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ON THE ALMOST SURE CONVERGENCE OF WEIGHTED SUMS OF NA RANDOM VARIABLES[†]

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ABSTRACT

Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed, negatively associated (NA) random variables and assume that $|X|^r$, $r > 0$, has a finite moment generating function. A strong law of large numbers is established for weighted sums of these variables.

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1. INTRODUCTION

A finite family $\{X_i, 1 \leq i \leq n\}$ of random variables is said to be negatively associated (NA) if, for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f(X_i, i \in A),\ g(X_j, j \in B)) \leq 0,$$

whenever f and g are coordinatewise nondecreasing functions such that their covariance exists. An infinite family of random variables is said to be NA if every finite subfamily is NA. This definition was introduced by Alam and Saxena (1981) and carefully studied by Joag-Dev and Proschan (1983). Because of its wide applications in multivariate statistical analysis and reliability, the notion of NA has been received with considerable attention recently. We refer to Joag-Dev and Proschan (1983) for their fundamental properties, Newman (1984) for

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central limit theorem, Matula (1992) for three series theorem, Shao (2000) for Rosenthal-type inequality and Kolmogorov exponential inequality, Shao and Su (1999) for the law of the iterated logarithm and Su *et al.* (1997) for moment inequality and weak convergence.

Studies of strong laws for weighted sums have provided significant progress in probability theory with applications in mathematical statistics. The almost sure limiting law of weighted sums $\sum_{i=1}^{n} a_{ni}X_i$, where $\{X, X_i, i \geq 1\}$ is a sequence of *iid* random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of weights, was investigated by many authors (see Bai and Cheng, 2000; Chow and Lai, 1973). For uniformly bounded weights $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$, Teicher (1985) obtained

$$\limsup_{n\to\infty} \frac{|\sum_{i=1}^{n} a_{ni}X_i|}{b_n} = 0 \quad a.s. \tag{1.1}$$

at a rate $b_n = n^{1/\alpha}\log n$ for $1 < \alpha \leq 2$, and Chow and Lai (1973) considered the case of $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(1)$ for some $\alpha > 0$. Strong laws of the form (1.1) with more general normalizing constants b_n were also obtained by Cuzick (1995) under a moment condition $A_{\alpha,n} = (n^{-1}\sum_{i=1}^{n} |a_{ni}|^{\alpha})^{1/\alpha} = O(1)$ and some additional conditions on the distribution of X. Recently, Bai and Cheng (2000) derived the strong laws of large numbers by considering a standard case when the moment generating function exists: Let $\{X, X_i, i \geq 1\}$ be a sequence of *iid* random variables with $EX = 0$ and

$$E\{\exp(h|X|^r)\} < \infty \quad \text{for some } h > 0 \text{ and } r > 0 \tag{1.2}$$

and let $\{a_{ni}, 1 \leq i \leq n\}$ be an array of real numbers such that

$$A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty, \quad A_{\alpha,n}^{\alpha} = n^{-1}\sum_{i=1}^{n} |a_{ni}|^{\alpha}. \tag{1.3}$$

They showed that if (1.2) holds and (1.3) holds for $\alpha \in (0, 2)$, then, for $0 < \alpha \leq 1$ and $b_n = n^{1/\alpha}(\log n)^{1/r}$, we have

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \frac{|a_{ni}X_i|}{b_n} \leq h^{-1/r}A_\alpha, \quad a.s. \tag{1.4}$$

Moreover, for $1 < \alpha < 2$ and $b_n = n^{1/\alpha}(\log n)^{(1/r)+r(\alpha-1)/\alpha(1+r)}$, we have

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \frac{|a_{ni}X_i|}{b_n} = 0 \quad a.s. \tag{1.5}$$

In this paper, we study the similar almost sure limiting behavior on the weighted sums of identically distributed NA random variables of the form (1.1) under stronger condition on the moment generating function

$$E\{\exp(h|X|^r)\} < \infty \quad \text{for any} \quad h > 0 \quad \text{and} \quad r > 0. \tag{1.6}$$

Note that our result is different from Cheng and Gan (1998) for the almost sure convergence of $\sum_{i=1}^{k_n} a_{ni}X_i$, where $\{X_n, n \geq 1\}$ is a sequence of negatively associated (NA) random variables which are stochastically bounded by a non-negative random variable X, and $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ is a double array of constants.

2. Preliminaries

We start with the properties of negatively associated (NA) random variables.

Property P_2 (Joag-Dev and Proschan, 1983). *Let $A_1, \ldots, A_k$ be disjoint subsets of $\{1, \ldots, n\}$ and $f_1, f_2, \ldots, f_k$ be increasing positive functions. If $\{X_i : 1 \leq i \leq n\}$ is NA, then*

$$E\prod_{i=1}^{k} f_i(X_j : j \in A_i) \leq \prod_{i=1}^{k} Ef_i(X_j : j \in A_i).$$

Property P_6 (Joag-Dev and Proschan, 1983). *Increasing functions defined on disjoint subsets of a set of NA random variables are NA.*

Lemma 2.1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables satisfying (1.6). Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise NA random variables with $EX_{ni} = 0$ for $1 \leq i \leq n$ and $n \geq 1$, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of positive constants. Assume that, for $1 \leq i \leq n$, some $0 < \beta \leq r$ and some constant $C > 0$*

$$|a_{ni}X_{ni}| \leq C\frac{|X_i|^\beta}{\log n} \quad a.s. \tag{2.1}$$

and, for some sequence $\{u_n\}$ of positive constants such that $\lim_{n\to\infty} u_n = 0$, and some $\delta > 0$,

$$X_{ni}^2 \sum_{i=1}^{n} a_{ni}^2 \leq \frac{u_n |X_i|^\delta}{\log n} \quad a.s. \tag{2.2}$$

Then

$$\lim_{n\to\infty}\sum_{i=1}^{n} a_{ni}X_{ni} = 0 \quad a.s. \tag{2.3}$$

PROOF. From the inequality

$$e^x \le 1 + x + \frac{1}{2}x^2 e^{|x|} \quad \text{for all} \quad x \in \mathbb{R}$$

we have

$$E\{\exp(ta_{ni}X_{ni})\} \le 1 + \frac{1}{2}t^2 a_{ni}^2 E\{X_{ni}^2 \exp(ta_{ni}|X_{ni}|)\} \quad \text{for any } t > 0.$$

Let $\epsilon > 0$ and put $t = 2\,(\log n)/\epsilon$. It follows from (2.1) and (2.2) that

$$\begin{aligned}
E\{\exp(ta_{ni}X_{ni})\} &\le 1 + \frac{1}{2}\left(\frac{2}{\epsilon}\right)^2 (\log n)^2 a_{ni}^2 E\left\{X_{ni}^2 \exp\left(\frac{2}{\epsilon}\log n a_{ni}|X_{ni}|\right)\right\}\\
&\le 1 + \frac{2}{\epsilon^2}u_n \log n\left(\frac{a_{ni}^2}{\sum_{i=1}^n a_{ni}^2}\right) E\left\{|X_i|^\delta \exp\left(\frac{2}{\epsilon}C|X_i|^\beta\right)\right\}\\
&\le 1 + \frac{2}{\epsilon^2}u_n \log n\left(\frac{a_{ni}^2}{\sum_{i=1}^n a_{ni}^2}\right) E\left\{\exp\left(\frac{2}{\epsilon}C'|X_i|^\beta\right)\right\}\\
&\le 1 + \frac{1}{2}\log n\left(\frac{a_{ni}^2}{\sum_{i=1}^n a_{ni}^2}\right)\\
&\le \exp\left\{\frac{1}{2}\log n\left(\frac{a_{ni}^2}{\sum_{i=1}^n a_{ni}^2}\right)\right\}
\end{aligned} \tag{2.4}$$

for all large n and some $C' > 0$. Note that

$$E\exp\left(t\sum_{i=1}^n a_{ni}X_{ni}\right) = E\prod_{i=1}^n \exp(ta_{ni}X_{ni}) \le \prod_{i=1}^n E\exp(ta_{ni}X_{ni}) \tag{2.5}$$

by Property P_2 and Property P_6. By Markov inequality, (2.4) and (2.5),

$$\begin{aligned}
P\left(\sum_{i=1}^n a_{ni}X_{ni} \ge \epsilon\right) &\le e^{-t\epsilon}E\left\{\exp\left(t\sum_{i=1}^n a_{ni}X_{ni}\right)\right\}\\
&\le e^{-t\epsilon}\prod_{i=1}^n E\exp(ta_{ni}X_{ni})\\
&\le e^{-2\log n}\prod_{i=1}^n \exp\left\{\frac{1}{2}\log n\left(\frac{a_{ni}^2}{\sum_{i=1}^n a_{ni}^2}\right)\right\}\\
&= n^{-3/2}.
\end{aligned} \tag{2.6}$$

which is summable. Since $-X_{ni}$'s are NA by Property P_6, by replacing X_{ni} with $-X_{ni}$ from the above statement, we obtain

$$P\left(\sum_{i=1}^{n} a_{ni}X_{ni} \leq -\epsilon\right) \leq n^{-\frac{3}{2}} \text{ for all large } n. \tag{2.7}$$

Hence, by (2.6), (2.7) and the Borel-Cantelli lemma the result (2.3) follows. □

REMARK 2.1. Lemma 2.1 can be extended to the case where $\{a_{ni}\}$ is an array of real numbers.

3. RESULTS

THEOREM 3.1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed and symmetric NA random variables with $EX = 0$ and satisfying (1.6) and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers such that (1.3) holds for some $1 < \alpha \leq 2$. Then, for $0 < r \leq 1$ and $b_n = n^{1/\alpha}(\log n)^{1/r}$,*

$$\sum_{i=1}^{n} \frac{a_{ni}X_i}{b_n} \to 0 \quad a.s. \tag{3.1}$$

PROOF. Define

$$\begin{aligned} X'_{ni} = {} & X_i I(|X_i| \leq (\log n)^{1/r}) - (\log n)^{1/r} I(X_i < -(\log n)^{1/r}) \\ & + (\log n)^{1/r} I(X_i > (\log n)^{1/r}) \end{aligned}$$

and

$$X''_{ni} = X_i - X'_{ni} \text{ for } 1 \leq i \leq n,\ n \geq 1.$$

First note that $\{X'_{ni}\}$ and $\{X''_{ni}\}$ are negatively associated by Property P_6. We also note that

$$E\exp|X|^r < \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} P(|X_n| > (\log n)^{1/r}) < \infty.$$

Hence, $\sum_{i=1}^{\infty} P(|X''_{ni}| > 0) < \infty$ and so by the Borel-Cantelli lemma, $P(|X''_{ni}| >$

$0\ i.o.) = 0$, that is, $\sum_{i=1}^{n} |X''_{ni}|$ is bounded almost surely. It follows that

$$\begin{aligned} b_n^{-1}\left|\sum_{i=1}^{n} a_{ni}X''_{ni}\right| &\le b_n^{-1}\sum_{i=1}^{n}\left|a_{ni}X''_{ni}\right| \\ &\le b_n^{-1}\max_{1\le i\le n}|a_{ni}|\sum_{i=1}^{n}\left|X''_{ni}\right| \\ &\le b_n^{-1}\left(\sum_{i=1}^{n}|a_{ni}|\right)\sum_{i=1}^{n}\left|X''_{ni}\right| \\ &\le A_{\alpha,n}\sum_{i=1}^{n}\frac{|X''_{ni}|}{(\log n)^{1/r}} \to 0 \quad a.s. \end{aligned} \tag{3.2}$$

We will apply Lemma 2.1 to the random variable X'_{ni} and weight $b_n^{-1}a_{ni}$. Note that

$$\begin{aligned} |b_n^{-1}a_{ni}X'_{ni}| &\le b_n^{-1}|a_{ni}|(\log n)^{(1-r)/r}|X_i|^r \\ &\le b_n^{-1}A_{\alpha,n}n^{1/\alpha}(\log n)^{(1-r)/r}|X_i|^r \\ &= \frac{A_{\alpha,n}|X_i|^r}{\log n} \end{aligned}$$

and

$$\begin{aligned} (X'_{ni})^2\sum_{i=1}^{n} b_n^{-2}a_{ni}^2 &\le X_i^2 b_n^{-2}\sum_{i=1}^{n} a_{ni}^2 \\ &\le \frac{A_{\alpha,n}^2 X_i^2}{(\log n)^{2/r}}. \end{aligned}$$

Hence, conditions (2.1) and (2.2) of Lemma 2.1 are satisfied and so

$$\sum_{i=1}^{n}\frac{a_{ni}X'_{ni}}{b_n} \to 0 \quad a.s. \tag{3.3}$$

Thus the desired result follows by (3.2) and (3.3). □

REMARK 3.1. If (3.1) holds for any array $\{a_{ni}\}$ satisfying (1.3), then $EX = 0$ and (1.6) holds. The proof is similar to that of Bai and Cheng (2000, pp. 108–109). Note that (1.6) is equivalent to the condition that $X_n/(\log n)^{1/r} \to 0\ a.s.$

THEOREM 3.2. *Let* $\{X, X_n, n \geq 1\}$ *be a sequence of identically distributed NA random variables satisfying* $EX = 0$ *and (1.2) and let* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be an array of constants satisfying (1.3) for some* $1 < \alpha \leq 2$. *Then, for* $0 < r \leq 1$ *and* $b_n = n^{1/\alpha}(\log n)^{(1/r)+\beta}$ $(\beta > 0)$,

$$\sum_{i=1}^{n} \frac{a_{ni}X_i}{b_n} \to 0 \quad a.s.$$

PROOF. Define

$$\begin{aligned} X'_{ni} &= X_i I(|X_i| \leq (h^{-1}\log n)^{1/r}) - (h^{-1}\log n)^{1/r} I(X_i < -(h^{-1}\log n)^{1/r}) \\ &\quad + (h^{-1}\log n)^{1/r} I(X_i > (h^{-1}\log n)^{1/r}) \end{aligned}$$

and $X''_{ni} = X_i - X'_{ni}$ for $1 \leq i \leq n$ and $n \geq 1$. The rest of the proof is similar to that of Theorem 3.1 and is omitted. □

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