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HARDY-LITTLEWOOD PROPERTY WITH THE INNER LENGTH METRIC

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ABSTRACT. A result of Hardy and Littlewood relates Hölder continuity of analytic functions in the unit disk with a bound on the derivative. Gehring and Martio extended this result to the class of uniform domains. We call it the Hardy-Littlewood property. Langmeyer further extended their result to the class of John disks in terms of the inner length metric. We call it the Hardy-Littlewood property with the inner length metric. In this paper we give several properties of a domain which satisfies the Hardy-Littlewood property with the inner length metric. Also we show some results on the Hölder continuity of conjugate harmonic functions in various domains.

1. Introduction

Suppose that D is a domain in the complex plane $\mathbb{C}$. Let $\mathbb{B}(z,r) = \{w : |w-z| < r\}$ for $z \in \mathbb{C}$ and $r > 0$ and let $\mathbb{B} = \mathbb{B}(0,1)$ be the unit disk in $\mathbb{C}$. Let $\ell(\gamma)$ denote the euclidean length of a curve γ, $\text{dia}(\gamma)$ be a diameter of γ and $\text{dist}(A,B)$ denote the euclidian distance from A to B for two sets $A, B \subset \overline{\mathbb{C}}$. Let $\alpha \in (0,1]$.

Suppose that f is a real or complex valued function defined in D. We say that f is in the *Lipschitz class*, $Lip_\alpha(D)$, $0 < \alpha \leq 1$, if there exists a constant m such that

$$|f(z_1) - f(z_2)| \leq m|z_1 - z_2|^\alpha \tag{1.1}$$

for all z_1 and z_2 in D, and we let $||f||_\alpha$ denote the infimum of the numbers m for which (1.1) holds. f is said to belong to the *local Lipschitz class*,

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$locLip_\alpha(D)$, if there is a constant m such that (1.1) holds whenever z_1, z_2 lie in any open disk which is contained in D. Let $||f||_\alpha^{loc}$ denote the infimum of the numbers m such that (1.1) holds in this situation.

DEFINITION 1.1. A domain D is called a *Lip_α-extension domain* if there exists a constant a depending on D and α such that $f \in locLip_\alpha(D)$ implies $f \in Lip_\alpha(D)$ with

$$||f||_\alpha \leq a||f||_\alpha^{loc}.$$

Suppose that f is analytic in D. If f is in $Lip_\alpha(D)$, then it is not difficult to show that

$$|f'(z)| \leq m dist(z, \partial D)^{\alpha-1}$$

in D. Conversely, we have the following well known result of Hardy and Littlewood.

THEOREM 1.2. [5] *If D is an open disk and f is analytic in D with*

$$|f'(z)| \leq m\text{dist}(z, \partial D)^{\alpha-1}$$

for all z in D and for every $\alpha \in (0, 1]$, then $f \in Lip_\alpha(D)$ with

$$||f||_\alpha \leq \frac{cm}{\alpha},$$

where c is an absolute constant.

The above theorem leads to the following notion, introduced in [3].

DEFINITION 1.3. A proper subdomain D in $\mathbb{C}$ is said to have the *Hardy-Littlewood property* if there exists a constant $c = c(D)$ such that whenever f is analytic in D and f satisfies the condition

$$|f'(z)| \leq m\text{dist}(z, \partial D)^{\alpha-1} \tag{1.2}$$

in D for every $\alpha \in (0, 1]$, then $f \in Lip_\alpha(D)$ with

$$||f||_\alpha \leq \frac{cm}{\alpha}.$$

Theorem 1.2 tells that each open disk has the Hardy-Littlewood property. In [3, Corollary 2.2] it is proved moreover that uniform domains, defined below, have the Hardy-Littlewood property and it is showed that there exist domains having the Hardy-Littlewood property without being uniform [9].

A domain D in $\mathbb{C}$ is said to be *b-uniform* if there exists a constant $b \geq 1$ such that each pair of points z_1 and z_2 in D can be joined by a rectifiable arc γ in D with

$$\ell(\gamma) \leq b|z_1 - z_2|$$

and with

$$\min(\ell(\gamma_1), \ell(\gamma_2)) \leq b\,\mathrm{dist}(z, \partial D) \tag{1.3}$$

for each $z \in \gamma$, where γ_1 and γ_2 are the components of $\gamma \setminus \{z\}$.

Next we say that a proper subdomain D in $\mathbb{C}$ has the *Hardy-Littlewood property of order* α for some $\alpha \in (0,1]$, if there exists a constant $k = k(D,\alpha)$ such that if f is analytic in D with (1.2) for all $z \in D$, then $f \in Lip_\alpha(D)$ with $||f||_\alpha \leq km$.

It is clear that if D has the Hardy-Littlewood property, then D has the Hardy-Littlewood property of order α for each $\alpha \in (0,1]$. But the opposite implication does not hold in general [1].

Next in [1] they give a characterization of a domain which has the Hardy-Littlewood property with order α as follows.

THEOREM 1.4. [1] *A simply connected domain D in $\mathbb{C}$ has the Hardy-Littlewood property with order $\alpha \in (0,1]$ if and only if D is a Lip_α-extension domain.*

On the other hands, in [8] it is showed that the Hardy-Littlewood property does not hold for John disks, defined below.

A simply connected bounded domain $D \subset \mathbb{C}$ is said to be a *b-John disk* if there exist a point $z_0 \in D$ and a constant $b \geq 1$ such that each point $z_1 \in D$ can be joined to z_0 by an arc γ in D satisfying

$$\ell(\gamma(z_1, z)) \leq b\,\mathrm{dist}(z, \partial D)$$

for each $z \in \gamma$, where $\gamma(z_1, z)$ is the subarc of γ with endpoints z_1, z. We call z_0 a *John center*, b a *John constant* and γ a *b-John arc.* A domain D in $\mathbb{C}$ is a b-John disk if and only if there is a constant $b \geq 1$ such that each pair of points $z_1, z_2 \in D$ can be joined by an arc γ in D which satisfies (1.3) [10]. Thus the class of simply connected bounded uniform domains is properly contained in the class of John disks. The converse is not true, for example, $\mathbb{B} \setminus [0,1)$.

But John disks hold analogues of the Hardy-Littlewood property [6], [8] and an analogue in [8] is explained in terms of the inner length metric.

THEOREM 1.5. [8] *If D is a b-John disk and f is analytic in D and f satisfies the condition (1.2) in D for some $\alpha \in (0, 1]$, then*

$$|f(z_1) - f(z_2)| \leq \frac{cm}{\alpha} \lambda_D(z_1, z_2)^\alpha,$$

for all z_1 and z_2 in D, where c is a constant which depends only on b,

$$\lambda_D(z_1, z_2) = \inf \ell(\beta).$$

Here infimum is taken over all open arcs β in D which join z_1 and z_2.

Suppose that f is a real or complex valued function defined in D. We say that f is in the *Lipschitz class with the inner length metric, $Lip^I_\alpha(D)$*, $0 < \alpha \leq 1$, if there exists a constant m_1 such that

$$|f(z_1) - f(z_2)| \leq m_1 \lambda_D(z_1, z_2)^\alpha \tag{1.4}$$

for all z_1 and z_2 in D, and we let $||f||^I_\alpha$ denote the infimum of the numbers m for which (1.4) holds.

By definition it is clear that if $f \in Lip_\alpha(D)$, then $f \in Lip^I_\alpha(D)$.

DEFINITION 1.6. A proper subdomain D in $\mathbb{C}$ is said to have the *Hardy-Littlewood property with the inner length metric of order α*, if there exists a constant $k = k(D, \alpha)$ such that whenever f is analytic in D and f satisfies the condition (1.2) in D for some $\alpha \in (0, 1]$, then f is in $Lip^I_\alpha(D)$ with

$$||f||^I_\alpha \leq km.$$

Clearly a proper subdomain D in $\mathbb{C}$ with the Hardy-Littlewood property of order α has the Hardy-Littlewood property with the inner length metric of order α. Also by Theorem 1.5 a John disk has the Hardy-Littlewood property with the inner length metric of order α.

One of the main subjects of this paper is to find properties of domains which have the Hardy-Littlewood property with the inner length metric of order α (see Section 2).

Also in Section 3 we show some results on the Hölder continuity of conjugate harmonic functions in domains introduced above.

2. The Hardy-Littlewood property with the inner length metric of order α

First of all we show that the converse of Theorem 1.5 is not true.

THEOREM 2.1. *There exists a domain D in $\mathbb{C}$ having the Hardy-Littlewood property with the inner length metric of order α which is not a John disk.*

PROOF. Let $G_j = \mathbb{B}(z_j, \frac{2^{-j}}{\sqrt{3}})$ where $z_j = |z_j|e^{i\theta_j}$ and

$$|z_j| = 1 - \frac{4^{-j}}{2} + \frac{2^{-j}}{\sqrt{3}}, \quad \theta_j = \frac{3\pi}{2}(1 - 2^{-j}), \quad j = 0, 1, 2, \dots .$$

Next let $D = \mathbb{B} \cup \bigcup_{j=0}^{\infty} G_j$. To show that D is not a John disk, let α_j, $j = 0, 1, 2, \dots$, be a straight crosscut of D joining two points at which $\mathbb{B}$ and G_j meet. Let A_j and B_j denote two subdomains of D divided by α_j with $0 \in A_j$ and $z_j \in B_j$. Then

$$\min(\mathrm{dia}(A_j), \mathrm{dia}(B_j)) = \mathrm{dia}(B_j) = 2a$$

and

$$\mathrm{dia}(\alpha_j) = \frac{4(b(a-b)(1-b)(1+a-b))^{\frac{1}{2}}}{1 - 2b + a},$$

where $a = \frac{2^{-j}}{\sqrt{3}}$ and $b = 4^{-j-1}$. By elementary calculation, we have

$$\frac{\mathrm{dia}(B_j)}{\mathrm{dia}(\alpha_j)} = k2^j$$

for some absolute constant k and hence there is no constant c such that

$$\mathrm{dia}(B_j) \le c\,\mathrm{dia}(\alpha_j)$$

for all $j = 0, 1, 2, \dots$. Thus by [2, Theorem 2.1], D is not a John disk. But by [9, Corollary 7.11] D satisfies the Hardy-Littlewood property of order α and thus it has the Hardy-Littlewood property with the inner length metric of order α. □

Now let us recall the distance functions k_α and δ_α on a domain D, introduced in [7]. For each $\alpha \in (0, 1]$ and for z_1, z_2 in D we define

$$k_\alpha(z_1, z_2) = \inf_\gamma \int_\gamma \mathrm{dist}(x, \partial D)^{\alpha-1} ds,$$

where the infimum is taken over all rectifiable arcs γ joining z_1 to z_2 in D. Furthermore,

$$\delta_\alpha(z_1, z_2) = \sup_f |f(z_1) - f(z_2)|,$$

where the supremum is taken over all analytic functions f on D satisfying

$$|f'(z)| \leq \operatorname{dist}(z, \partial D)^{\alpha-1}$$

for all $z \in D$.

The next theorem characterizes a domain which satisfies the Hardy-Littlewood property with the inner length metric of order α.

THEOREM 2.2. *A domain D in $\mathbb{C}$ has the Hardy-Littlewood property with the inner length metric of order α if and only if there is a constant $M < \infty$ such that for all z_1, $z_2 \in D$ there exists a rectifiable curve γ joining z_1 to z_2 in D with*

$$\int_\gamma \operatorname{dist}(x, \partial D)^{\alpha-1} ds \leq M \lambda_D(z_1, z_2)^\alpha. \tag{2.1}$$

To prove Theorem 2.2 we need the following Lemma 2.3 which shows that δ_α is connected to the metric k_α.

LEMMA 2.3. [7] *In a simply connected bounded domain $D \subset \mathbb{C}$ we have*

$$\delta_\alpha \leq k_\alpha \leq c_1 \delta_\alpha, \tag{2.2}$$

where $\alpha \in (0, 1]$ and c_1 is an absolute constant.

PROOF OF THEOREM 2.2. Assume that D has the Hardy-Littlewood property with the inner length metric of order α. By the definition of δ_α, the second inequality of (2.2) and

$$|f(z_1) - f(z_2)| \leq \frac{c(D)}{\alpha} \lambda_D(z_1, z_2)^\alpha$$

for f analytic in D and $|f'(z)| \leq \operatorname{dist}(z, \partial D)^{\alpha-1}$ in D, we obtain

$$k_\alpha(z_1, z_2) \leq c_1 \delta_\alpha(z_1, z_2) = c_1 \sup_f |f(z_1) - f(z_2)| \leq \frac{c_1 c(D)}{\alpha} \lambda_D(z_1, z_2)^\alpha.$$

Hence there exists a rectifiable curve γ joining z_1 and z_2 in D such that (2.1) is satisfied with $M = \frac{2c_1 c(D)}{\alpha}$. Conversely, assume that there exists a constant $M < \infty$ such that for all z_1, $z_2 \in D$ there exists a rectifiable curve γ joining z_1 to z_2 in D with (2.1). Then by the first inequality of (2.2) for all f analytic in D with $|f'(z)| \leq \mathrm{dist}(z, \partial D)^{\alpha-1}$ in D, we obtain

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq \sup_f |f(z_1) - f(z_2)| \\ &\leq \inf_\gamma \int_\gamma \mathrm{dist}(x, \partial D)^{\alpha-1} ds \\ &\leq M \lambda_D(z_1, z_2)^\alpha. \end{aligned}$$

□

THEOREM 2.4. *If a domain D in $\mathbb{C}$ is a Lip_α-extension domain, then it has the Hardy-Littlewood property with the inner length metric of order α.*

PROOF. In [4, Theorem 2.2] it is showed that a domain D in $\mathbb{C}$ is a *Lip_α-extension domain* if and only if there is a constant $M < \infty$ such that for all z_1, $z_2 \in D$ there exists a rectifiable curve γ joining z_1 to z_2 in D with (2.1) replaced $\lambda_D(z_1, z_2)^\alpha$ by $|z_1 - z_2|^\alpha$. Therefore we get the conclusion. □

REMARK 2.5. By Theorem 2.1, Theorem 2.4 and definition of the Hardy-Littlewood property of order α, we observe that the classes of John domains, *Lip_α-extension domain* and domains which satisfies the Hardy-Littlewood property of order α are properly contained in the class of domains which satisfies the Hardy-Littlewood property with the inner length metric of order α.

3. The Hölder continuity of conjugate harmonic functions in domains

Let

$$|\partial f(z)| = \lim_{|h| \to 0} \sup \frac{|f(z+h) - f(z)|}{|h|}, \quad \text{for } z \in D.$$

LEMMA 3.1. [3, Theorem 1.1] *If f is harmonic and in $Lip_\alpha(D)$, then*

$$|\partial f(z)| \leq \frac{4}{\pi} ||f||_\alpha \mathrm{dist}(z, \partial D)^{\alpha-1} \tag{3.1}$$

in D.

In [3] combining Lemma 3.1 and the fact that an uniform domain has the Hardy-Littlewood property yields the following extension of a result due to Privaloff on the continuity of conjugate harmonic functions in the unit disk.

LEMMA 3.2. [3, Corollary 2.2] *If D is b-uniform and if f is analytic with $Re(f)$ in $Lip_\alpha(D)$, then f is in $Lip_\alpha(D)$ with*

$$||f||_\alpha \leq \frac{c}{\alpha}||Re(f)||_\alpha, \tag{3.2}$$

where c is a constant which depends only on the constant b.

Now we extend the above result to the class of domains which has the Hardy-Littlewood property of order α.

THEOREM 3.3. *If a domain D in $\mathbb{C}$ satisfies the Hardy-Littlewood property of order α and if f is analytic with $Re(f)$ in $Lip_\alpha(D)$, then f is in $Lip_\alpha(D)$ with (3.2) where $c = c(D)$.*

PROOF. Let $u = Re(f)$. Then u is harmonic in D,

$$|f'(z)| = |\frac{\partial u}{\partial x}(z) - i\frac{\partial u}{\partial y}(z)| \leq 2|\partial u(z)| \leq \frac{8}{\pi}||u||_\alpha \text{dist}(z, \partial D)^{\alpha-1}$$

by the Cauchy-Riemann equations and Lemma 3.1. Then since D satisfies the Hardy-Littlewood property of order α, we obtain that f is in $Lip_\alpha(D)$ with (3.2) where $c = c_1(D)\frac{8}{\pi}$. □

By Theorem 1.4 and Theorem 3.3 gives the following.

COROLLARY 3.4. *If a domain D in $\mathbb{C}$ is a Lip_α-extension domain and if f is analytic with $Re(f)$ in $Lip_\alpha(D)$, then f is in $Lip_\alpha(D)$ with (3.2) where c is the same constant as the above Theorem 3.3.*

But the above result does not hold for a John disk.

THEOREM 3.5. *There exists an analytic function f on a John disk such that $Re(f)$ is in $Lip_\alpha(D)$, but f is not in $Lip_\alpha(D)$*

PROOF. Let $D = \mathbb{B} \setminus (-1, 0]$ and define a function f on D by $f(z) = Logz$ which is an analytic branch of $logz$. Then clearly D is a John Disk. Also $f(z) = log|z| + iArg(z)$ and $Re(f) = log|z|$ is differentiable on D, thus $Re(f)$ is in $Lip_\alpha(D)$ for $0 < \alpha \leq 1$. But $Arg(z)$ is not in $Lip_\alpha(D)$. For let

$$z_n = \frac{1}{4}e^{i\pi n/(n+1)}, w_n = \frac{1}{4}e^{-i\pi n/(n+1)},$$

where $n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} |Arg(z_n) - Arg(w_n)| = 2\pi,$$

while

$$\lim_{n\to\infty} |z_n - w_n|^\alpha = 0.$$

Thus $Arg(z)$ is not in $Lip_\alpha(D)$, therefore f is not in $Lip_\alpha(D)$. □

To obtain an analogous result of Lemma 3.2 for a John disk, we need a following analogous result of Lemma 3.1 for $f \in Lip^I_\alpha(D)$. The proof is similar to the proof of Lemma 3.1 [3, Theorem 1.1].

THEOREM 3.6. *If f is harmonic and in $Lip^I_\alpha(D)$, then for $z \in D$*

$$|\partial f(z)| \le \frac{4}{\pi}||f||^I_\alpha \mathrm{dist}(z, \partial D)^{\alpha-1}. \tag{3.3}$$

PROOF. For $z \in \mathbb{C}$ and $0 < r < \infty$ let $B(z,r)$ denote the open disk with center z and radius r. If $z \in D$ and $r < \mathrm{dist}(z,\partial D)$, then $\overline{\mathbb{B}}(z,r) \subset D$ and with the Poisson integral formula we obtain

$$\begin{aligned} f(z+h) - f(z) &= \frac{1}{2\pi}\int_0^{2\pi} \Big(\frac{r^2 - |h|^2}{|re^{i\theta} - h|^2} - 1\Big)(f(z+re^{i\theta}) - f(z))d\theta \\ &= \frac{|h|}{\pi}\int_0^{2\pi} \frac{rcos(\theta-\phi) - |h|}{|re^{i\theta} - h|^2}(f(z+re^{i\theta}) - f(z))d\theta \end{aligned}$$

for $|h| < r$ where $h = |h|e^{i\phi}$. Thus by (1.4),

$$\frac{|f(z+h) - f(z)|}{|h|} \le \frac{1}{\pi}\int_0^{2\pi} \frac{r|cos(\theta-\phi)| + |h|}{(r-|h|)^2} m\lambda_D(z+re^{i\theta}, z)^\alpha d\theta.$$

Then since $\lambda_D(z + re^{i\theta}, z) = r$, we have

$$|\partial f(z)| \le \frac{4}{\pi} m r^{\alpha-1}.$$

Letting $r \to \mathrm{dist}(z, \partial D)$ and $m \to ||f||^I_\alpha$ then yields (3.3). □

THEOREM 3.7. *If a domain D in $\mathbb{C}$ has the Hardy-Littlewood property with the inner length metric of order α and if f is analytic with $Re(f) \in Lip^I_\alpha(D)$, then f is in $Lip^I_\alpha(D)$ with*

$$||f||^I_\alpha \le \frac{8}{\pi}\frac{c(D)}{\alpha}||Re(f)||^I_\alpha. \tag{3.4}$$

PROOF. Let $u = Re(f)$. Then u is harmonic in D,

$$|f'(z)| = |\frac{\partial u}{\partial x}(z) - i\frac{\partial u}{\partial y}(z)| \leq 2|\partial u(z)| \leq \frac{8}{\pi}||u||^I_\alpha \text{dist}(z,\, \partial D)^{\alpha-1}$$

by the Cauchy-Riemann equations and Theorem 3.6. Then since D satisfies the Hardy-Littlewood property with the inner length metric of order α, we obtain that f is in $Lip^I_\alpha(D)$ with (3.4). □

Now Theorem 1.5 and Theorem 3.7 give an analogous result of Lemma 3.2 for a John disk.

COROLLARY 3.8. *If a domain D in $\mathbb{C}$ is a b-John domain and if f is analytic with $Re(f)$ in $Lip^I_\alpha(D)$, then f is in $Lip^I_\alpha(D)$ with* (3.4) *replaced $c(D)$ by $c(b)$.*

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