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DERIVATIONS ON CR MANIFOLDS

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ABSTRACT. We studied the relation between the tangential Cauchy-Riemann operator $\overline{\partial}_b$ on CR-manifolds and the derivation $d^{\pi^{0,1}}$ associated to the natural projection map $\pi^{0,1} : TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1} \to T^{0,1}$. We found that these two differential operators agree only on the space of functions $\Omega^0(M)$, unless $T^{1,0}$ is involutive as well. We showed that the difference is a derivation, which vanishes on $\Omega^0(M)$, and it is induced by the Nijenhuis tensor associated to $\pi^{0,1}$.

1. Introduction

On a smooth manifold M the most important differential operator is the exterior differential operator d, and it is the only "natural" first order operator up to a multiplicative scalar ([3]). But if we consider manifolds with more structure, we are naturally led to other differential operators. For example, we consider *Cauchy-Riemann operator* $\overline{\partial}$ for complex manifolds and *tangential Cauchy-Riemann operator* $\overline{\partial}_b$ for CR-manifolds. These are derivations of degree 1.

H.-J. Kim considered derivations of degree 1 on the set of smooth functions $\Omega^0(M) := C^\infty(M, \mathbb{R})$ ([1]). For each endomorphism $F : TM \to TM$, he associated a derivation d^F on $\Omega^0(M)$,

$$d^F f(X) := F(X)f = df(F(X)),$$

and showed that the set of all derivations of degree 1 on $\Omega^0(M)$ is isomorphic to the set of all endomorphisms of TM. Thus $\overline{\partial}$ and $\overline{\partial}_b$ on $\Omega^0(M)$ can be understood as a derivation $d^{\pi^{0,1}}$ associated to the natural projection map

$$\pi^{0,1} : TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1} \to T^{0,1}.$$

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Any derivation D on $\Omega^0(M)$ can be extended to a derivation on differential forms with the property $D \circ d + d \circ D = 0$ and its extension is unique ([1], [2]). Thus we may ask whether $\overline{\partial}$ (or $\overline{\partial}_b$) and $d^{\pi^{0,1}}$ agree as derivations on differential forms. However, for $D = \overline{\partial}$ (or $\overline{\partial}_b$), the condition $D \circ d + d \circ D = 0$ involves the integrability of $T^{0,1}$ and $T^{1,0}$ as well. Thus $\overline{\partial}_b$ on CR-manifolds may not agree with $d^{\pi^{0,1}}$ unless $T^{1,0}$ is involutive, which is not in general.

In this paper we will compare $\overline{\partial}_b$ and $d^{\pi^{0,1}}$ on CR-manifolds and relate the difference of them, which is a derivation vanishing on $\Omega^0(M)$, to the Nijenhuis tensor associated to $\pi^{0,1}$.

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2. Derivations

Let M be a smooth manifold. We use $\Omega^k(M)$ to denote the space of smooth sections $C^\infty(M, \wedge^k T^*M)$ and consider the graded commutative algebra $\Omega(M) = \oplus \Omega^k(M)$ of differential forms on M. We denote by $\mathrm{Der}_k \Omega(M)$ the space of all graded derivations of degree k, that is, all linear mappings $D : \Omega(M) \to \Omega(M)$ such that $D(\Omega^l(M)) \subset \Omega^{k+l}(M)$ and

$$D(\phi \wedge \psi) = D(\phi) \wedge \psi + (-1)^{kl} \phi \wedge D(\psi),$$

for $\phi \in \Omega^l(M)$, $\psi \in \Omega^m(M)$. Then the space $\mathrm{Der}\, \Omega(M) := \oplus \mathrm{Der}_k\, \Omega(M)$ is a graded Lie algebra with the graded commutator

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1.$$

This bracket satisfies the graded Jacobi identity

$$[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{k_1 k_2} [D_2, [D_1, D_3]]$$

so that $ad(D_1) := [D_1, \ ]$ is a derivation of degree k_1.

A derivation $D \in \mathrm{Der}_k \Omega(M)$ is called *algebraic* if $D \equiv 0$ on $\Omega^0(M)$. Then $D(f\omega) = fD(\omega)$ for $f \in \Omega^0(M)$. So it induces a derivation $D_x \in \mathrm{Der}_k(\wedge T_x^* M)$ for each $x \in M$. It is uniquely determined by its restriction to 1-forms $D_x|_{T_x^*M} : T_x^*M \to \wedge^{k+1} T_x^* M$, which may be viewed as an element $K_x \in \wedge^{k+1} T_x^* M \otimes T_x M$ depending smoothly on $x \in M$. To express this dependence we write $D = \iota_K$, where $K \in \Omega^{k+1}(M, TM) := C^\infty(M, \wedge^{k+1} T^*M \otimes TM)$. Note the defining equation $\iota_K(\omega) = \omega \circ K$ for $\omega \in \Omega^1(M)$. By applying it to an exterior product of 1-forms, one can derive the formula for l-forms;

$$\iota_K(\omega)(X_1,\dots,X_{k+l}) = \frac{1}{(k+1)!(l-1)!}\sum_{\sigma\in S_{k+l}} \mathrm{sign}(\sigma)\omega(K(X_{\sigma(1)},\dots,X_{\sigma(k+1)}),X_{\sigma(k+2)},\dots).$$

The exterior derivative d is a derivation of degree 1. We define the *Lie derivation* $\mathcal{L}_K := [\iota_K, d]$ for a tensor field $K \in \Omega^k(M,TM)$ similar to $\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X = [\iota_X, d]$ for the vector field X. Then the mapping $\mathcal{L} : \Omega(M,TM) \to \mathrm{Der}\Omega(M)$ is injective, since $\mathcal{L}_K f = \iota_K \circ df = df \circ K$ for $f \in C^\infty(M,\mathbb{R})$. However $\mathcal{L}$ is not surjective, which is explained in the following;

THEOREM 2.1. ([2]) *For any graded derivation $D \in \mathrm{Der}_k\Omega(M)$ there are unique $K \in \Omega^k(M,TM)$ and $L \in \Omega^{k+1}(M,TM)$ such that*

$$D = \mathcal{L}_K + \iota_L.$$

Moreover, $L = 0$ if and only if $[D,d] = 0$. D is algebraic if and only if $K = 0$.

We denote by d^F the derivation of degree 1 on $\Omega^0(M)$ associated to an endomorphism $F : TM \to TM$. It can be extended to a derivation on differential forms with the property $d^F \circ d + d \circ d^F = 0$. Then the extension of d^F can be written explicitly ([1]);

$$\begin{aligned} d^F\omega(X_0,\cdots,X_k) &= \sum_i (-1)^i F(X_i)\omega(X_0,\cdots,\widehat{X_i},\cdots,X_k) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j]_F, X_0,\cdots,\widehat{X_i},\cdots,\widehat{X_i},\cdots,X_k), \end{aligned}$$

where

$$[X,Y]_F := [FX,Y] + [X,FY] - F[X,Y].$$

Since $d^F \circ d + d \circ d^F = [d^F, d] = 0$, d^F is in fact $\mathcal{L}_F$ in theorem 2.1.

3. Tangential Cauchy-Riemann operator

We consider a $(2n+1)$-dimensional real smooth manifold M and an n-dimensional complex subbundle $\mathcal{Z}$ of the complexified tangent bundle $TM \otimes \mathbb{C}$. The pair $(M,\mathcal{Z})$ is called a *CR-manifold* if

1. $\mathcal{Z}_p \cap \overline{\mathcal{Z}_p} = 0$, for all $p \in M$,
2. $\mathcal{Z}$ is involutive, that is, $[Z_1,\ Z_2]$ belongs to $\mathcal{Z}$ whenever $Z_1,\ Z_2 \in \mathcal{Z}$.

There is a complex structure map J defined on the real subbundle $H \subset TM$ so that $H \otimes \mathbb{C} = \mathcal{Z} \oplus \overline{\mathcal{Z}}$ and $\mathcal{Z}$, $\overline{\mathcal{Z}}$ are the eigenspaces of extension of J to $H \otimes \mathbb{C}$ with the eigenvalues i, $-i$, respectively. The *tangential Cauchy-Riemann operator* $\overline{\partial}_b$ is defined by

$$\overline{\partial}_b f := df|_{\overline{\mathcal{Z}}}.$$

It will be necessary to choose a complementary subbundle to $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ to extend $\overline{\partial}_b$ to differential forms. In order to do this, we choose a Hermitian metric on $TM \otimes \mathbb{C}$ so that $\mathcal{Z}$ is orthogonal to $\overline{\mathcal{Z}}$ and let X_p be the orthogonal complement of $\mathcal{Z}_p \oplus \overline{\mathcal{Z}_p}$ in $T_pM \otimes \mathbb{C}$, for each $p \in M$. Then $X(M) = \cup_{p \in M} X_p$ forms a subbundle of $TM \otimes \mathbb{C}$.

Define subbundles $T^{0,1}M := \overline{\mathcal{Z}}$ and $T^{1,0}M := \mathcal{Z} \oplus X(M)$. We denote by $T^{*\,0,1}M$ and $T^{*\,0,1}M$ the dual spaces of $T^{1,0}M$ and $T^{0,1}M$, respectively. Denote $\wedge^{p,q}T^*M := \wedge^p(T^{*\,1,0}M)\hat{\otimes}\wedge^q(T^{*\,0,1}M)$, then we have the orthogonal decomposition

$$\wedge^r T^*M \otimes \mathbb{C} = \wedge^{r,0}T^*M \oplus \wedge^{r-1,1}T^*M \oplus \cdots \oplus \wedge^{0,r}T^*M.$$

We denote the natural projection map by

$$\pi^{p,q} : \wedge^r T^*M \otimes \mathbb{C} \to \wedge^{p,q}T^*M,$$

for $p + q = r$ and the space of smooth sections of $\wedge^{p,q}T^*M$ by $\Omega^{p,q}$. We extend $\overline{\partial}_b$ to $\Omega^{p,q}$ by $\overline{\partial}_b = \pi^{p,q+1} \circ d$.

4. Theorem

We consider a CR-manifold M and denote the natural projection to $T^{0,1}$ by

$$F = \pi^{0,1} : TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1} \to T^{0,1}.$$

Note that, for a CR-manifold in general, $T^{0,1}$ is involutive but $T^{1,0}$ may not be involutive. The lack of integrability of $T^{1,0}$ appears in the difference between $\overline{\partial}_b$ and d^F.

THEOREM 4.1. *For a CR-manifold M, $d^F - \overline{\partial}_b$ is a nontrivial algebraic derivation unless $T^{1,0}$ is involutive. More explicitly, there is a $(2,1)$ tensor L such that $d^F = \overline{\partial}_b + \iota_L$.*

PROOF. It is easy to see that $\overline{\partial}_b f = d^F f$ for a function $f \in \Omega^0(M)$. For 1-forms, we consider 6 cases, according to the types of forms and vector fields. Among them we will explain following 3 cases, since $\overline{\partial}_b \equiv d^F$ as zero in the other cases.

Case 1. $\eta \in \Omega^{1,0}$, $X_1 \in T^{1,0}M$ and $X_2 \in T^{0,1}M$.

Note that $F(X_1) = 0$, $F(X_2) = X_2$, $\eta(X_2) = 0$ and $\eta(F[X_1, X_2]) = 0$.

$$\begin{aligned} d^F\eta(X_1, X_2) &= F(X_1)\eta(X_2) - F(X_2)\eta(X_1) \\ &\quad - \eta([F(X_1), X_2] + [X_1, F(X_2)] - F([X_1, X_2])) \\ &= 0 - X_2\eta(X_1) - \eta([0, X_2] + [X_1, X_2] - F([X_1, X_2])) \\ &= X_1\eta(X_2) - X_2\eta(X_1) - \eta((1 - F)([X_1, X_2])) \\ &= X_1\eta(X_2) - X_2\eta(X_1) - \eta([X_1, X_2]) \\ &= d\eta(X_1, X_2) \\ &= \overline{\partial}_b\eta(X_1, X_2). \end{aligned}$$

Case 2. $\eta \in \Omega^{0,1}$, $X_i \in T^{0,1}M$.
Note that $F(X_i) = X_i$ and $\eta((1 - F)[X_1, X_2]) = 0$.

$$\begin{aligned} d^F\eta(X_1, X_2) &= F(X_1)\eta(X_2) - F(X_2)\eta(X_1) \\ &\quad - \eta([F(X_1), X_2] + [X_1, F(X_2)] - F([X_1, X_2])) \\ &= X_1\eta(X_2) - X_2\eta(X_1) - \eta([X_1, X_2] + (1 - F)([X_1, X_2])) \\ &= X_1\eta(X_2) - X_2\eta(X_1) - \eta([X_1, X_2]) \\ &= d\eta(X_1, X_2) \\ &= \overline{\partial}_b\eta(X_1, X_2). \end{aligned}$$

Case 3. $\eta \in \Omega^{0,1}$, $X_i \in T^{1,0}M$.
Note that $F(X_i) = 0$.

$$\begin{aligned} d^F\eta(X_1, X_2) &= F(X_1)\eta(X_2) - F(X_2)\eta(X_1) \\ &\quad - \eta([F(X_1), X_2] + [X_1, F(X_2)] - F([X_1, X_2])) \\ &= \eta(F[X_1, X_2]). \end{aligned}$$

Note that, in Case 3, $\eta(F[X_1, X_2]) \neq 0$ unless $T^{1,0}$ is involutive, while $\overline{\partial}_b\eta(X_1, X_1) = 0$ for all $\eta \in \Omega^{0,1}$ and X_1, $X_2 \in T^{1,0}$. Therefore we can write

$$d^F\eta(X_1, X_2) = \overline{\partial}_b\eta(X_1, X_2) + \eta(F[(1 - F)X_1, (1 - F)X_2]).$$

We now show that $L(X,Y) := F[(1-F)X, (1-F)Y]$ is a tensor of type (2,1). Note that $F(1-F) = 0$, for F is a projection.

$$\begin{aligned} L(fX,Y) &= F[(1-F)fX,(1-F)Y] \\ &= F[f(1-F)X,(1-F)Y] \\ &= F\{f[(1-F)X,(1-F)Y] - (((1-F)Y)f)(1-F)X\} \\ &= fF[(1-F)X,(1-F)Y] - F(1-F)(((1-F)Y)f)X \\ &= fF[(1-F)X,(1-F)Y], \end{aligned}$$

for a function f. Thus L is a tensor. □

We denote the Nijenhuis tensor associated to F by

$$N_F(X,Y) := [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y].$$

REMARK 4.2. Since $T^{0,1}$ is involutive, L is in fact the Nijenhuis tensor. Note that $F^2 = F$, $[FX,FY] \in T^{0,1}$. Thus

$$\begin{aligned} L(X,Y) &= F[(1-F)X,(1-F)Y] \\ &= F\{[X,Y] - [FX,Y] - [X,FY] + [FX,FY]\} \\ &= F[X,Y] - F[FX,Y] - F[X,FY] + F[FX,FY] \\ &= F^2[X,Y] - F[FX,Y] - F[X,FY] + [FX,FY] \\ &= N_F(X,Y). \end{aligned}$$

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