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A STUDY ON EQUIVALENT FORMS OF THE AXIOM OF CHOICE IN AN WELL-POINTED TOPOS

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ABSTRACT. There are various forms of the axiom of choice and also various weak forms of the axiom of choice in a topos. This paper give equivalent forms of the axiom of choice in a well-pointed topos.

1. Introduction

In a topos, the *axiom of choice* can be expressed as following.

(AC1) Every epimorphism is a retraction.

(AC2) For any noninitial object A and $f : A \to B$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$.

(AC3) For any noninitial object A, there exists $\sigma : \Omega^A \to A$ such that for all $f : 1 \to \Omega^A$, we have $\sigma \circ f \in f'$ where $f' : A' \to A$ is a monomorphism, provided that $ev \circ (f \times i_A)$ is not the characteristic morphism of $0 \to A$.

(WO) For any P and $q : U \to \Omega^P$, if there exist $\alpha : V \to U$ and $p : V \to P$ such that $(q\alpha, p)$ factors through $\in_P \rightarrow \Omega^P \times P$, then there exists $\alpha_0 : V_0 \to U$ and $p_0 : V_0 \to P$ such that $(q\alpha_0, p_0)$ factors through $\in_P$, and such that for all $\beta : W \to V_0$ and all $p_1 : W \to P$, if $(q\alpha_0\beta, p_1)$ factors through $\in_P$, then $(p_0\beta, p_1)$ factors through a monomorphism $P_1 \to P \times P$.

(ASC) Every separated epimorphism of $\mathcal{E}$ is a retraction.

(IAC1) The functor $(-)^A : \mathcal{E} \to \mathcal{E}$ preserves epimorphisms for every object A.

(IAC2) For any epimorphism $u : C \to D$, there exists an object V with epimorphism $V \to 1$ such that $V^*(u)$ is a retraction in $\mathcal{E}/V$.

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(IAC3) If $f : X \to Y$ is an epimorphism in $\mathcal{E}$, then $\pi_Y(f)$ has global support.

A. M. Penk [7] showed that (AC2) and (AC3) are equivalent in a well-pointed topos, M. M. Mawanda [5] showed that (AC1), (AC2) and (ASC) are equivalent in a well-pointed topos and P. T. Johnstone [2] showed that (IAC1), (IAC2) and (IAC3) are equivalent in any topos. Hence we know that (AC1), (AC2), (AC3) and (ASC) are equivalent in a well-pointed topos. In this paper, we show that (AC2) is equivalent to (WO) in a well-pointed topos and also show that (IAC1) is equivalent to (ASC) in a well-pointed topos. Therefore we know that (AC1), (AC2), (AC3), (IAC1), (IAC2), (IAC3), (ASC) and (WO) are equivalent in a well-pointed topos.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

DEFINITION 2.1. An *elementary topos* is a category $\mathcal{E}$ that satisfies the following;

(T1) $\mathcal{E}$ is finitely complete,

(T2) $\mathcal{E}$ has exponentiation,

(T3) $\mathcal{E}$ has subobject classifier.

(T2) means that for every object A in $\mathcal{E}$, endofunctor $(-) \times A$ has its right adjoint $(-)^A$. Hence for every object A in $\mathcal{E}$, there exists an object B^A, and a morphism $ev_A : B^A \times A \to B$, called the evaluation map of A, such that for any Y and $f : Y \times A \to B$ in $\mathcal{E}$, there exists a unique morphism g such that $ev_A \circ (g \times i_A) = f$;

$$\begin{array}{ccc} Y \times A & \xrightarrow{f} & B \\ {\scriptstyle g \times i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\ B^A \times A & \xrightarrow{ev_A} & B \end{array}$$

And subobject classifier in (T3) is an $\mathcal{E}$-object Ω, together with a morphism $\top : 1 \to \Omega$ such that for any monomorphism $h : D \to C$, there is unique morphism $\chi_h : C \to \Omega$, called the character of $h : D \to C$ that makes the following diagram a pull-back;

$$\begin{array}{ccc} D & \xrightarrow{!} & 1 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \top} \\ C & \xrightarrow{\chi_h} & \Omega \end{array}$$

DEFINITION 2.2. A topos $\mathcal{E}$ is called *Boolean* if for every object D in $\mathcal{E}$, $(\mathrm{Sub}(D), \in)$ is a Boolean algebra where $\mathrm{Sub}(D)$ is the class of monomorphism with common codomain D, and we say $g \in f$ if there exists a morphism $h : B \to A$ such that $f \circ h = g$ where $f : A \to D$ and $g : B \to D$ are monomorphisms.

LEMMA 2.3. *For any topos $\mathcal{E}$, the following statements are equivariant;*

(1) *$\mathcal{E}$ is Boolean.*
(2) *$Sub(\Omega)$ is a Boolean algebra.*
(3) *$\top : 1 \to \Omega$ has a complement in $Sub(\Omega)$.*
(4) *$\perp : 1 \to \Omega$ is the complement of $\top$ in $Sub(\Omega)$.*
(5) *$\top \cup \perp \simeq 1_\Omega$ in $Sub(\Omega)$.*
(6) *$\mathcal{E}$ is classical.*
(7) *$i_1 : 1 \to 1 + 1$ is a subobject classifier.*

PROOF. See Goldblatt [1]. □

EXAMPLE 2.4. The category $M - Set$ is a non-Boolean topos.

PROOF. See Goldblatt [1], Madanshekaf [4] and Mehdi Ebahimi [6].□

DEFINITION 2.5. A topos is called *well-pointed* if it satisfies the extensionality principle for morphism, i.e., If $f, g : A \to B$ are a pair of distinct parallel morphisms, then there is an element $a : 1 \to A$ of A such that $f \circ a \neq g \circ a$.

LEMMA 2.6. *In a well-pointed topos,* (AC2) *is equivalent to* (AC3).

PROOF. See Penk [7]. □

LEMMA 2.7. *In a well-pointed topos,* (AC1), (AC2) *and* (ASC) *are equivalent.*

PROOF. See Mawanda [5]. □

LEMMA 2.8. *In any topos,* (IAC1), (IAC2) *and* (IAC3) *are equivalent.*

PROOF. See Johnstone [2]. □

3. Main parts

THEOREM 3.1. *In a well-pointed topos* $\mathcal{E}$, (WO) *implies* (AC2).

PROOF. Let $f : A \to B$ be a morphism in $\mathcal{E}$, then there exist an epimorphism $e : A \to X$ and a monomorphism $m : X \to B$ such that $f = m \circ e$. By hypothesis, there exists a morphism $t : B \to X$ such that $t \circ m = i_X$. We only show that there is a morphism $s : X \to A$ such that $f = f \circ (s \circ t) \circ f = f$. Since $e : A \to X$ is epimorphism, there is a morphism $q : X \to \Omega^A$ which is the interpretation of the term $\{a | e(a) = x\}$. By definition of (WO), we can find an epimorphism $r : V \to X$ and a morphism $n : V \to A$ such that n is a minimal choice of qr. Since every epimorphism is coequalizer, there are morphisms $u, v : W \to V$ such that the following square commutes.

$$\begin{array}{ccc} W & \xrightarrow{u} & V \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle r} \\ V & \xrightarrow[r]{} & X \end{array}$$

Thus we get $q \circ r \circ v = q \circ r \circ u$. Also nu, nv are both minimal choice of $q \circ r \circ v = q \circ r \circ u$. By definition of (WO), we can find $nu = nv$. Since every epimorphism is coequalizer, there is a morphism $s : X \to A$ such that $s \circ r = n$. Also there is a morphism $c : X \to \in_A$ such that $k \circ s = c$ where $k : A \to \in_A$ and $\in_A$ is the subobject classified by $ev : \Omega^A \times A \to \Omega$. Then we have $(q, s) = l \circ c = l \circ k \circ s = (q \circ e \circ s, s)$ where $l : \in_A \to \Omega^A \times A$. Since q is a monomorphism, we have that $f = f \circ (s \circ t) \circ f = f$. □

THEOREM 3.2. *In a well-pointed topos* $\mathcal{E}$, (AC2) *implies* (WO).

PROOF. Let X_0 be noninitial object in $\mathcal{E}$. Since $\mathcal{E}$ satisfies (AC2), there is a morphism $\psi : NX_0 \to X_0$ such that $\psi \circ g_i \in g_i'$ where NX_0 is the object of noninitial subobjects of X_0 with the usual ordering, $g_i : U \to NX_0$ is a morphism and $g_i' : X_0' \to X_0$ is a monomorphism. Since $\mathcal{E}$ is Boolean, we get that $-(\psi \circ g_0) \equiv g_1$ where the pullbacks of $\psi \circ g_0$ and $-(\psi \circ g_0)$ is the initial object, $-(\psi \circ g_1) \equiv g_2$ where the pullbacks of $\psi \circ g_1$ and $-(\psi \circ g_1)$ is the initial object, etc. Generally, we get that $-(\psi \circ g_{n-1}) \equiv g_n$ where the pullbacks of $\psi \circ g_{n-1}$ and $-(\psi \circ g_{n-1})$ is the initial object.

Thus we construct $\phi : X_0 \to NX_0$ such that $Im(\phi)$ is a subobject of NX_0 consists of $g_0, -(\psi \circ g_0), -(\psi \circ g_1), \dots$ and $-(\psi \circ g_{m-1})$, where $-g_m$

is an initial object, and $\psi \circ \phi = i_{X_0}$. Then $Im(\phi)$ is a linear ordered with minimal choice. Since ϕ is a monomorphism, X_0 has an ordering with minimal choice. □

THEOREM 3.3. *In a well-pointed topos $\mathcal{E}$,* (ASC) *implies* (IAC1).

PROOF. Since $\mathcal{E}$ is well-pointed, $\mathcal{E}$ satisfies (ASC) if and only if every epimorphism in $\mathcal{E}$ has a right inverse. Let $e : A \to B$ be an epimorphism. We claim that, for any object X, $e^X : A^X \to B^X$ is an epimorphism. For any $v : X \to B$, since epimorphism $e : A \to B$ in $\mathcal{E}$ has a right inverse, there is a morphism $r : B \to A$ such that $e \circ r = i_B$.

$$\begin{array}{ccc} X & \xrightarrow{i_X} & X \\ {\scriptstyle rov}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow{e} & B \end{array}$$

Hence we get a morphism $rov : X \to A$ such that $e^X(rov) = eorov = v$. Thus e^X is an epimorphism.

THEOREM 3.4. *In a well-pointed topos $\mathcal{E}$,* (IAC1) *implies* (ASC).

PROOF. Let $f : A \to B$ be an epimorphism. Since $\mathcal{E}$ is finitely complete category with exponentiation, we have a pull-back diagram.

$$\begin{array}{ccc} \Pi_B(f) & \xrightarrow{u} & 1 \\ \downarrow & & \downarrow \\ A^B & \xrightarrow{f^B} & B^B \end{array}$$

Since B is internally projective, f^B is an epimorphism. By the property of pull-back, $u : \Pi_B(f) \to 1$ is an epimorphism . Since $\mathcal{E}$ is well-pointed, $u : \Pi_B(f) \to 1$ has a right inverse $v : 1 \to \Pi_B(f)$ such that $u \circ v = i_1$. Since the pull-back functor has a right adjoint, there is a morphism $v' : B^*(1) \to f$ such that $f \circ v' = i_B$. Thus $\mathcal{E}$ satisfies (AC1). Since (AC1) implies Booleanness, every separated epimorphism is an epimorphism by Lemma 2.3. Hence $\mathcal{E}$ satisfies (ASC). □

COROLLARY 3.5. *In a well-pointed topos $\mathcal{E}$,* (AC1), (AC2), (AC3), (IAC1), (IAC2), (IAC3), (ASC) *and* (WO) *are equivalent.*

PROOF. By Lemma 2.6, Lemma 2.7, Lemma 2.8, Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4. □

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