma|1,2}^s)(c_{\sigma|2,1}^s + c_{\sigma|2,2}^s)\cdots(c_{\sigma|k,1}^s + c_{\sigma|k,2}^s).$$

We recall the s-dimensional Hausdorff measure of F :

$$H^s(F) = \lim_{\delta\to 0} H^s_\delta(F),$$

where $H^s_\delta(F) = \inf\{\sum_{n=1}^\infty \mid U_n \mid^s : \{U_n\}_{n=1}^\infty$ is a δ-cover of $F\}$, and the Hausdorff dimension of F :

$$\begin{aligned}&\dim_H(F)\\ &= \sup\{s > 0 : H^s(F) = \infty\}(= \inf\{s > 0 : H^s(F) = 0\})([5],[6]).\end{aligned}$$

Also we recall the s-dimensional packing measure of F :

$$p^s(F) = \inf\{\sum_{n=1}^\infty P^s(F_n) : \bigcup_{n=1}^\infty F_n = F\},$$

where $P^s(E) = \lim_{\delta\to 0} P^s_\delta(E)$ and $P^s_\delta(E) = \sup\{\sum_{n=1}^\infty \mid U_n \mid^s : \{U_n\}_{n=1}^\infty$ is a δ-packing of E $\}$, and the packing dimension of F :

$$\begin{aligned}&\dim_p(F)\\ &= \sup\{s > 0 : p^s(F) = \infty\}(= \inf\{s > 0 : p^s(F) = 0\})([5],[6],[7]).\end{aligned}$$

We note that a deranged Cantor set satisfying $c_{\tau,1} = a_{n+1}$ and $c_{\tau,2} = b_{n+1}$ for all $\tau \in \{1,2\}^n$, for each $n = 0, 1, 2, \cdots$ is called a perturbed Cantor set ([1]).

In a perturbed Cantor set we have the same s-dimensional local Hausdorff measure $h^s(\sigma)$ for all $\sigma \in F$ and the same s-dimensional local packing measure $q^s(\sigma)$ for all $\sigma \in F$. So we have the same local Hausdorff dimension $f(\sigma)$ for all $\sigma \in F$,which we ([1]) call the lower Cantor dimension($\equiv \dim_{\underline{C}}(F)$) of F, and the same local packing dimension $g(\sigma)$ for all $\sigma \in F$, which we call the upper Cantor dimension ($\equiv \dim_{\overline{C}}(F)$) of F.

3. Main results

In this section, F means a deranged Cantor set determined by $\{c_\tau\}$ with $\tau \in \{1,2\}^n$ where $n = 1, 2, \cdots$. Hereafter we only consider a deranged Cantor set whose contraction ratios $\{c_\tau\}$ and gap ratios $\{d_\tau(= 1 - (c_{\tau,1} + c_{\tau,2}))\}$ are uniformly bounded away from 0.

Before going into our principal results Theorems 6 and 7, it is fruitful to know some properties(Lemma 1-Corollary 4) of the ratios $\{a_n\}, \{b_n\}$ of a perturbed Cantor set F.

LEMMA 1 ([1]). *For the solutions s_n satisfying $a_n^{s_n} + b_n^{s_n} = 1$,*

$$0 < \liminf_{n\to\infty} s_n \le \dim_{\underline{C}}(F) \le \dim_{\overline{C}}(F) \le \limsup_{n\to\infty} s_n < 1.$$

The following theorem gives an improved form of the definition of Cantor dimensions.

THEOREM 2. $\dim_{\underline{C}}(F) = \liminf_{n\to\infty} x_n$ *and* $\dim_{\overline{C}}(F) = \limsup_{n\to\infty} x_n$, *where x_n is the solution s of the equation $\prod_{k=1}^n (a_k^s + b_k^s) = 1$ for each $n \in N$.*

Proof. Let $\liminf_{n\to\infty} x_n = \underline{x}$. Let $\epsilon > 0$ such that $\underline{x} - \epsilon > 0$. Then $0 < \underline{x} - \epsilon < x_n$ for all but finitely many $n \in N$. Hence $\prod_{k=1}^n (a_k^{\underline{x}-\epsilon} + b_k^{\underline{x}-\epsilon}) > \prod_{k=1}^n (a_k^{x_n} + b_k^{x_n}) = 1$ for all but finitely many $n \in N$. So $\liminf_{n\to\infty} \prod_{k=1}^n (a_k^{\underline{x}-\epsilon} + b_k^{\underline{x}-\epsilon}) \ge 1$, which gives $\underline{x} - \epsilon \le \dim_{\underline{C}}(F)$. Clearly $x_n < \underline{x} + \epsilon$ for infinitely many $n \in N$. Then $\prod_{k=1}^n (a_k^{\underline{x}+\epsilon} + b_k^{\underline{x}+\epsilon}) < \prod_{k=1}^n (a_k^{x_n} + b_k^{x_n}) = 1$ for infinitely many $n \in N$. Hence $\liminf_{n\to\infty} \prod_{k=1}^n (a_k^{\underline{x}+\epsilon} + b_k^{\underline{x}+\epsilon}) \le 1$, which gives $\underline{x} + \epsilon \ge \dim_{\underline{C}}(F)$.

Similar arguments give $\dim_{\overline{C}}(F) = \limsup_{n\to\infty} x_n$. □

COROLLARY 3. $0 < \liminf_{n\to\infty} s_n \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} s_n < 1$, *where* s_n *is the solution* s *of the equation* $a_n^s + b_n^s = 1$ *and* x_n *is the solution* s *of the equation* $\prod_{k=1}^{n}(a_k^s + b_k^s) = 1$ *for each* $n \in N$.

Proof. It is immediate from Lemma 1 and Theorem 2. □

From Corollary 3, we have an interesting information about the solutions of the power equations induced from contraction ratios of a perturbed Cantor set.

COROLLARY 4. *If* (s_n) *converges to some* s, *then* (x_n) *converges to* s.

COROLLARY 5. $\liminf_{k\to\infty} s_{\sigma|k} \le \liminf_{k\to\infty} x_{\sigma|k} = f(\sigma)$ *and* $g(\sigma) = \limsup_{k\to\infty} x_{\sigma|k} \le \limsup_{k\to\infty} s_{\sigma|k}$ *for all* σ *in* F *where* $f(\sigma)$ *is the local Hausdorff dimension of* σ *and* $g(\sigma)$ *is the local packing dimension of* σ *and* $x_{\sigma|k}$ *is the solution* s *of* $\prod_{i=0}^{k}(c_{\sigma|i,1}^s + c_{\sigma|i,2}^s) = 1$ *and* $s_{\sigma|k}$ *is the solution* s *of* $c_{\sigma|k,1}^s + c_{\sigma|k,2}^s = 1$.

Proof. It is immediate from the same arguments of Lemma 1 and Theorem 2 applied to a deranged Cantor set. □

The following theorem is a dual form of the theorem 1 in [3] noting that $\liminf_{k\to\infty} x_{\sigma|k} = f(\sigma)$.

THEOREM 6. *Let* $x_{\sigma|k}$ *satisfy* $\prod_{i=0}^{k}(c_{\sigma|i,1}^{x_{\sigma|k}} + c_{\sigma|i,2}^{x_{\sigma|k}}) = 1$. *If positive* s *is given, and for any* $0 < t < s$, $\mu_t(\{\sigma \in \{1,2\}^N : \liminf_{k\to\infty} x_{\sigma|k} \ge s\}) > 0$, *where* μ_t *is the Borel measure on* F *satisfying*

$$\mu_t(I_\tau) = \frac{|I_\tau|^t}{(c_1^t + c_2^t)(c_{i_1,1}^t + c_{i_1,2}^t)\cdots(c_{i_1,i_2,\cdots,i_{k-1},1}^t + c_{i_1,i_2,\cdots,i_{k-1},2}^t)}$$

for each $\tau = i_1, i_2, \cdots, i_{k-1}, i_k$ *where* $i_j \in \{1,2\}$, *then* $\dim_H(\{\sigma \in \{1,2\}^N : \liminf_{k\to\infty} x_{\sigma|k} \ge s\}) \ge s$. *Hence if* $\liminf_{k\to\infty} x_{\sigma|k} \ge s$ *for all except for countable* σ *in* F *then* $\dim_H(F) \ge s$.

Proof. Let $\sigma \in \{\sigma \in \{1,2\}^N : \liminf_{k\to\infty} x_{\sigma|k} \ge s\}$. Then for a given natural number m, $x_{\sigma|k} > s - \frac{1}{m}$ for all but finitely many k.

Then $h^{s-\frac{2}{m}}(\sigma) = \liminf_{k\to\infty} \prod_{i=0}^{k}(c_{\sigma|i,1}^{s-\frac{2}{m}} + c_{\sigma|i,2}^{s-\frac{2}{m}}) = \infty$.

Since $\mu_t(\{\sigma \in \{1,2\}^N : \liminf_{k\to\infty} x_{\sigma|k} \ge s\}) > 0$ for any $t < s$, following the density theorem argument used in the proof of the theorem 1 in [3], we get $H^{s-\frac{2}{m}}(\{\sigma \in \{1,2\}^N : \liminf_{k\to\infty} x_{\sigma|k} \ge s\}) = \infty$ for each m. Thus $\dim_H(\{\sigma \in \{1,2\}^N : \liminf_{k\to\infty} x_{\sigma|k} \ge s\}) \ge s$.

Now assume that $\liminf_{k\to\infty} x_{\sigma|k} \geq s$ for all except for countable σ in F. Noting that every countable subset of F has μ_t-measure 0 and $\mu_t(F) = 1$ for each t, we easily see that $\dim_H(F) \geq s$. □

We remark that second statement in Theorem 6 also follows from the theorem 1 in [3]. Now in packing dimension case, the first statement of the following theorem is essentially the same as the corollary 7 in [3] noting that $\limsup_{k\to\infty} x_{\sigma|k} = g(\sigma)$. But we rewrite it and give a proof to give a contrast with the corollary 8 in [3].

THEOREM 7. *Let $x_{\sigma|k}$ satisfy $\prod_{i=0}^{k}(c_{\sigma|i,1}^{x_{\sigma|k}} + c_{\sigma|i,2}^{x_{\sigma|k}}) = 1$. If positive s is given, then $\dim_p(\{\sigma \in \{1,2\}^N : \limsup_{k\to\infty} x_{\sigma|k} \leq s\}) \leq s$.*

Hence $\dim_p(F) \leq s$ if $\limsup_{k\to\infty} x_{\sigma|k} \leq s$ for all except for countable σ in F.

Proof. Let $\sigma \in \{\sigma \in \{1,2\}^N : \limsup_{k\to\infty} x_{\sigma|k} \leq s\}$. Then for a given natural number m, $x_{\sigma|k} < s + \frac{1}{m}$ for all but finitely many k.

Then $q^{s+\frac{2}{m}}(\sigma) = \limsup_{k\to\infty} \prod_{i=0}^{k}(c_{\sigma|i,1}^{s+\frac{2}{m}} + c_{\sigma|i,2}^{s+\frac{2}{m}}) = 0$.

Following the dual density theorem argument related to packing measure used in the proof of the corollary 3 in [3], we get

$$p^{s+\frac{2}{m}}(\{\sigma \in \{1,2\}^N : \limsup_{k\to\infty} x_{\sigma|k} \leq s\}) = 0$$

for each m.

Thus $\dim_p(\{\sigma \in \{1,2\}^N : \limsup_{k\to\infty} x_{\sigma|k} \leq s\}) \leq s$.

Now assume that $\limsup_{k\to\infty} x_{\sigma|k} \leq s$ for all except for countable σ in F. Since the packing dimension of a set is invariant under an increment of a countable set, we have $\dim_p(F) \leq s$. □

COROLLARY 8. *Let $x_{\sigma|k}$ be the solution s of $\prod_{i=0}^{k}(c_{\sigma|i,1}^{s} + c_{\sigma|i,2}^{s}) = 1$.*

If $\liminf_{k\to\infty} x_{\sigma|k} = \underline{x}$ for all $\sigma \in \{1,2\}^N$, then $\dim_H(F) = \underline{x}$. Similarly if $\limsup_{k\to\infty} x_{\sigma|k} = \overline{x}$ for all $\sigma \in \{1,2\}^N$, then $\dim_p(F) = \overline{x}$.

Proof. $\dim_H(F) = \underline{x}$ follows from our Theorem 6 and the corollary 3 in [3]. Similarly $\dim_p(F) = \overline{x}$ follows from our Theorem 7 and the theorem 5 in [3]. □

COROLLARY 9 ([1]). *Let F be a perturbed Cantor set. If $\liminf_{k\to\infty} x_n = \underline{x}$ then $\dim_H(F) = \underline{x}$ where x_n is the solution s of the equation $\prod_{k=1}^{n}(a_k^s + b_k^s) = 1$ for each $n \in N$. Similarly if $\limsup_{k\to\infty} x_n = \overline{x}$, then $\dim_p(F) = \overline{x}$.*

We remark that easily we can find many examples of deranged Cantor sets satisfying the conditions of Corollary 8 but Corollary 9.

COROLLARY 10 ([2]). *Let* $s_{\sigma|k}$ *satisfy* $c_{\sigma|k,1}^{s_{\sigma|k}} + c_{\sigma|k,2}^{s_{\sigma|k}} = 1$. *If* $\lim_{k\to\infty} s_{\sigma|k} = s$ *for all* $\sigma \in \{1,2\}^N$, *then* $\dim_H(F) = \dim_p(F) = s$.

Proof. It is immediate from Corollaries 5 and 8. □

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