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A MINIMUM THEOREM FOR THE RELATIVE ROOT NIELSEN NUMBER

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ABSTRACT. In [1], a relative root Nielsen number $N_{rel}(f;c)$ is introduced which is a homotopy invariant lower bound for the number of roots at $c \in Y$ for a map of pairs of spaces $f : (X, A) \to (Y, B)$. In this paper, we obtain a minimum theorem for $N_{rel}(f;c)$ under some new assumptions on the spaces and maps which are different from those in [1].

1. Introduction

Let $f : X \to Y$ be a map and $c \in Y$ a point. A root of f at c is a point $x \in X$ that is a solution to the equation $f(x) = c$. Denote the set of roots by $root(f;c)$ and let $\sharp root(f;c)$ be the cardinality of that set. Nielsen root theory is concerned with

$$MR[f;c] = min\{\sharp root(g;c) : g \sim f\},$$

where $g \sim f$ means that the minimum is taken over all maps g homotopic to f.

By analogy with Nielsen fixed point and coincidence theory, a Nielsen number of roots $N(f;c)$ was defined by Hopf and Brooks which is a lower bound for $MR[f;c]$. Hopf showed that there exist maps f between closed oriented surfaces for which $N(f;c)$ is strictly less than $MR[f;c]$ but if $f : X \to Y$ is a map between closed oriented n-manifolds with $n \neq 2$, then $N(f;c) = MR[f;c]$, which is the first minimum theorem.

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In the setting of maps of pairs, i.e. maps of the form $f : (X, A) \to (Y, B)$, instead of the minimum number $MR[f; c]$, we are concerned with

$$MR_{rel}[f; c] = min\{\#root(g; c) : g \approx f\},$$

where $g \approx f$ means that the minimum is taken over all maps g homotopic, as maps of pairs, to f, i.e. there is a map of pairs $H : (X \times I, A \times I) \to (Y, B)$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

In [7], Yang defined a relative Nielsen number $N(f; X, A, c)$ for roots of relative map $f : (X, A) \to (Y, B)$ at $c \in Y$ and showed that this number is a homotopy invariant lower bound of $MR_{rel}[f; c]$. Later, Brown and Schirmer presented more precise lower bound $N_{rel}(f; c)$, and obtained some minimum theorems. "No local cut point" and "by passing" lie in the sufficient conditions in the minimum theorems of classical and relative Nielsen numbers. In [8], Zhao introduced a new concept "local cut set" and showed the relation between the "no local cut set" and "by passing" conditions in Nielsen fixed point theory.

The purpose of this paper is to prove a new minimum theorem for the relative root Nielsen number $N_{rel}(f; c)$ under some new assumption on the space pairs and maps which are different from those in [1]. In some case, the "by passing" condition may be replaced by "no local cut set".

The definitions for Nielsen root numbers in this paper are based on [1].

2. The relative root Nielsen number

Throughout this paper, we always assume that X and Y are compact connected polyhedra, and that A and B be closed subpolyhedra of X and Y, respectively. Consider the relative of the form $f : (X, A) \to (Y, B)$, we shall denote by $\bar{f} : A \to B$ the restriction of f to A. Let $c \in Y$ be any point.

We recall from [5] that the *root classes* of $f : X \to Y$ are the equivalence classes of $\mathrm{root}(f; c)$ under the following equivalence relation. Points $x, x' \in root(f; c)$ are equivalent if there is a path w in X from x to x' such that $\langle f \circ w \rangle = 1 \in \pi_1(Y, c)$. The definition of root class applies as well to the restriction $\bar{f} : A \to B$, keeping in mind that there are no root classes if $c \notin B$ and that if $x, x' \in A$ are not in the same path component, they cannot be members of the same root class. Each root class of $\bar{f}$ is contained in some root class of $f : X \to Y$. By [5; p.126] the maps $f : X \to Y$ and $\bar{f} : A \to B$ have only finitely many root classes at c.

Suppose $K : X \times I \to Y$ is a relative homotopy, so we have the maps $k_t : (X, A) \to (Y, B)$ defined by $k_t(x) = K(x, t)$. Given a root class R of k_0 at $c \in Y$, there is a root class $\mathbb{R}$ of K at c containing R, and each nonempty t-slice $[\mathbb{R}]_t$ is a root class of k_t so, in particular $[\mathbb{R}]_0 = R$. A root class R of $f : X \to Y$ is *inessential* if there is a homotopy $K : X \times I \to Y$ such that $k_0 = f$ with $[\mathbb{R}]_0 = R$ and $[\mathbb{R}]_1 = \varnothing$. A root class that is not inessential is said to be *essential* and the *Nielsen number* $N(f; c)$ of roots of f at c is defined to be the number of essential root classes for f at c.

In the setting of maps of pairs, there is a modification of the concept of essential root class as follows.

Definition 2.1. ([1; Definition 2.1]) Let $f : (X, A) \to (Y, B)$ be a relative map and $c \in Y$ a point. A root class R of $f : X \to Y$ at c is *relatively inessential* if there is a homotopy of pairs $K : (X \times I) \to (Y, B)$ such that $k_0 = f$ and the root class $\mathbb{R}$ of K with $[\mathbb{R}]_0 = R$ and $[\mathbb{R}]_1 = \varnothing$. Otherwise, we say that the root class R of f is *relatively essential.* We will use the notation $N^+(f; c)$ to denote the number of relatively essential root classes of f at c.

An essential root class is relatively essential, so $N^+(f; c) \geq N(f; c)$. They are not always equal. Consider the identity map $f : (D, \partial D) \to (D, \partial D)$. For any $c \in D$, there is a single root class for $f : D \to D$. It is inessential, thus $N(f; c) = 0$, but it is relatively essential so $N^+(f; c) = 1$.

If, for a map $f : (X, A) \to (Y, B)$ and $c \in Y$, a root class R of $f : X \to Y$ at c contains an essential root class $\bar{R}$ of $\bar{f} : A \to B$ at c, then R is called a *common root class* of f and $\bar{f}$ at c. We denote the number of relatively essential common root classes by $N^+(f, \bar{f}; c)$.

Definition 2.2. ([1; Definition 2.2]) Let $f : (X, A) \to (Y, B)$ be a map and choose $c \in Y$. Define $N_{rel}(f; c)$, the relative Nielsen number of roots of $f : (X, A) \to (Y, B)$ at $c \in Y$, as follows:

$$N_{rel}(f; c) = N(\bar{f}; c) + N(f; c) - N^+(f, \bar{f}; c).$$

It follows easily from the definition that $N_{rel}(f; c) \geq N(f; c)$. We also note that $N_{rel}(f; c) = N(f; c)$ if $B = \varnothing$, so the relative Nielsen number specializes to the root Nielsen number when f is not a map of pairs. Although the definition of $N_{rel}(f;c)$ applies to any point $c \in Y$, if $c \notin B$ there are no root classes of $\bar{f}$, so the definition simplifies to $N_{rel}(f;c) = N^{+}(f;c)$.

By definition, we know that $N_{rel}(f;c)$ has the relatively homotopic invariance, and that any map relatively homotopic to $f : (X, A) \to (Y, B)$ has at least $N_{rel}(f;c)$ roots at c, i.e. $N_{rel}(f;c) \leq MR_{rel}[f;c]$ (cf. [1; Theorem 2.5]).

3. Local cut set

A local cut point x in X is the point at which there is a connected neighborhood U such that $U - \{x\}$ is not connected. For a space X with local cut points, $MR[f]$ may be larger than $N(f)$ even if f is a deformation. Zhao, in [8], introduces a new concept "local cut set", which is a generalization of the "local cut point".

DEFINITION 3.1. ([8; Definition 3.2]) A connected subpolyhedron A of X is said to be a *local cut set* of X if there is an open connected neighborhood $N(A)$ of A in X such that the set $N(A) - A$ is not connected.

It was shown in [8] that the property of "no local cut set" played the similar role as the "by-passing".

The next lemma is a root version of [8; Lemma 3.5].

LEMMA 3.2. *Let $f : (X, A) \to (Y, B)$ be a relative map. If for each component A_k component of A, A_k is not a local cut set of X and the restriction $f|_{A_k}$ of f on A_k induces a trivial homomorphism $\pi_1(A_k) \to \pi_1(B)$. Then two roots x_0 and x_1 of f on $X - A$ are in the same root class if and only if there exists a path w in $X - A$ from x_0 to x_1 such that $\langle f \circ w \rangle = 1 \in \pi_1(Y, c)$.*

PROOF. "If". It is trivial.

"Only if". Since x_0 and x_1 are in the same root class of f, there is a path $p : I \to X$ from x_0 to x_1 such that $\langle f \circ p \rangle = 1 \in \pi_1(Y, c)$. By general position technique, we can assume that $p(I) \cap A$ has finitely many components, denoted by $c_1, c_2, \ldots, c_m$. Because every component of A is not a local cut set, we can also assume that, for each k, $k = 1, 2, \ldots, m$, c_k is not a constant path. Hence $p = b_0c_1b_1c_2b_2 \cdots c_mb_m$, where $b_0, b_1, \ldots, b_m$ are in $X - A$ but for their endpoints,

From [8, proposition 3.2], the boundary $Bd(A_j)$ of each component A_j of A is connected. Thus, for each k, there exists a path d_k in $Bd(A) =$

$Bd(X - A)$ such that c_k and d_k have the same endpoints, where $k = 1, 2, \ldots, m$. Set

$$w_i = b_0 d_1 b_1 d_2 b_2 \cdots d_{i-1} b_{i-1}, \ i = 1, 2, \ldots, m+1.$$

Then each w_i is a path in $Cl(X - A)$, and

$$p = (w_1 c_1 d_1^{-1} w_1^{-1})(w_2 c_2 d_2^{-1} w_2^{-1}) \cdots (w_m c_m d_m^{-1} w_m^{-1}) w_{m+1}.$$

Thus, we have, in $\pi_1(Y, c)$, that

$$\begin{aligned} 1 &= \langle f \circ p \rangle \\ &= \langle (f(w_1) f(c_1) f(d_1^{-1}) f(w_1^{-1}) (f(w_2) f(c_2) f(d_2{}^{-1}) f(w_2^{-1}) \cdots \\ &\quad (f(w_m) f(c_m) f(d_m^{-1}) f(w_m^{-1}) f(w_{m+1})) \rangle \\ &= \langle f(w_1) f(c_1 d_1^{-1}) f(w_1)^{-1} f(w_2) f(c_2 d_2{}^{-1}) f(w_2)^{-1} \cdots \\ &\quad f(w_m) f(c_m d_m^{-1}) f(w_m)^{-1} f(w_{m+1}) \rangle. \end{aligned}$$

Since $c_k d_k^{-1}$ is a loop in A ($k = 1, 2, \ldots, m$), from the assumption of the behavior of f on A_k, we have $\langle f(c_k d_k^{-1}) \rangle = 1 \in \pi_1(Y, c)$.

So,

$$\begin{aligned} 1 &= \langle f(w_1) f(w_1)^{-1} f(w_2) f(w_2)^{-1} \cdots f(w_m) f(w_m)^{-1} f(w_{m+1}) \rangle \\ &= \langle f \circ w_{m+1} \rangle, \end{aligned}$$

where w_{m+1} is a path in $Cl(X - A)$ from x_0 to x_1. From [8, proposition 3.3], we know that $Bd(A)$ can be by-passed in $Cl(X - A)$, and hence there exists a path w from x_0 to x_1 such that $w \sim w_{m+1}$. Thus, $\langle f \circ w \rangle = 1 \in \pi_1(Y, c)$. □

With the same method as in the proof of the lemma, we have

COROLLARY 3.3. *Let $f : (X, A) \to (Y, B)$ be a relative map. If for each component A_k component of A, A_k is not a local cut set of X and the restriction $f|_{A_k}$ of f on A_k induces a trivial homomorphism $\pi_1(A_k) \to \pi_1(B)$. Then, for any two roots x_0 and x_1 of f on $X - A$, where $x_0 \in X - A$ and $x_1 \in Bd(A)$, x_0 and x_1 are in the same root class if and only if there exists a path w from x_0 to x_1 such that $w([0, 1)) \subset X - A$ and $\langle f \circ w \rangle = 1 \in \pi_1(B, c)$.*

4. The minimum theorem for $N_{rel}(f;c)$

The root Nielsen number $N(f;c)$ of a map $f : X \to Y$ is called *sharp* if it is a sharp lower bound for the number of roots for all maps in the homotopy class of f, that is, if $N(f;c) = MR[f;c]$. Sharpness of $N(f;c)$ is established by constructing, in the proof of a "minimum theorem", a map $g \sim f$ with precisely $N(f;c)$ roots at c. In general this can be done only in a manifold setting. It is known that the root Nielsen number $N(f;c)$ of $f : X \to Y$ is sharp if both X and Y are closed , connected, oriented PL-manifolds and $n \neq 2$.

In [1], the definition of sharpness is extended to maps of pairs, and say that the relative root Nielsen number $N_{rel}(f;c)$ of a map $f : (X, A) \to (Y, B)$ is *sharp* if $N_{rel}(f;c) = MR_{rel}[f;c]$. And we can establish sharpness of $N_{rel}(f;c)$ only in manifold settings. Thus, we assume that X and Y are connected oriented manifolds of the same dimension. In [1], the sharpness of $N_{rel}(f;c)$ established by proving a minimum theorem where X and Y are closed oriented n-manifolds. The minimum theorem needs that A can be by-passed in X, which is an assumption frequently needed in relative Nielsen theory. This assumption can be relaxed in some sense.

The key point in the proof of minimum is to united roots in the same root class. We write this procedure as a lemma, which is based on [4, Theorem 2.4].

LEMMA 4.1. *Let $f : X \to Y$ be a map between smooth n-manifolds, where $n \geq 3$, and let $c \in Y$. Let x' and x'' be two roots of f at c and w a path from x' to x'' with $\langle f(w)\rangle = 1 \in \pi_1(Y, c)$ and $w(I) \cap root(f, c) = \{x', x''\}$. Then for any open subset W of X with $w((0, 1]) \subset W$ and $x' = w(0) \in Cl(W) - W$, there is map $f' : X \to Y$ such that*

1) $f' \sim f$ rel $X - W$,

2) $root(f', c) = root(f, c) - \{x''\}$.

PROOF. Because $dimX \geq 3$, by a small homotopy, we may change homotopically w into a smooth arc w', i.e. a path without self-intersection such that $w \sim w'$ rel $\{0, 1\}$, $w'((0, 1]) \subset W$, and $w'(I) \cap root(f, c) = \{x', x''\}$. We still have $\langle f(w')\rangle = 1 \in \pi_1(Y, c)$.

Using the tubular neighborhood of $w'(I)$ in X, there is an open subset U in X which is, up to a homeomorphism, considered as a subset of R^n such that

1) $U = \{(x_1, \dots, x_n) | 0 < x_1 < 2,\ -2 < x_i < 2,\ i = 2, \dots, n\} \subset W$,

2) $w'(t) = (t, 0, \dots, 0)$ for all $t \in I$,

3) $Cl(U) - \{x'\} \subset W$.

Take a coordinate neighborhood V of c. We may consider V as the standard united open ball in R^n with $c = 0$ the original point. Since $f(x') = f(x'') = c$, there a small ε $(0 < \varepsilon < \frac{1}{2})$ such that $f(w'([0,\varepsilon])) \cup f(w'([1-\varepsilon,1])) \subset V$. Pick an arc v in V such that $v(t) = f(w'(t))$ for $t \in [0,\varepsilon] \cup [1-\varepsilon, 1]$ and that $v([\varepsilon, 1-\varepsilon])$ is a line segment from $f(w'(\varepsilon))$ to $f(w'(1-\varepsilon))$. Since v lies in the contractible ball V, $\langle v \rangle = 1 \in \pi_1(Y,c)$. Notice that $\langle f(w') \rangle = 1 \in \pi_1(Y,c)$, we have that $f(w') \sim v$ rel $\{0,1\}$. It is no difficulty to see that $f(w') \sim v$ rel $w'([0,\varepsilon] \cup [1-\varepsilon, 1])$. We write $H : w'(I) \times I \to Y$ for this homotopy. Notice that $dimY \geq 3$. The homotopy H is chosen so that $c \notin H(w'([\varepsilon, 1-\varepsilon]) \times I)$.

Define a map $F : X \times \{0\} \cup (K' \cup w'(I) \cup K'' \cup (X - U)) \times I \to Y$ by

$$F(x,t) = \begin{cases} H(x,t) & \text{if } \ x \in w'([\varepsilon, 1-\varepsilon]) \\ f(x) & \text{otherwise,} \end{cases}$$

where $K' = \{(x_1, \dots, x_n) | 0 \leq x_1 \leq \varepsilon, \ -1 < x_i < 1, \ i = 2, \dots, n\}$ and $K'' = \{(x_1, \dots, x_n) | 1 - \varepsilon \leq x_1 \leq 1 + \varepsilon, \ -1 < x_i < 1, \ i = 2, \dots, n\}$.

By the well-known retraction map $r : X \times I \to (K' \cup w'(I) \cup K'' \cup (X - U)) \times I$ (see p.31, Ex. O of [3]). We can extend F into a map $F : X \times I \to Y$. As $F(x,t) \neq c$ for all points (x,t) on the boundary of the extended set $(U - (K' \cup w'(I) \cup K'')) \times I$, we have that $F^{-1}(c) \cap cl(U) = \{x', x''\} \times I$, i.e. $root(F_t, c) \cap Cl(U) = \{x', x''\}$ for all t-slice of F.

Consider the 1-slice F_1 of F. As $F_1(w'(I)) \subset V$, there is a convex open set Q with $w'((0,1]) \subset Q$, $x' = w'(0) \in Bd(Q)$ and $F_1(Q) \subset V$. Notice that in V, the point c is identified with zero point 0. We may homotope F_1 to a map f' defined by

$$f'(x) = \begin{cases} (1-t)F_1(y) & \text{if } \ x = tx' + (1-t)y, \ y \in Bd(Q) - \{x'\} \\ F_1(x) & \text{otherwise.} \end{cases}$$

With the same argument as in [4], f' is the desired map. □

THEOREM 4.2. (*Minimum theorem for closed manifolds*) *Let X and Y be closed oriented smooth n-manifolds, where $n \neq 2$, and let $c \in Y$. Let A be a disjoint union of submanifolds of X, which is not a local cut set of X, and let B be a disjoint union of submanifolds of Y. If $f : (X,A) \to (Y,B)$ is a map of pairs such that each component A_k of A, $f|_{A_k}$ induces a trivial homomorphism from $\pi_1(A_k)$ to $\pi_1(B)$ and if $N(\bar{f};c)$ is sharp, then $N_{rel}(f;c)$ is sharp.*

PROOF. Clearly, this theorem is trivial if the dimension n of X and Y is 1 or 0. In the following proof, we assume that $n \geq 3$.

As $N(\bar{f}; c)$ is sharp, we may assume that f has $N(\bar{f}; c)$ roots on A. If $Int(A) \neq \varnothing$, then A will be an n-manifold with boundary. In this case, we claim that $N(\bar{f}, c) = 0$. In fact, if $N(\bar{f}, c) \neq 0$, $\bar{f}$ has a root z at c because $N(\bar{f}; c)$ is sharp. As in the proof of Theorem 2.1 in [2], we can take a subset $Q \cong D^{\dim A - 1} \times [0,1]$, which is a tubular neighborhood of a path from a point on ∂A to z. Notice that $A - Int_A(Q)$ is a strong deformation retractor of A, where Int_A means the relative interior. We have a retraction $k_z : A \to A - Int_A(Q)$ such that

1) $k_z(a) = a$ for $a \in A - Int_A(Q)$;

2) $k_z(a) \in A - Int_A(Q)$ for all $a \in A$.

It is clear that k_z is homotopic relative $A - Int_A(Q)$ to the identity map on A with $z \notin k_z(A)$. As $root(\bar{f}; c)$ is finite, we may choose Q so that it does not meet $root(\bar{f}; c)$. Thus, $\bar{f}$ is homotopic to the map $\bar{f}k_z$ with $root(\bar{f}k_z; c) = root(\bar{f}; c) - \{z\}$. This contradicts to the fact that $N(\bar{f}; c)$ is sharp.

Thus, all roots lie in $Bd(X - A) = A$ if $N(\bar{f}; c) \neq 0$.

Using homotopy extension theorem, we can extend the map $\bar{f}$ to a new $f : (X, A) \to (Y, B)$. Using the transversality, we can deform the map relative A so that c is the regular value of $f|_{X-A}$. Thus, we may assume that f has finitely many roots at c.

Let x' and x'' be two roots in $X - A$ lying in the same class. By Lemma 3.2, there is a path α in $X - A$ from x' to x'' such that $\langle f(\alpha) \rangle = 1 \in \pi_1(Y, c)$. Using Lemma 4.1, we can combine the root x'' into x' so that the map changing happens on a small neighborhood W of $\alpha((0,1])$. Because $\alpha(I) \in X - A$, we may choose W so that it lies in $X - A$. Thus, the new map is relatively homotopic to f. Repeat the procedure above, we get a map so that each root class has at most one root in $X - A$. We still call it f.

Let y be a root of f which is contained in a relatively essential common root class. If $y \in X - A$, then there is a root $z \in A$ such that y and z are in the same root class. Similarly, by Corollary 3.3 and Lemma 4.1, we can unite y into z. Thus, the result map will have $N_{rel}(f, c)$ roots.□

Our minimum theorem is different from the Theorem 3.3 in [1]. Consider the following example.

EXAMPLE 4.3. Let $X = T^3 \# T^3$ be a connected sum of two 3-dimensional tori and A a connected summand, which is homeomorphic to a punctured 3-dimensional torus. Let $Y = T^3$ and $B = D^3$ a three-ball. The $f : (X, A) \to (Y, B)$ is a pinch on A, i.e. f is homotopic to the natural quotient map $X \to X/A \cong Y$.

By Theorem 4.2, the root number $N_{rel}(f; c)$ in this example is sharp. But [1; Theorem 3.3] cannot be applied here, because A is not by-passed in X.

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