Commun. Korean Math. Soc. 19 (2004), No. 1, pp. 119–127

PERTURBATION OF WAVELET FRAMES AND RIESZ BASES I

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ABSTRACT. Suppose that $\psi \in L^2(\mathbb{R})$ generates a wavelet frame (resp. Riesz basis) with bounds A and B. If $\phi \in L^2(\mathbb{R})$ satisfies $|\widehat{\psi}(\xi) - \widehat{\phi}(\xi)| < \lambda \frac{|\xi|^\alpha}{(1+|\xi|)^\gamma}$ for some positive constants α, γ, λ such that $1 < 1 + \alpha < \gamma$ and $\lambda^2 M < A$, then ϕ also generates a wavelet frame (resp. Riesz basis) with bounds $A\left(1 - \lambda\sqrt{M/A}\right)^2$ and $B\left(1 + \lambda\sqrt{M/B}\right)^2$, where M is a constant depending only on α, γ, the dilation step a, and the translation step b.

1. Introduction

A wavelet frame is some generalization of a wavelet. While a C^∞-function with exponential decay can never generate an orthonormal wavelet basis, it can generate a wavelet frame. A well-known example is the Mexican hat function, the second derivative of the Gaussian $e^{-x^2/2}$; if we normalize it so that $\|\psi\|_2 = 1$ and $\psi(0) > 0$, then

$$\psi(x) = \frac{2}{\sqrt{3}}\pi^{-1/4}(1 - x^2)e^{-x^2/2}$$

(see Daubechies [5] p.75). Wavelets and their generalizations provide very efficient building blocks for most of the classical function spaces. They offer flexible methods for studying integral operators and partial differential equations. They are particularly effective for analyzing functions with multifractal structure, and indispensable tools for signal and image processing.

In practical applications, approximations are often used, whence the stability problem occurs. Some guarantee that small perturbation does

Received December 11, 2002.
2000 Mathematics Subject Classification: 42C15, 41A30.
Key words and phrases: wavelet, frame, Riesz basis, perturbation, stability.
The second author was supported by Ajou University Research Fund.

not alter the main property of the wavelet and their generalizations is thus needed.

For given $a > 1, b > 0$, and $\psi \in L^2(\mathbb{R})$ a family of functions $\psi_{m,n}$ is defined by

$$\psi_{m,n}(x) = a^{-m/2}\psi(a^{-m}x - bn), \quad m, n \in \mathbb{Z},$$

where the constant a is called the dilation step, and the constant b the translation step. An L^2-function ψ is said to generate a wavelet frame or a wavelet Riesz basis if the family $\{\psi_{m,n}\}$ is a frame or a Riesz basis for $L^2(\mathbb{R})$, respectively.

R. Balan showed in [1] that if ψ generates a wavelet frame and $\widehat{\psi}$ and $\widehat{\psi}'$ have some smoothness and boundedness, then ψ with slightly perturbed translation step still generates a wavelet frame. This is not an exact answer to the stability problem. But, a result by S. Z. Favier and R. A. Zalik gives a partial answer. They showed in [6] that if $\widehat{\psi}$ has a compact support and ψ with an integer dilation step generates a wavelet frame (resp. Riesz basis), then a function ϕ, slightly perturbed from ψ in their Fourier transforms, also generates a wavelet frame(resp. Riesz basis). We obtain the following results which extend this result by removing the restrictions on the dilation step and the support.

THEOREM. *Suppose that $\psi \in L^2(\mathbb{R})$ generates a wavelet frame with bounds A and B. If $\phi \in L^2(\mathbb{R})$ satisfies $|\widehat{\psi}(\xi) - \widehat{\phi}(\xi)| < \lambda \frac{|\xi|^\alpha}{(1+|\xi|)^\gamma}$ for some $1 < 1 + \alpha < \gamma$ and $\lambda^2 M < A$, then ϕ also generates a wavelet frame with bounds $A(1 - \lambda\sqrt{M/A})^2$, $B(1 + \lambda\sqrt{M/B})^2$.*

THEOREM. *Suppose that $\psi \in L^2(\mathbb{R})$ generates a wavelet Riesz basis with bounds A and B. If $\phi \in L^2(\mathbb{R})$ satisfies $\left|\widehat{\psi}(\xi) - \widehat{\phi}(\xi)\right| < \lambda \frac{|\xi|^\alpha}{(1+|\xi|)^\gamma}$ for some $1 < 1 + \alpha < \gamma$ and $\lambda^2 M < A$, then ϕ also generates a wavelet Riesz basis with bounds $A(1 - \lambda\sqrt{M/A})^2$, $B(1 + \lambda\sqrt{M/B})^2$.*

2. Definitions and preliminary results

In this paper the Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx,$$

and so $\|f\|_2 = \|\widehat{f}\|_2$ for all $f \in L^2(\mathbb{R})$. Now a frame is defined as follows.

DEFINITION 1. A family $\{\psi_j\}_{j\in J}$ of elements in a Hilbert space $\mathcal{H}$ is called a frame if there exist positive constants A and B such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \le \sum_{j\in J} |\langle f, \psi_j\rangle|^2 \le B\|f\|^2.$$

The constants A and B are called frame bounds.

A frame can be thought of as a kind of generalized basis. It contains more than enough elements. But, such redundancy allows flexibility. The frame condition can be split into the upper frame condition and the lower frame condition. The upper frame condition gives Bessel sequences.

DEFINITION 2. A sequence $\{\psi_j\}_{j\in J}$ of elements in a Hilbert space $\mathcal{H}$ is called a Bessel sequence if there exists a positive constant B such that for all $f \in \mathcal{H}$,

$$\sum_{j\in J} |\langle f, \psi_j\rangle|^2 \le B\|f\|^2.$$

The constant B is called a Bessel bound.

A Bessel sequence $\{\psi_j\}_{j\in J}$ can be characterized in terms of the so called pre-frame operator $T : l^2 \to \mathcal{H}$, defined by $T(\{c_j\}) = \sum_{j\in J} c_j\psi_j$. If the series $\sum_{j\in J} c_j\psi_j$ converges (i.e., $T(\{c_j\})$ is well-defined) for all $\{c_j\} \in l^2$, then it follows that T is bounded, and the adjoint $T^* : \mathcal{H} \to l^2$ of T is given by $T^*f = \{\langle f, \psi_j\rangle\}$. Since the boundedness of T^* corresponds to the Bessel condition of the sequence $\{\psi_j\}_{j\in J}$, the following theorem is obtained.

THEOREM 3 ([7]). *A sequence $\{\psi_j\}_{j\in J}$ of elements in a Hilbert space $\mathcal{H}$ is a Bessel sequence with a bound B if and only if $\left\|\sum_{j\in J} c_j\psi_j\right\|^2 \le B\sum_{j\in J} |c_j|^2$ for all $\{c_j\} \in l^2$.*

There are several equivalent definitions for a Riesz basis. We use in this paper the following definition.

DEFINITION 4. A family $\{\psi_j\}_{j\in J}$ of elements in a Hilbert space $\mathcal{H}$ is a Riesz basis if $\{\psi_j\}_{j\in J}$ is complete, and there exist positive constants A and B such that

$$A\sum_{j\in J} |c_j|^2 \le \left\|\sum_{j\in J} c_j\psi_j\right\|_2^2 \le B\sum_{j\in J} |c_j|^2$$

for all $\{c_j\} \in l^2$. The constants A and B are called Riesz bounds.

The following theorem is used in the next section to show that a frame with a certain property is, in fact, a Riesz basis.

THEOREM 5 ([4]). *Let $\{\psi_j\}_{j\in J}$ be a family of elements in a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*
(i) *$\{\psi_j\}_{j\in J}$ is a Riesz basis with bounds A and B.*
(ii) *$\{\psi_j\}_{j\in J}$ is an l^2-linearly independent frame with bounds A and B.*

Perturbation of a frame can be treated with the following theorem.

THEOREM 6 ([6], p.164). *Let $\{\psi_j\}_{j\in J}$ be a frame in a Hilbert space $\mathcal{H}$ with bounds A and B. Assume $\{\phi_j\}_{j\in J}\subset\mathcal{H}$ is such that $\{\psi_j-\phi_j\}_{j\in J}$ is a Bessel sequence with a bound $M<A$. Then $\{\phi_j\}_{j\in J}$ is a frame with bounds $A\left(1-\sqrt{M/A}\right)^2$ and $B\left(1+\sqrt{M/B}\right)^2$.*

3. Main results and proofs

In the sequel, the dilated and translated versions $\psi_{m,n}$ of a given function $\psi\in L^2(\mathbb{R})$ are always considered with some fixed dilation step $a>1$ and translation step $b>0$. I. Daubechies showed in [5] an estimate which enables us to get bounds of wavelet frames in $L^2(\mathbb{R})$: For any $\psi, f\in L^2(\mathbb{R})$

$$\begin{aligned}&\sum_{m,n\in\mathbb{Z}}|\langle f,\psi_{m,n}\rangle|^2\\ (1)\qquad &\le \frac{2\pi}{b}\|f\|_2^2\left[\sup_{1\le|\xi|\le a}\sum_{m\in\mathbb{Z}}|\widehat{\psi}(a^m\xi)|^2\right.\\ &\qquad\left.+\sum_{k\in\mathbb{Z},k\neq 0}\left[\beta\left(\frac{2\pi k}{b}\right)\beta\left(-\frac{2\pi k}{b}\right)\right]^{1/2}\right],\end{aligned}$$

where $\beta(s)=\beta_{\widehat{\psi}}(s)=\sup_\xi\sum_{m\in Z}|\widehat{\psi}(a^m\xi)||\widehat{\psi}(a^m\xi+s)|$. We will use this estimate to derive frame bounds.

The results of the following lemmas will be used in the proofs of the main theorems. We first define a nonnegative function η by

$$\eta(t)=\eta_{\alpha,\gamma}(t)=\frac{|t|^\alpha}{(1+|t|)^\gamma},$$

where α and γ are positive constants.

LEMMA 7. *If* $0 < \alpha < \gamma$ *and* $a > 1$, *then*

$$\sum_{m\in\mathbb{Z}} \eta(a^m t) \leq \frac{a^\alpha}{a^\alpha - 1} + \frac{a^{\gamma-\alpha}}{a^{\gamma-\alpha} - 1}$$

for all $1 \leq |t| \leq a$.

PROOF. Let $1 \leq |t| \leq a$. Then,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \eta(a^m t) &= \sum_{m<0} \frac{|a^m t|^\alpha}{(1+|a^m t|)^\gamma} + \sum_{m\geq 0} \frac{|a^m t|^\alpha}{(1+|a^m t|)^\gamma} \\ &\leq \sum_{m<0} |a^m t|^\alpha + \sum_{m\geq 0} |a^m t|^{\alpha-\gamma} \\ &\leq \sum_{m<0} a^{\alpha(m+1)} + \sum_{m\geq 0} a^{(\alpha-\gamma)m} \\ &= \frac{a^\alpha}{a^\alpha - 1} + \frac{a^{\gamma-\alpha}}{a^{\gamma-\alpha} - 1}.\end{aligned}$$

□

COROLLARY 8. *If* $0 < \alpha < \gamma$ *and* $a > 1$, *then*

$$\sup_{1\leq|t|\leq a} \sum_{m\in\mathbb{Z}} \eta(a^m t)^2 \leq \frac{a^{2\alpha}}{a^{2\alpha} - 1} + \frac{a^{2(\gamma-\alpha)}}{a^{2(\gamma-\alpha)} - 1}.$$

PROOF. Observe that $\eta^2 = \eta^2_{\alpha,\gamma} = \eta_{2\alpha,2\gamma}$. Since $0 < 2\alpha < 2\gamma$ by the hypothesis, the result is then followed from Lemma 7. □

LEMMA 9. *If* $0 < \alpha < \gamma$ *and* $a > 1$, *then*

$$\beta_\eta(s) = \sup_t \sum_{m\in\mathbb{Z}} \eta(a^m t)\eta(a^m t + s) \leq B(1+|s|)^{\alpha-\gamma} \tag{2}$$

where $B = B_{a;\alpha,\gamma}$ *is a positive constant depending only on* a, α, *and* γ.

PROOF. The inequality is obvious when $t = 0$. If $t \neq 0$, we can find an integer k such that $1 \leq |t|/a^k \leq a$. Hence, if necessary, by replacing t/a^k and a^{m+k} with t and m, respectively, it suffices to consider the inequality only for the case $1 \leq |t| \leq a$.

Assume $1 \leq |t| \leq a$. For each fixed t and s, an arbitrary integer m falls into exactly one of the following three cases:
(i) $|a^m t| \geq a|s|$, (ii) $|a^m t| < |s|/a$, (iii) $|s|/a \leq |a^m t| < a|s|$.

If $|a^m t| \geq a|s|$, then

$$\begin{aligned}\eta(a^m t+s) &= \frac{|a^m t+s|^\alpha}{(1+|a^m t+s|)^\gamma} \leq (1+|a^m t+s|)^{\alpha-\gamma} \\ &\leq (1+|a^m t|-|s|)^{\alpha-\gamma} \leq (1+a|s|-|s|)^{\alpha-\gamma} \\ &= (1+(a-1)|s|)^{\alpha-\gamma} \leq B_1(1+|s|)^{\alpha-\gamma},\end{aligned}$$

where $B_1 = 1$ if $a \geq 2$ or $B_1 = (a-1)^{\alpha-\gamma}$ if $1 < a < 2$. Hence,

$$\sum_{\text{(i)}} \eta(a^m t)\eta(a^m t+s) \leq B_1(1+|s|)^{\alpha-\gamma} \sum_{\text{(i)}} \eta(a^m t). \tag{3}$$

Now, if $|a^m t| \leq |s|/a$, then

$$\begin{aligned}\eta(a^m t+s) &= \frac{|a^m t+s|^\alpha}{(1+|a^m t+s|)^\gamma} \leq (1+|a^m t+s|)^{\alpha-\gamma} \\ &\leq (1+|s|-|a^m t|)^{\alpha-\gamma} \leq \left[1+|s|-\frac{|s|}{a}\right]^{\alpha-\gamma} \\ &= \left[1+\left(1-\frac{1}{a}\right)|s|\right]^{\alpha-\gamma} \leq B_2(1+|s|)^{\alpha-\gamma},\end{aligned}$$

where $B_2 = (1-1/a)^{\alpha-\gamma}$. Hence,

$$\sum_{\text{(ii)}} \eta(a^m t)\eta(a^m t+s) \leq B_2(1+|s|)^{\alpha-\gamma} \sum_{\text{(ii)}} \eta(a^m t). \tag{4}$$

Finally, if $|s|/a \leq |a^m t| < a|s|$, then

$$\eta(a^m t) \leq (1+|a^m t|)^{\alpha-\gamma} \leq \left[1+\frac{1}{a}|s|\right]^{\alpha-\gamma} \leq a^{\gamma-\alpha}(1+|s|)^{\alpha-\gamma}.$$

Observe that if $r \in [|s|/a, |s|)$, then $a^k r \notin [|s|/a, a|s|)$ for $k \geq 2$ or $k \leq -1$, and if $r \in [|s|, a|s|)$, then $a^k r \notin [|s|/a, a|s|)$ for $k \geq 1$ or $k \leq -2$. Hence there are at most two integers m satisfying $|s|/a \leq |a^m t| < a|s|$. Since $\eta(a^m t+s) \leq 1$, we now get

$$\sum_{\text{(iii)}} \eta(a^m t)\eta(a^m t+s) \leq 2a^{\gamma-\alpha}(1+|s|)^{\alpha-\gamma}. \tag{5}$$

From the inequalities (3), (4), (5), and the fact that $B_1 \leq B_2$, we thus obtain

$$\begin{aligned}
&\sum_{min\mathbb{Z}} \eta(a^m t)\eta(a^m t + s) \\
&\leq \left[B_1 \sum_{\text{(i)}} \eta(a^m t) + B_2 \sum_{\text{(ii)}} \eta(a^m t) + 2a^{\gamma-\alpha} \right] (1 + |s|)^{\alpha-\gamma} \\
&\leq \left[B_2 \sum_{m\in\mathbb{Z}} \eta(a^m t) + 2a^{\gamma-\alpha} \right] (1 + |s|)^{\alpha-\gamma}.
\end{aligned}$$

We can now complete the proof by Lemma 7. □

Now one can easily see by the previous results that if $\alpha > 0, \gamma > \alpha + 1, a > 1$, and $b > 0$, then

$$\begin{aligned}
(6)\quad M &= M_{a,b;\alpha,\gamma} \\
&= \frac{2\pi}{b} \left[\sup_{1\leq|t|\leq a} \sum_{m\in\mathbb{Z}} \eta(a^m t)^2 + \sum_{k\in\mathbb{Z},k\neq 0} \left[\beta_\eta\left(\frac{2\pi k}{b}\right) \beta_\eta\left(-\frac{2\pi k}{b}\right) \right]^{1/2} \right]
\end{aligned}$$

is finite. In the following theorems, the constant M always refers to the same constant defined in (6).

THEOREM 10. *Suppose that $\psi \in L^2(\mathbb{R})$ generates a wavelet frame with bounds A and B. If $\phi \in L^2(\mathbb{R})$ satisfies $|\widehat{\psi}(\xi) - \widehat{\phi}(\xi)| < \lambda \frac{|\xi|^\alpha}{(1+|\xi|)^\gamma}$ for some $1 < 1 + \alpha < \gamma$ and $\lambda^2 M < A$, then ϕ also generates a wavelet frame with bounds $A(1 - \lambda\sqrt{M/A})^2$, $B(1 + \lambda\sqrt{M/B})^2$.*

PROOF. By Daubechies's estimate (1), we have

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} |\langle f, \psi_{m,n} - \phi_{m,n}\rangle|^2 \\
&\leq \frac{2\pi}{b} \|f\|_2^2 \times \left[\sup_{1\leq|\xi|\leq a} \sum_{m\in\mathbb{Z}} \left|\widehat{\psi}(a^m\xi) - \widehat{\phi}(a^m\xi)\right|^2 \right. \\
&\quad \left. + \sum_{k\in\mathbb{Z},k\neq 0} \left[\beta\left(\frac{2\pi k}{b}\right) \beta\left(-\frac{2\pi k}{b}\right) \right]^{1/2} \right].
\end{aligned}$$

From the assumption,

$$\sup_{1\leq|\xi|\leq a} \sum_{m\in\mathbb{Z}} \left|\widehat{\psi}(a^m\xi) - \widehat{\phi}(a^m\xi)\right|^2 \leq \lambda^2 \sup_{1\leq|\xi|\leq a} \sum_{m\in\mathbb{Z}} \eta(a^m\xi)^2$$

and

$$\begin{aligned}\beta(s) &= \sup_{\xi} \sum_{m\in\mathbb{Z}} \left|\widehat{\psi}(a^m\xi) - \widehat{\phi}(a^m\xi)\right| \left|\widehat{\psi}(a^m\xi + s) - \widehat{\phi}(a^m\xi + s)\right| \\ &\leq \lambda^2 \sup_{\xi} \sum_{m\in\mathbb{Z}} \eta(a^m\xi)\eta(a^m\xi + s) = \lambda^2\beta_\eta(s).\end{aligned}$$

Hence, by the constant M defined in (6) we have

$$\begin{aligned}&\sum_{m,n\in\mathbb{Z}} |\langle f, \psi_{m,n} - \phi_{m,n}\rangle|^2 \\ &\leq \lambda^2 \frac{2\pi}{b} \|f\|_2^2 \Bigg[\sup_{1\leq|\xi|\leq a} \sum_{m\in\mathbb{Z}} \eta(a^m\xi)^2 \\ &\quad + \sum_{k\in\mathbb{Z},k\neq 0} \left[\beta_\eta\left(\frac{2\pi k}{b}\right)\beta_\eta\left(\frac{2\pi k}{b}\right)\right]^{1/2}\Bigg] \\ &\leq \lambda^2 M\|f\|_2^2.\end{aligned}$$

Therefore $\{\psi_{m,n} - \phi_{m,n}\}$ is a Bessel sequence with bound $\lambda^2 M$, and the proof is now completed by Theorem 6. □

THEOREM 11. *Suppose that $\psi \in L^2(\mathbb{R})$ generates a wavelet Riesz basis with bounds A and B. If $\phi \in L^2(\mathbb{R})$ satisfies $\left|\widehat{\psi}(\xi) - \widehat{\phi}(\xi)\right| < \lambda\frac{|\xi|^\alpha}{(1+|\xi|)^\gamma}$ for some $1 < 1 + \alpha < \gamma$ and $\lambda^2 M < A$, then ϕ also generates a wavelet Riesz basis with bounds $A(1 - \lambda\sqrt{M/A})^2$, $B(1 + \lambda\sqrt{M/B})^2$.*

PROOF. By Theorem 5, ψ generates a wavelet frame with bounds A and B, so by the assumption and Theorem 10, ϕ also generates a wavelet frame with bounds $A(1 - \lambda\sqrt{M/A})^2$ and $B(1 + \lambda\sqrt{M/B})^2$. Hence, by Theorem 5 again, it suffices to show that $\{\phi_{m,n}\}$ is l^2-linearly independent.

In the proof of Theorem 10, we have shown that $\{\psi_{m,n} - \phi_{m,n}\}$ is a Bessel sequence with bound $\lambda^2 M$. Let $\{c_{m,n}\} \in l^2$. Then, by Theorem 3,

$$\left\|\sum_{m,n\in\mathbb{Z}} c_{m,n}(\psi_{m,n} - \phi_{m,n})\right\|_2^2 \leq \lambda^2 M \sum_{m,n\in\mathbb{Z}} |c_{m,n}|^2.$$

Hence,

$$\begin{aligned}\left\|\sum_{m,n\in\mathbb{Z}} c_{m,n}\phi_{m,n}\right\|_2 &\geq \left\|\sum_{m,n\in\mathbb{Z}} c_{m,n}\psi_{m,n}\right\|_2 - \left\|\sum_{m,n\in\mathbb{Z}} c_{m,n}(\psi_{m,n}-\phi_{m,n})\right\|_2 \\ &\geq \sqrt{A}\left[\sum_{m,n\in\mathbb{Z}} |c_{m,n}|^2\right]^{1/2} - \lambda\sqrt{M}\left[\sum_{m,n\in\mathbb{Z}} |c_{m,n}|^2\right]^{1/2} \\ &\geq \left(\sqrt{A}-\lambda\sqrt{M}\right)\left[\sum_{m,n\in\mathbb{Z}} |c_{m,n}|^2\right]^{1/2}\end{aligned}$$

It now follows that $\left\|\sum_{m,n} c_{m,n}\phi_{m,n}\right\|_2 = 0$ implies $c_{m,n} = 0$ for all $m, n \in \mathbb{Z}$ because $\sqrt{A} - \lambda\sqrt{M} > 0$ by the assumption. Therefore, $\{\phi_{m,n}\}$ is l^2-linearly independent. □

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