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WEAK SMOOTH α-STRUCTURE OF SMOOTH TOPOLOGICAL SPACES

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ABSTRACT. In [3] and [6] the concepts of smooth closure, smooth interior, smooth α-closure and smooth α-interior of a fuzzy set were introduced and some of their properties were obtained. In this paper, we introduce the concepts of several types of weak smooth compactness and weak smooth α-compactness in terms of these concepts introduced in [3] and [6] and investigate some of their properties.

1. Introduction

Badard [1] introduced the concept of a smooth topological space which is a generalization of Chang's fuzzy topological space [2]. Many mathematical structures in smooth topological spaces were introduced and studied. In particular, Gayyar, Kerre and Ramadan [5] and Demirci [3, 4] introduced the concepts of smooth closure and smooth interior of a fuzzy set and several types of compactness in smooth topological spaces and obtained some of their properties. In [6] we introduced the concepts of smooth α-closure and smooth α-interior of a fuzzy set which are generalizations of smooth closure and smooth interior of a fuzzy set defined in [3] and obtained some of their properties. In this paper, we introduce the concepts of several types of weak smooth compactness and weak smooth α-compactness in terms of these concepts introduced in [3] and [6] and investigate some of their properties.

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2. Preliminaries

Let X be a set and $I = [0,1]$ be the unit interval of the real line. I^X will denote the set of all fuzzy sets of X. 0_X and 1_X will denote the characteristic functions of ϕ and X, respectively.

A smooth topological space (s.t.s.) [7] is an ordered pair (X, τ), where X is a non-empty set and $\tau : I^X \to I$ is a mapping satisfying the following conditions:

(O1) $\tau(0_X) = \tau(1_X) = 1$;

(O2) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(O3) for every subfamily $\{A_i : i \in J\} \subseteq I^X, \tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \tau(A_i)$.

Then the mapping $\tau : I^X \to I$ is called a smooth topology on X. The number $\tau(A)$ is called the degree of openness of A.

A mapping $\tau^* : I^X \to I$ is called a smooth cotopology [7] if the following three conditions are satisfied:

(C1) $\tau^*(0_X) = \tau^*(1_X) = 1$;

(C2) $\forall A, B \in I^X,\ \tau^*(A \cup B) \geq \tau^*(A) \wedge \tau^*(B)$;

(C3) for every subfamily $\{A_i : i \in J\} \subseteq I^X, \tau^*(\cap_{i\in J} A_i) \geq \wedge_{i\in J} \tau^*(A_i)$.

If τ is a smooth topology on X, then the mapping $\tau^* : I^X \to I$, defined by $\tau^*(A) = \tau(A^c)$ where A^c denotes the complement of A, is a smooth cotopology on X. Conversely, if τ^* is a smooth cotopology on X, then the mapping $\tau : I^X \to I$, defined by $\tau(A) = \tau^*(A^c)$, is a smooth topology on X [7].

For the s.t.s. (X, τ) and $\alpha \in [0,1]$, the family $\tau_\alpha = \{A \in I^X : \tau(A) \geq \alpha\}$ defines a Chang's fuzzy topology (CFT) on X [2]. The family of all closed fuzzy sets with respect to τ_α is denoted by τ_α^* and we have $\tau_\alpha^* = \{A \in I^X : \tau^*(A) \geq \alpha\}$. For $A \in I^X$ and $\alpha \in [0,1]$, the τ_α-closure (resp., τ_α-interior) of A, denoted by $cl_\alpha(A)$ (resp., $int_\alpha(A)$), is defined by $cl_\alpha(A) = \cap\{K \in \tau_\alpha^* : A \subseteq K\}$ (resp., $int_\alpha(A) = \cup\{K \in \tau_\alpha : K \subseteq A\}$).

Demirci [3] introduced the concepts of smooth closure and smooth interior in smooth topological spaces as follows:

Let (X, τ) be a s.t.s. and $A \in I^X$. Then the τ-smooth closure (resp., τ-smooth interior) of A, denoted by $\overline{A}$ (resp., A^o), is defined by $\overline{A} = \cap\{K \in I^X : \tau^*(K) > 0, A \subseteq K\}$ (resp., $A^o = \cup\{K \in I^X : \tau(K) > 0, K \subseteq A\}$).

Let (X, τ) and (Y, σ) be two smooth topological spaces. A function $f : X \to Y$ is called smooth continuous with respect to τ and σ [7] if $\tau(f^{-1}(A)) \geq \sigma(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is called weakly smooth continuous with respect to τ and σ [7] if $\sigma(A) > 0 \Rightarrow \tau(f^{-1}(A)) > 0$ for every $A \in I^Y$. In this paper, a weakly smooth continuous function with respect to τ and σ is called a quasi-smooth continuous function with respect to τ and σ.

A function $f : X \to Y$ is smooth continuous with respect to τ and σ if and only if $\tau^*(f^{-1}(A)) \geq \sigma^*(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is weakly smooth continuous with respect to τ and σ if and only if $\sigma^*(A) > 0 \Rightarrow \tau^*(f^{-1}(A)) > 0$ for every $A \in I^Y$ [7].

THEOREM 2.1 [3]. *Let (X, τ) and (Y, σ) be two smooth topological spaces. If a function $f : X \to Y$ is quasi-smooth continuous with respect to τ and σ, then*

(a) $f(\overline{A}) \subseteq \overline{f(A)}$ for every $A \in I^X$,

(b) $\overline{f^{-1}(A)} \subseteq f^{-1}(\overline{A})$ for every $A \in I^Y$,

(c) $f^{-1}(A^o) \subseteq (f^{-1}(A))^o$ for every $A \in I^Y$.

A function $f : X \to Y$ is called smooth open (resp., smooth closed) with respect to τ and σ [7] if $\tau(A) \leq \sigma(f(A))$(resp., $\tau^*(A) \leq \sigma^*(f(A))$) for every $A \in I^X$.

THEOREM 2.2 [3]. *Let (X, τ) and (Y, σ) be two smooth topological spaces and $A \in I^X$. If a function $f : X \to Y$ is smooth open with respect to τ and σ, then $f(A^o) \subseteq (f(A))^o$.*

A function $f : X \to Y$ is called smooth preserving (resp., strict smooth preserving) with respect to τ and σ [5] if $\sigma(A) \geq \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) \geq \tau(f^{-1}(B))$ (resp., $\sigma(A) > \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) > \tau(f^{-1}(B))$) for every $A, B \in I^Y$.

If $f : X \to Y$ is a smooth preserving function (resp., a strict smooth preserving function) with respect to τ and σ, then $\sigma^*(A) \geq \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) \geq \tau^*(f^{-1}(B))$ (resp., $\sigma^*(A) > \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) > \tau^*(f^{-1}(B))$) for every $A, B \in I^Y$ [5].

A function $f : X \to Y$ is called smooth open preserving (resp., strict smooth open preserving) with respect to τ and σ [5] if $\tau(A) \geq \tau(B) \Rightarrow \sigma(f(A)) \geq \sigma(f(B))$ (resp., $\tau(A) > \tau(B) \Rightarrow \sigma(f(A)) > \sigma(f(B))$) for every $A, B \in I^X$.

3. Types of weak smooth compactness

Demirci [4] defined the families $W(\tau) = \{A \in I^X : A = A^o\}$ and $W^*(\tau) = \{A \in I^X : A = \overline{A}\}$, where (X, τ) is a s.t.s. Note that $A \in W(\tau) \Leftrightarrow A^c \in W^*(\tau)$. In this section, we introduce topological concepts of a s.t.s. in terms of the family $W(\tau)$ and investigate some of their properties.

DEFINITION 3.1 [4]. Let (X,τ) and (Y,σ) be two smooth topological spaces. A function $f : X \to Y$ is called weak smooth continuous with respect to τ and σ if $A \in W(\sigma) \Rightarrow f^{-1}(A) \in W(\tau)$ for every $A \in I^Y$.

Let (X,τ) and (Y,σ) be two smooth topological spaces. A function $f : X \to Y$ is weak smooth continuous with respect to τ and σ if and only if $A \in W^*(\sigma) \Rightarrow f^{-1}(A) \in W^*(\tau)$ for every $A \in I^Y$ [4].

DEFINITION 3.2 [4]. Let (X,τ) and (Y,σ) be two smooth topological spaces. A function $f : X \to Y$ is called weak smooth open (resp., weak smooth closed) with respect to τ and σ if $A \in W(\tau) \Rightarrow f(A) \in W(\sigma)$ (resp., $A \in W^*(\tau) \Rightarrow f(A) \in W^*(\sigma)$) for every $A \in I^X$.

DEFINITION 3.3 [4]. A s.t.s. (X,τ) is called weak smooth compact if every family in $W(\tau)$ covering X has a finite subcover.

Note that a weak smooth compact s.t.s. (X,τ) is smooth compact.

THEOREM 3.4 [4]. *Let (X,τ) and (Y,σ) be two smooth topological spaces and $f : X \to Y$ a surjective and weak smooth continuous function with respect to τ and σ. If (X,τ) is weak smooth compact, then so is (Y,σ).*

THEOREM 3.5. *Let (X,τ) and (Y,σ) be two smooth topological spaces. If a function $f : X \to Y$ is quasi-smooth continuous with respect to τ and σ, then $f : X \to Y$ is weak smooth continuous with respect to τ and σ.*

PROOF. Let $f : X \to Y$ be a quasi-smooth continuous function with respect to τ and σ. Then by Theorem 2.1 $f^{-1}(A^o) \subseteq (f^{-1}(A))^o$ for every $A \in I^Y$. Let $A \in W(\sigma)$, i.e., $A = A^o$. Then $f^{-1}(A) = f^{-1}(A^o) \subseteq (f^{-1}(A))^o$. From the definition of smooth interior we have $(f^{-1}(A))^o \subseteq f^{-1}(A)$. Hence $f^{-1}(A) = (f^{-1}(A))^o$, i.e., $f^{-1}(A) \in W(\tau)$. Therefore $f : X \to Y$ is weak smooth continuous with respect to τ and σ. □

We obtain the following corollary from Theorem 3.4 and 3.5.

COROLLARY 3.6. *Let (X,τ) and (Y,σ) be two smooth topological spaces and $f : X \to Y$ a surjective and quasi-smooth continuous function with respect to τ and σ. If (X,τ) is weak smooth compact, then so is (Y,σ).*

DEFINITION 3.7. A s.t.s. (X,τ) is called weak smooth nearly compact if for every family $\{A_i : i \in J\}$ in $W(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}(\overline{A_i})^o = 1_X$.

DEFINITION 3.8. A s.t.s. (X,τ) is called weak smooth almost compact if for every family $\{A_i : i \in J\}$ in $W(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}\overline{A_i} = 1_X$.

THEOREM 3.9. *A weak smooth compact s.t.s. (X,τ) is weak smooth nearly compact.*

PROOF. Let $\{A_i : i \in J\}$ be a family in $W(\tau)$ covering X. Since (X,τ) is weak smooth compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}A_i = 1_X$. Since $A_i \in W(\tau)$ for each $i \in J$, we have $A_i = A_i^o$ for each $i \in J$. From Proposition 3.1 [3] we have $A_i = A_i^o \subseteq (\overline{A_i})^o$ for each $i \in J$. Thus $1_X = \cup_{i\in J_0}A_i \subseteq \cup_{i\in J_0}(\overline{A_i})^o$, i.e., $\cup_{i\in J_0}(\overline{A_i})^o = 1_X$. Hence (X,τ) is weak smooth nearly compact. □

THEOREM 3.10. *A weak smooth nearly compact s.t.s. (X,τ) is weak smooth almost compact.*

PROOF. Let $\{A_i : i \in J\}$ be a family in $W(\tau)$ covering X. Since (X,τ) is weak smooth nearly compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}(\overline{A_i})^o = 1_X$. Since $(\overline{A_i})^o \subseteq \overline{A_i}$ for each $i \in J$ by Proposition 3.2 [3], $1_X = \cup_{i\in J_0}(\overline{A_i})^o \subseteq \cup_{i\in J_0}\overline{A_i}$. So $\cup_{i\in J_0}\overline{A_i} = 1_X$. Hence (X,τ) is weak smooth almost compact. □

THEOREM 3.11. *Let (X,τ) and (Y,σ) be two smooth topological spaces and $f : X \to Y$ a surjective and quasi-smooth continuous function with respect to τ and σ. If (X,τ) is weak smooth almost compact, then so is (Y,σ).*

PROOF. Let $\{A_i : i \in J\}$ be a family in $W(\sigma)$ covering Y, i.e., $\cup_{i\in J}A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i\in J}f^{-1}(A_i)$. Since f is quasi-smooth continuous with respect to τ and σ, f is weak smooth continuous with respect to τ and σ by Theorem 3.5. Hence $f^{-1}(A_i) \in W(\tau)$ for each $i \in J$. Since (X,τ) is weak smooth almost compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}\overline{f^{-1}(A_i)} = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0}\overline{f^{-1}(A_i)}) = \cup_{i\in J_0}f(\overline{f^{-1}(A_i)})$. Since $f : X \to Y$ is quasi-smooth continuous with respect to τ and σ, from Theorem 2.1 we have $\overline{f^{-1}(A_i)} \subseteq f^{-1}(\overline{A_i})$ for each $i \in J$. Hence $1_Y = \cup_{i\in J_0}f(\overline{f^{-1}(A_i)}) \subseteq \cup_{i\in J_0}f(f^{-1}(\overline{A_i})) = \cup_{i\in J_0}\overline{A_i}$, i.e., $\cup_{i\in J_0}\overline{A_i} = 1_Y$. Thus (Y,σ) is weak smooth almost compact. □

THEOREM 3.12 Let (X,τ) and (Y,σ) be two smooth topological spaces and $f : X \to Y$ a surjective, quasi-smooth continuous and smooth open function with respect to τ and σ. If (X, τ) is weak smooth nearly compact, then so is (Y, σ).

PROOF. Let $\{A_i : i \in J\}$ be a family in $W(\sigma)$ covering Y, i.e., $\cup_{i\in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i\in J} f^{-1}(A_i)$. Since f is quasi-smooth continuous with respect to τ and σ, f is weak smooth continuous with respect to τ and σ by Theorem 3.5. Hence $f^{-1}(A_i) \in W(\tau)$ for each $i \in J$. Since (X, τ) is weak smooth nearly compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} (\overline{f^{-1}(A_i)})^o = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0} (\overline{f^{-1}(A_i)})^o) = \cup_{i\in J_0} f((\overline{f^{-1}(A_i)})^o)$. Since $f : X \to Y$ is smooth open with respect to τ and σ, from Theorem 2.2 we have $f((\overline{f^{-1}(A_i)})^o) \subseteq (f(\overline{f^{-1}(A_i)}))^o$ for each $i \in J$. Since $f : X \to Y$ is quasi-smooth continuous with respect to τ and σ, from Theorem 2.1 we have $\overline{f^{-1}(A_i)} \subseteq f^{-1}(\overline{A_i})$ for each $i \in J$. Hence $1_Y = \cup_{i\in J_0} f((\overline{f^{-1}(A_i)})^o) \subseteq \cup_{i\in J_0} (f(\overline{f^{-1}(A_i)}))^o \subseteq \cup_{i\in J_0} (f(f^{-1}(\overline{A_i})))^o = \cup_{i\in J_0} (\overline{A_i})^o$, i.e., $\cup_{i\in J_0} (\overline{A_i})^o = 1_Y$. Thus (Y, σ) is weak smooth nearly compact. □

4. Types of weak smooth α-compactness

In this section, we introduce topological concepts of a s.t.s. in terms of the family $W_\alpha(\tau)$ and investigate some of their properties.

DEFINITION 4.1 [6]. Let (X, τ) be a s.t.s., $\alpha \in [0, 1)$ and $A \in I^X$. The τ-smooth α-closure (resp., τ-smooth α-interior) of A, denoted by $\overline{A}_\alpha$ (resp., A^o_α), is defined by $\overline{A}_\alpha = \cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A), A \subseteq K\}$ (resp., $A^o_\alpha = \cup\{K \in I^X : \tau(K) > \alpha\tau(A), K \subseteq A\}$).

We define the families $W_\alpha(\tau) = \{A \in I^X : A = A^o_\alpha\}$ and $W^*_\alpha(\tau) = \{A \in I^X : A = \overline{A}_\alpha\}$, where (X, τ) is a s.t.s. Note that $A \in W_\alpha(\tau) \Leftrightarrow A^c \in W^*_\alpha(\tau)$.

DEFINITION 4.2. Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. A function $f : X \to Y$ is called weak smooth α-continuous with respect to τ and σ if $A \in W_\alpha(\sigma) \Rightarrow f^{-1}(A) \in W_\alpha(\tau)$ for every $A \in I^Y$.

THEOREM 4.3. *Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. A function $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ if and only if $A \in W^*_\alpha(\sigma) \Rightarrow f^{-1}(A) \in W^*_\alpha(\tau)$ for every $A \in I^Y$.*

PROOF. The proof follows directly from the definitions of $W_\alpha(\tau)$, $W^*_\alpha(\tau)$ and Definition 4.2. □

DEFINITION 4.4. Let (X,τ) and (Y,σ) be two smooth topological spaces and let $\alpha \in [0,1)$. A function $f : X \to Y$ is called weak smooth α-open (resp., weak smooth α-closed) with respect to τ and σ if $A \in W_\alpha(\tau) \Rightarrow f(A) \in W_\alpha(\sigma)$ (resp., $A \in W^*_\alpha(\tau) \Rightarrow f(A) \in W^*_\alpha(\sigma)$) for every $A \in I^X$.

DEFINITION 4.5. Let $\alpha \in [0,1)$. A s.t.s. (X,τ) is called weak smooth α-compact if every family in $W_\alpha(\tau)$ covering X has a finite subcover.

Note that a weak smooth α-compact s.t.s. (X,τ) is smooth compact.

THEOREM 4.6. *Let $\alpha \in [0,1)$. Then a s.t.s. (X,τ) is weak smooth α-compact if and only if every family in $W^*_\alpha(\tau)$ having the finite intersection property has a non-empty intersection.*

PROOF. Let (X,τ) be a weak smooth α-compact s.t.s. and let $\{A_i : i \in J\}$ be a family in $W^*_\alpha(\tau)$ having the finite intersection property, i.e., for any finite subset $J_0 \subseteq J$, $\cap_{i\in J_0} A_i \neq 0_X$. Now suppose that $\cap_{i\in J} A_i = 0_X$. Then $\cup_{i\in J} A_i^c = 1_X$. Since $\{A_i : i \in J\} \subseteq W^*_\alpha(\tau)$, i.e., $\{A_i^c : i \in J\} \subseteq W_\alpha(\tau)$ and (X,τ) is a weak smooth α-compact s.t.s., there exists a finite subset $J_0 \subseteq J$ such that $\cup_{i\in J_0} A_i^c = 1_X$. Hence $\cap_{i\in J_0} A_i = 0_X$, which is a contradiction.

Conversely, suppose that every family in $W^*_\alpha(\tau)$ having the finite intersection property has a non-empty intersection and (X,τ) is not a weak smooth α-compact s.t.s. Then there exists a family $\{A_i : i \in J\}$ in $W_\alpha(\tau)$ covering X such that for any finite subset $J_0 \subseteq J$, $\cup_{i\in J_0} A_i \neq 1_X$, i.e., $\cap_{i\in J_0} A_i^c \neq 0_X$. Since $\{A_i : i \in J\} \subseteq W_\alpha(\tau)$, we have $\{A_i^c : i \in J\} \subseteq W^*_\alpha(\tau)$. Hence the family $\{A_i^c : i \in J\}$ has the finite intersection property. From the hypothesis we have $\cap_{i\in J} A_i^c \neq 0_X$. Hence $\cup_{i\in J} A_i \neq 1_X$, which is a contradiction. □

THEOREM 4.7. *Let (X,τ) and (Y,σ) be two smooth topological spaces, $\alpha \in [0,1)$ and $f : X \to Y$ a surjective and weak smooth α-continuous function with respect to τ and σ. If (X,τ) is weak smooth α-compact, then so is (Y,σ).*

PROOF. Let $\{A_i : i \in J\}$ be a family in $W_\alpha(\sigma)$ covering Y, i.e., $\cup_{i\in J} A_i = 1_Y$. Then $\cup_{i\in J} f^{-1}(A_i) = f^{-1}(1_Y) = 1_X$. Since $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, $\{f^{-1}(A_i) : i \in J\} \subseteq W_\alpha(\tau)$. Since (X,τ) is weak smooth α-compact, there exists a finite subset $J_0 \subseteq J$ such that $\cup_{i\in J_0} f^{-1}(A_i) = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0} f^{-1}(A_i)) = \cup_{i\in J_0} f(f^{-1}(A_i)) = \cup_{i\in J_0} A_i$. Therefore (Y, σ) is weak smooth α-compact. □

DEFINITION 4.8 [6]. Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. A function $f : X \to Y$ is called smooth α-preserving (resp., strict smooth α-preserving) with respect to τ and σ if $\sigma(A) \geq \alpha\sigma(B) \Leftrightarrow \tau(f^{-1}(A)) \geq \alpha\tau(f^{-1}(B))$ (resp., $\sigma(A) > \alpha\sigma(B) \Leftrightarrow \tau(f^{-1}(A)) > \alpha\tau(f^{-1}(B))$) for every $A, B \in I^Y$.

A function $f : X \to Y$ is called smooth open α-preserving (resp., strict smooth open α-preserving) with respect to τ and σ if $\tau(A) \geq \alpha\tau(B) \Rightarrow \sigma(f(A)) \geq \alpha\sigma(f(B))$ (resp., $\tau(A) > \alpha\tau(B) \Rightarrow \sigma(f(A)) > \alpha\sigma(f(B))$) for every $A, B \in I^X$.

THEOREM 4.9. *Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. If a function $f : X \to Y$ is strict smooth α-preserving with respect to τ and σ, then $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ.*

PROOF. Let $f : X \to Y$ be a strict smooth α-preserving function with respect to τ and σ. Then by Theorem 3.13 [6] $f^{-1}(A^o_\alpha) \subseteq (f^{-1}(A))^o_\alpha$ for every $A \in I^Y$. Let $A \in W_\alpha(\sigma)$, i.e., $A = A^o_\alpha$. Then by the above result $f^{-1}(A) = f^{-1}(A^o_\alpha) \subseteq (f^{-1}(A))^o_\alpha$. From Theorem 3.5 [6] we have $f^{-1}(A) = (f^{-1}(A))^o_\alpha$, i.e., $f^{-1}(A) \in W_\alpha(\tau)$. Therefore $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ. □

We obtain the following corollary from Theorem 4.7 and 4.9.

COROLLARY 4.10. *Let (X, τ) and (Y, σ) be two smooth topological spaces, $\alpha \in [0, 1)$ and $f : X \to Y$ a surjective and strict smooth α-preserving function with respect to τ and σ. If (X, τ) is weak smooth α-compact, then so is (Y, σ).*

DEFINITION 4.11. Let $\alpha \in [0, 1)$. A s.t.s. (X, τ) is called weak smooth nearly α-compact if for every family $\{A_i : i \in J\}$ in $W_\alpha(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}(\overline{(A_i)_\alpha})^o_\alpha = 1_X$.

DEFINITION 4.12. Let $\alpha \in [0, 1)$. A s.t.s. (X, τ) is called weak smooth almost α-compact if for every family $\{A_i : i \in J\}$ in $W_\alpha(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}\overline{(A_i)_\alpha} = 1_X$.

A smooth topology $\tau : I^X \to I$ on X is called monotonic increasing (resp., monotonic decreasing) if $A \subseteq B \Rightarrow \tau(A) \leq \tau(B)$ (resp., $A \subseteq B \Rightarrow \tau(A) \geq \tau(B)$) for every $A, B \in I^X$ [6].

THEOREM 4.13. *Let (X,τ) be a s.t.s., $\alpha \in [0,1)$ and τ a monotonic decreasing smooth topology. If (X,τ) is weak smooth α-compact, then (X,τ) is weak smooth nearly α-compact.*

PROOF. Let $\{A_i : i \in J\}$ be a family in $W_\alpha(\tau)$ covering X. Since (X,τ) is weak smooth α-compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} A_i = 1_X$. Since $A_i \in W_\alpha(\tau)$ for each $i \in J$, $A_i = (A_i)^o_\alpha$ for each $i \in J$. Since τ is monotonic decreasing and $A_i \subseteq \overline{(A_i)}_\alpha$ for each $i \in J$, $\tau(A_i) \geq \tau(\overline{(A_i)}_\alpha)$ for each $i \in J$. Hence from Theorem 3.2 [6] we have $A_i = (A_i)^o_\alpha \subseteq (\overline{(A_i)}_\alpha)^o_\alpha$ for each $i \in J$. Thus $1_X = \cup_{i\in J_0} A_i \subseteq \cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha$, i.e., $\cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha = 1_X$. Hence (X,τ) is weak smooth nearly α-compact. □

THEOREM 4.14. *Let $\alpha \in [0,1)$. Then a weak smooth nearly α-compact s.t.s. (X,τ) is weak smooth almost α-compact.*

PROOF. Let $\{A_i : i \in J\}$ be a family in $W_\alpha(\tau)$ covering X. Since (X,τ) is weak smooth nearly α-compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha = 1_X$. Since $(\overline{(A_i)}_\alpha)^o_\alpha \subseteq \overline{(A_i)}_\alpha$ for each $i \in J$ by Theorem 3.5 [6], $1_X = \cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha \subseteq \cup_{i\in J_0}\overline{(A_i)}_\alpha$. So $\cup_{i\in J_0}\overline{(A_i)}_\alpha = 1_X$. Hence (X,τ) is weak smooth almost α-compact. □

THEOREM 4.15. *Let (X,τ) and (Y,σ) be two smooth topological spaces, $\alpha \in [0,1)$ and $f : X \to Y$ a surjective and strict smooth α-preserving function with respect to τ and σ. If (X,τ) is weak smooth almost α-compact, then so is (Y,σ).*

PROOF. Let $\{A_i : i \in J\}$ be a family in $W_\alpha(\sigma)$ covering Y, i.e., $\cup_{i\in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i\in J} f^{-1}(A_i)$. Since f is strict smooth α-preserving with respect to τ and σ, f is weak smooth α-continuous with respect to τ and σ by Theorem 4.9. Hence $f^{-1}(A_i) \in W_\alpha(\tau)$ for each $i \in J$. Since (X,τ) is weak smooth almost α-compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}\overline{(f^{-1}(A_i))}_\alpha = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0}\overline{(f^{-1}(A_i))}_\alpha) = \cup_{i\in J_0} f(\overline{(f^{-1}(A_i))}_\alpha)$. Since $f : X \to Y$ is strict smooth α-preserving with respect to τ and σ, from Theorem 3.13 [6] we have $\overline{(f^{-1}(A_i))}_\alpha \subseteq$

$f^{-1}(\overline{(A_i)}_\alpha)$ for each $i \in J$. Hence

$$1_Y = \cup_{i\in J_0} f(\overline{(f^{-1}(A_i))}_\alpha) \subseteq \cup_{i\in J_0} f(f^{-1}(\overline{(A_i)}_\alpha)) = \cup_{i\in J_0}\overline{(A_i)}_\alpha,$$

i.e., $\cup_{i\in J_0}\overline{(A_i)}_\alpha = 1_Y$. Thus (Y,σ) is weak smooth almost α-compact.□

THEOREM 4.16. *Let (X,τ) and (Y,σ) be two smooth topological spaces, $\alpha \in [0,1)$ and $f : X \to Y$ a surjective, strict smooth α-preserving and strict smooth open α-preserving function with respect to τ and σ. If (X,τ) is weak smooth nearly α-compact, then so is (Y,σ).*

PROOF. Let $\{A_i : i \in J\}$ be a family in $W_\alpha(\sigma)$ covering Y, i.e., $\cup_{i\in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i\in J} f^{-1}(A_i)$. Since f is strict smooth α-preserving with respect to τ and σ, f is weak smooth α-continuous with respect to τ and σ by Theorem 4.9. Hence $f^{-1}(A_i) \in W_\alpha(\tau)$ for each $i \in J$. Since (X,τ) is weak smooth nearly α-compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0}(\overline{(f^{-1}(A_i))}_\alpha)^o_\alpha = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0}(\overline{(f^{-1}(A_i))}_\alpha)^o_\alpha)$ $= \cup_{i\in J_0} f((\overline{(f^{-1}(A_i))}_\alpha)^o_\alpha)$. Since $f : X \to Y$ is strict smooth open α-preserving with respect to τ and σ, from Theorem 3.14 [6] we have $f((\overline{(f^{-1}(A_i))}_\alpha)^o_\alpha) \subseteq (f(\overline{(f^{-1}(A_i))}_\alpha))^o_\alpha$ for each $i \in J$. Since $f : X \to Y$ is strict smooth α-preserving with respect to τ and σ, from Theorem 3.13 [6] we have $\overline{(f^{-1}(A_i))}_\alpha \subseteq f^{-1}(\overline{(A_i)}_\alpha)$ for each $i \in J$. Hence

$$\begin{aligned} 1_Y = & \cup_{i\in J_0} f((\overline{(f^{-1}(A_i))}_\alpha)^o_\alpha) \subseteq \cup_{i\in J_0}(f(\overline{(f^{-1}(A_i))}_\alpha))^o_\alpha \\ & \subseteq \cup_{i\in J_0}(f(f^{-1}(\overline{(A_i)}_\alpha)))^o_\alpha = \cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha, \end{aligned}$$

i.e.,

$$\cup_{i\in J_0}(\overline{(A_i)}_\alpha)^o_\alpha = 1_Y.$$

Thus (Y,σ) is weak smooth nearly α-compact. □

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