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HOPF FIBRATIONS ON LENS SPACES

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ABSTRACT. We give a certain uniqueness properties for the fiber of the Hopf fibration on lens spaces.

1. Introduction

Let S^3 be the unit sphere in $\mathbb{C}\times\mathbb{C}$ endowed with the geometry associated with the natural action of $O(4)$. Let p, q be relatively prime positive integers. The map $\rho : S^3 \to S^3$ defined by $\rho(u,v) = (e^{2\pi iq/p}u, e^{-2\pi i/p}v)$ is an isometry which generates a free $\mathbb{Z}_p$ action on S^3. The quotient space $S^3/ < \rho >$ is the lens space $L(p,q)$. We let $\mu : S^3 \to L(p,q)$ denote the quotient map. The 3 sphere S^3 is the union of two solid tori $V_1 = \{(u,v) \in S^3 : \mid u \mid^2 \geq 1/2\}$ and $V_2 = \{(u,v) \in S^3 : \mid u \mid^2 \leq 1/2\}$ whose intersection is the torus $T = \{(u,v) \in S^3 : \mid u \mid^2 = 1/2\}$. This decomposition of S^3 is invariant under ρ and descends to give a decomposition of $L(p,q)$ into solid tori $\mu(V_1)$, $\mu(V_2)$ whose intersection is the torus $\mu(T)$. Choose integers r and s so that $rq - ps = -1$, and f be the affine diffeomorphism on $S^1 \times S^1$ given by $f(u,v) = (u^r v^p, u^s v^q)$. Then $L(p,q)$ can also be described as the 3-manifold $V \cup_f V$ obtained by identifying the boundaries of a solid torus $V = S^1 \times D^2$ using $f : \partial V \to \partial V$ as attaching map. For more details on these definitions see [2], [3] and [5].

An embedded torus which separates $L(p,q)$ into two solid tori V_1 and V_2 is called a *Heegaard torus*, and the associated decomposition of the lens space into two solid tori is a Heegaard decomposition. Bonahon [1] has shown that any two Heegaard tori of a lens space are isotopic. Hence every diffeomorphism of $L(p,q)$ is isotopic to a diffeomorphism which preserves the Heegaard torus and Bonahon used this idea to calculate the mapping class group of $L(p,q)$, namely π_0 Diff$(L(p,q))$.

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A *sweepout* of $L(p,q)$ is a smooth map $\sigma : T \times [0,1] \to L(p,q)$, where T is a torus such that

1. $\Sigma_0 = \sigma(T \times \{0\})$ and $\Sigma_1 = \sigma(T \times \{1\})$ are imbedded circles in $L(p,q)$,
2. $\sigma |_{T\times(0,1)}: T \times (0,1) \to L(p,q)$ is a diffeomorphism onto $L(p,q) - (\Sigma_0 \cup \Sigma_1)$,
3. Near $T \times \partial I$, σ gives a mapping cylinder neighborhood of $\Sigma_0 \cup \Sigma_1$.

Associated to any t with $0 < t < 1$, we denote $\sigma(T \times t)$ by P_t, and call it a level surface of σ.

The standard elliptic geometry on the 3-sphere is the geometry associated with the orthogonal group under its natural action on the unit sphere in $\mathbb{R}^4$. A 3-manifold M is elliptic if it admits a covering map $S^3 \to M$ whose covering transformations act freely on S^3 as a subgroup of $\text{Isom}(S^3) = O(4)$. The Hopf fibering on S^3 is an S^1-bundle structure with projection map $H : S^3 \to S^2 = \mathbb{C} \cup \{\infty\}$ defined by $H(z_0, z_1) = z_0/z_1$. The left action of S^1 on S^3 takes each Hopf fiber to itself, so preserves Hopf fibering, namely the fibers are the orbits of the left action of S^1 on S^3. If G is a subgroup of $\text{Isom}(S^3)$ which preserves the Hopf fibering then S^3/G has an induced fibration from S^3. We call it the *Hopf fibering* of S^3/G. In this way, we may obtain Hopf fibrations on lens spaces. Basic details and background concerning elliptic structures and Hopf fibrations on elliptic manifolds may be found in [6] and [4] (section 3).

For $(z,w) \in S^1 \times S^1$, we define $(z,w)\cdot(u,v) = (zu, wv)$ when $(u,v) \in V_1$, and $(z,w)\cdot(u,v) = (z^r w^p u, z^s w^q v)$ when $(u,v) \in V_2$. This defines a torus action on L. Suppose a and b are relatively prime integers then $z \mapsto (z^a, z^b)$ is an embedding of S^1 into $S^1 \times S^1$, and composing with the torus action defines an S^1-action on L. This action determines a Seifert fibering on L in which V_1 and V_2 are union of fibers. On the solid torus V_1 the fibering has type (a,b), and it has type $(ra+pb, sa+qb)$ on V_2. The associated Seifert fibration $\phi : L \to B = L/S^1$ has orbit space B which is a 2-sphere with cone points of order $\mid a \mid$ and $\mid ra+pb \mid$. By making different choices of the type (a,b) of the fibering on V_1, we may obtain infinitely many distinct Heegard fibering on L. Analyzing the orders $\mid a \mid$ and $\mid ra+pb \mid$ of the cone points on B with the condition $rq - ps = -1$, one can easily deduce that $L(p,q)$ has a Heegaard fibering with no exceptional fibers if and only if $q = \pm 1 \pmod p$.

In this paper, we give a certain uniqueness properties for the fiber of the Hopf fibration on lens spaces.

2. Hopf fibration

From now on, we endow $L(p,q)$ with the Hopf fibering and assume that our sweepout of L is selected so that each P_t is a union of fibers.

LEMMA 1. *Let $L(p,q)$ be a lens space with $1 \leq q \leq p/2$, which is Seifert-fibered with Hopf fibering. Let P be a Heegaard torus which is a union of fibers, bounding solid tori V and W. Suppose that a loop in P is a longitude for V and W. Then $q = 1$. If $p > 2$, then the loop is isotopic in P to a fiber.*

PROOF. Let ℓ and m be loops in P which are respectively a longitude and a meridian of V, and so that $p\ell + qm$ is a meridian of W. If c is any loop in P which is a longitude for V, then (with one of its two orientations) c has the form $\ell + km$ in $H_1(P)$ for some k. The intersection number of c with $p\ell + qm$ is $q - kp$. Since $1 \leq q \leq p/2$, this can equal ± 1 only if $(p, q, k) = (2, 1, 1)$ or $(p, 1, 0)$, so $q = 1$. When $p > 2$, $k = 0$ and so $c = \ell$. Since $q = 1$, the Hopf fibering is nonsingular, so the fiber is a longitude for both V and W. Since $p > 2$, c is the only longitude of V that has intersection number ± 1 with the meridian of W, so it must be isotopic in P to a fiber. □

THEOREM 2. *Let $h : L(p,q) \to L(p,q)$ be a diffeomorphism isotopic to the identity with $h(P_s) = P_t$ where $0 < s, t < 1$. If $p > 2$, then the image of a fiber of P_s is isotopic in P_t to a fiber.*

PROOF. Conjugating by a fiber-preserving diffeomorphism of $L(p,q)$ that moves P_s to P_t, we may assume that $s = t$. Writing P for P_t, let V and W be the solid tori that P bounds. Let ℓ and m be loops in P as in the proof of lemma 1, and write $h_* : H_1(P) \to H_1(P)$ for the induced isomorphism.

Suppose first that $h(V) = V$. Since the meridian disk of V is unique up to isotopy, we have $h_*(m) = \pm m$. Since h is isotopic to the identity on $L(p,q)$ and $p > 2$, h is orientation preserving and induces the identity on $\pi_1(V)$. This implies that $h_*(m) = m$. Similar consideration for W show that $h_*(p\ell + qm) = p\ell + qm$, so $h_*(\ell) = \ell$. Thus h_* is the identity on $H_1(P)$ and the theorem follows for this case.

Suppose now that $h(V) = W$. Since h is isotopic to the identity and reverses the sides of P, h is orientation-reversing on P. Since h must take a meridian of V to one of W, $h_*(m) = \epsilon_1(p\ell + qm)$ where $\epsilon_1 = \pm 1$. Writing $h_*(\ell) = r\ell + sm$, we find that $1 = \ell \cdot m = -h_*(\ell) \cdot h_*(m) = -\epsilon_1(qr - ps)$. The facts that h is isotopic to the identity on L, ℓ generates $\pi_1(L)$, and m is 0 in $\pi_1(V)$ imply that $r \equiv 1 \pmod{p}$, so modulo p we have $1 \equiv -\epsilon_1 q$, forcing $q = 1$, $\epsilon_1 = -1$, and $h_*(m) = -p\ell - m$. Since h carries a meridian of W to one of V, we also have $h_*(p\ell + m) = \epsilon_2 m$ where $\epsilon_2 = \pm 1$. Subtracting, we find $h_*(p\ell) = p\ell + (1 + \epsilon_2)m$. Since $p > 2$ and $\epsilon_2 = \pm 1$, we have $1 + \epsilon_2 = 0$, so $h_*(\ell) = \ell$. Since $q = 1$, ℓ has intersection number 1 with the meridian $p\ell + qm$ of W. Lemma 1 shows that ℓ is homotopic in P to the fiber of the Hopf fibering. This proves the theorem. □

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