

BOUNDARIES FOR AN ALGEBRA OF BOUNDED HOLOMORPHIC FUNCTIONS

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Dedicated to the memory of Klaus Floret (1941-2002)

ABSTRACT. Let $A_b(B_E)$ be the Banach algebra of all complex valued bounded continuous functions on the closed unit ball B_E of a complex Banach space E , and holomorphic in the interior of B_E , endowed with the sup norm. We present some sufficient conditions for a set to be a boundary for $A_b(B_E)$ in case E belongs to a class of Banach spaces that includes the pre-dual of a Lorentz sequence space studied by Gowers in [6]. We also prove the non-existence of the Shilov boundary for $A_b(B_E)$ and give some examples of boundaries.

Introduction

Let $A_b(B_E)$ be the Banach space of all complex valued functions defined on the closed unit ball B_E of a Banach space E which are bounded on B_E and holomorphic in the interior of B_E . It is clear that $A_b(B_E)$ is a Banach algebra when given the norm $\|f\| = \sup_{x \in B_E} |f(x)|$. If E is finite dimensional, $A_b(B_E)$ coincides with its closed subalgebra $A_u(B_E)$ of those functions of $A_b(B_E)$ which are uniformly continuous on B_E .

Following Globevnik [5], a subset F of B_E is a boundary for $\mathcal{A} = A_b(B_E)$ (or $\mathcal{A} = A_u(B_E)$) if $\|f\| = \sup_{x \in F} |f(x)|$ for all $f \in \mathcal{A}$. If E is a finite dimensional space, \mathcal{A} is a uniform algebra and it is well known that the intersection of all closed boundaries for \mathcal{A} is again a closed boundary for \mathcal{A} , called the Shilov boundary for \mathcal{A} (see [12], p.38). If E

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is an infinite dimensional space, the existence of the Shilov boundary for \mathcal{A} is not guaranteed.

If $E = \mathcal{C}^n$ with the sup norm, $\mathcal{A} = A_b(B_E) = A_u(B_E)$ is the polydisc algebra and the boundaries for \mathcal{A} have been studied extensively (see [9] or [7]). The cases $E = c_0$ and $E = l_p$ ($1 \leq p \leq \infty$) have been studied in [4] and [1] respectively. Globevnik showed that the Shilov boundary for $A_u(B_{c_0})$ doesn't exist. Latter Aron, Choi, Lourenço and Paques showed that in case $E = l_p$ ($1 \leq p < \infty$) the Shilov boundary for $A_u(B_{l_p})$ exists and coincides with the unit sphere S_{l_p} . In the same paper they show that the Shilov boundary for $A_u(B_{l_\infty})$ doesn't exist.

In this paper we will be interested in the algebra $A_b(B_G)$ where G is the pre-dual of the Lorentz sequence space $d(\{\frac{1}{n}\}, 1)$ considered by Gowers in [6]. Gowers was the first one to observe that this space is useful when we are interested in studying problems related with norm attaining functions. Afterwards this space has received attention in some recent papers (see [2], [3] and [11]). In [8] we studied the boundaries for $A_u(B_G)$. Since $A_u(B_G) \subset A_b(B_G)$, every boundary for $A_b(B_G)$ is a boundary for $A_u(B_G)$. But it is not true that every boundary for $A_u(B_G)$ is a boundary for $A_b(B_G)$ (see Example 2.7). In cases $E = c_0$ and $E = G$ it was possible to give a complete description of the boundaries for the algebra $A_u(B_E)$ (see [5] for $E = c_0$ and [8] for $E = G$), but the study of the boundaries for $A_b(B_E)$ seems to be harder. In [5] Globevnik got sufficient conditions for a set to be a boundary for $A_b(B_{c_0})$. In this work we establish sufficient conditions for a set to be a boundary for $A_b(B_G)$. We also investigate existence of the Shilov boundary for $A_b(B_G)$ (i.e. the smallest closed boundary for $A_b(B_G)$). It was known that the intersection of all closed boundaries for $A_b(B_E)$ is empty when $E = c_0$ or l_∞ (see [5] and [1]) and coincides with the unit sphere S_{l_p} when $E = l_p$ ($1 \leq p < \infty$) (see [1]). We are going to show that the intersection of all closed boundaries for $A_b(B_G)$ is empty.

First of all we recall some definitions and results established in [8]. If we fix $p \in \mathbb{N} = \{1, 2, 3, \dots\}$, for each complex sequence $z = (z_j)_{j=1}^\infty$ we define

$$\phi_{p,n}(z) = \sup_{|J|=n} \frac{\sum_{j \in J} |z_j|}{\sum_{j=1}^n \frac{1}{p+j-1}}$$

where $J \subset \mathbb{N}$ and $|J|$ denotes the cardinal of the set J . We denote by G_p the complex Banach space of the complex sequences $z = (z_j)_{j=1}^\infty$ such

that $\lim_{n \rightarrow \infty} \phi_{p,n}(z) = 0$, endowed with the norm given by

$$\|z\|_p = \sup_{n \in \mathbb{N}} \phi_{p,n}(z)$$

for all $z \in G_p$. The space considered by Gowers in [6] corresponds to the case $p = 1$. For this reason G_p will be called the Gowers space with characteristic p . All the spaces G_p are clearly isomorphic but our sufficient conditions for a set to be a boundary for $A_b(B_G)$ depend on the introduction of these spaces.

It is clear that we can identify the set \mathcal{C}^n with the set of the sequences $z = (z_i)_{i=1}^\infty \in G_p$ such that $z_i = 0$ for all $i > n$. We will denote $(\mathcal{C}^n, \|\cdot\|_p)$ by $G_{p,n}$ and we will say that $G_{p,n}$ is the Gowers space with characteristic p and finite dimension n . We remark that

$$(F1) \quad \|z\|_p = \sup_{1 \leq k \leq n} \sup_{|J|=k} \frac{\sum_{j \in J} |z_j|}{\sum_{j=1}^k \frac{1}{p+j-1}} \quad \text{for all } z \in G_{p,n}.$$

For each $J \subset \mathbb{N}$, let \mathcal{C}^J denote the set of all $z = (z_i)_{i=1}^\infty$ such that $z_i = 0$ for all $i \notin J$. If $|J| = k$ we have that \mathcal{C}^J is isomorphic to a subset of \mathcal{C}^n for all $n \geq k$ (we will write $\mathcal{C}^J \subset \mathcal{C}^n$). Given $z = (z_i)_{i=1}^\infty \in G_p$, let $J \subset \mathbb{N}$ such that $z_i = 0$ for all $i \notin J$. If $z_J = \sum_{j \in J} z_j e_j$ it is clear

that $\|z\|_p = \|z_J\|_p$. In this sense, if $z \in G_p$ has at most k coordinates different from zero we will say that $z \in G_{p,k}$. If $z = (z_i)_{i=1}^\infty \in G_p$, the set of all $j \in \mathbb{N}$ such that $z_j \neq 0$ is called the support of z and is denoted by $\text{supp}(z)$. An element $z \in G_p$ is a finite vector if $\text{supp}(z)$ is a finite set.

Let S_p and B_p denote, respectively, the unit sphere $\{z \in G_p : \|z\|_p = 1\}$ and the closed unit ball $\{z \in G_p : \|z\|_p \leq 1\}$. By Proposition 3.2 of [8] the set of the finite vectors of the unit sphere S_p is dense in B_p . For each $n \in \mathbb{N}$, let $S_{p,n}$ and $B_{p,n}$ denote $S_p \cap G_{p,n}$ and $B_p \cap G_{p,n}$, respectively.

DEFINITION. Let J be a finite subset of \mathbb{N} . The torus in G_p associated to J is the set T_p^J of all $z = (z_j)_{j=1}^\infty \in G_p$ such that

- (t1) $z_j \neq 0$ if and only if $j \in J$
- (t2) $\sum_{j \in J} |z_j| = \sum_{j=1}^k \frac{1}{p+j-1}$
- (t3) $\sum_{j \in L} |z_j| \leq \sum_{j=1}^t \frac{1}{p+j-1}$ for all $L \subset \mathbb{N}$ satisfying $L \subset J$.

In case $J = \{1, \dots, n\}$, we will denote T_p^J by T_p^n . We call any vector belonging to a torus a toroidal vector.

1. Boundaries for $A_b(B_p)$

PROPOSITION 1.1. *Let $(J_n)_{n=1}^\infty$ be a sequence of finite subsets of \mathbb{N} such that any finite subset of \mathbb{N} is contained in J_n for some $n \in \mathbb{N}$. Then $S = \bigcup_{n=1}^\infty T_p^{J_n}$ is a boundary for $A_b(B_p)$.*

Proof. Take $f \in A_b(B_p)$ with $\|f\| = 1$ arbitrary. Since the set of the finite vectors of S_p is dense in S_p , we can find a sequence of finite vectors $(y_k)_{k=1}^\infty \subset S_p$ such that

$$1 - \frac{1}{k} < |f(y_k)| \leq 1.$$

By hypothesis, for each k we have $\text{supp}(y_k) \subset J_n$ for some n and consequently $y_k \in B_{p,k_n}$ where $k_n = |J_n|$. By Theorem 2.4 of [8], $T_p^{J_n}$ is a boundary for $A_u(B_{p,k_n}) = A_b(B_{p,k_n})$ and so there exists $x_k \in T_p^{J_n}$ such that

$$1 - \frac{1}{k} < |f(y_k)| \leq |f(x_k)| = 1.$$

So there exists a sequence $(x_k)_{k=1}^\infty \in S = \bigcup_{n=1}^\infty T_p^{J_n}$ such that

$$\lim_{k \rightarrow \infty} |f(x_k)| = 1.$$

□

DEFINITION 1.1. Let $S \subset B_p$ such that $S \cap B_{p+k} \neq \emptyset$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ and $0 < \epsilon < 1$ we define

$$C_k(S, \epsilon) = \sup\{|f(0)| : f \in A_b(B_{p+k}), \|f\| \leq 1, |f(x)| < 1 - \epsilon, \text{ for all } x \in S \cap B_{p+k}\}.$$

We say that S is a 0-boundary for $A_b(B_p)$ if $\sup_{k \in \mathbb{N}} C_k(S, \epsilon) < 1$ for all $\epsilon > 0$. A family $\{S_\gamma\}_{\gamma \in \Gamma}$ of subsets of B_p such that $S_\gamma \cap B_{p+k} \neq \emptyset$ for all $\gamma \in \Gamma$ and for all $k \in \mathbb{N}$ is said to be a uniform family of 0-boundaries for $A_b(B_p)$ if

$$\sup_{\gamma \in \Gamma} \sup_{k \in \mathbb{N}} C_k(S_\gamma, \epsilon) < 1 \text{ for all } \epsilon > 0.$$

DEFINITION 1.2. Let $M = \{m_1, m_2, \dots, m_k\} \subset \mathbb{N}$ with $m_1 < m_2 < \dots < m_k$ and let $\phi : \mathbb{N} \rightarrow \mathbb{N} \setminus M$ be the canonical bijection that

preserves the order. We define a mapping Π_M from $H = \{x \in G_p : \text{supp}(x) \cap M = \emptyset\}$ into G_p by

$$\Pi_M(x) = \sum_{i=1}^{\infty} x_{\phi(i)} e_i \quad \text{for all } x \in H.$$

Take $S \subset B_p$. With each finite vector $x \in B_p$ we may associate the set

$$S(x) = \{\Pi_{\text{supp}(x)}(y) : x + y \in S, \text{supp}(y) \cap \text{supp}(x) = \emptyset\}.$$

THEOREM 1.2. *A subset S of B_p is a boundary for $A_b(B_p)$ whenever there exists a set V of finite vectors of B_p dense in some boundary for $A_b(B_p)$ and such that $\{S(v)\}_{v \in V}$ is a uniform family of 0-boundaries.*

Proof. Suppose that S is not a boundary for $A_b(B_p)$. In this case there exist $f \in A_b(B_p)$ and $\delta > 0$ such that $\|f\| = 1$ and $|f(x)| < 1 - \delta$ for all $x \in S$.

Let V be as in the hypothesis and let T be a boundary for $A_b(B_p)$ such that V is dense in T . It is clear that there exists a sequence $(v_n)_{n=1}^{\infty} \subset V$ such that $\lim_{n \rightarrow \infty} |f(v_n)| = 1$.

For each $n \in \mathbb{N}$, let m_n be the cardinal of $\text{supp}(v_n)$ and define a mapping ϕ_n from B_{p+m_n} into \mathcal{C} by

$$\phi_n(x) = f(v_n + \pi_{\text{supp}(v_n)}^{-1}(x))$$

for all $x \in B_{p+m_n}$. It is routine to verify that

$$v_n + \pi_{\text{supp}(v_n)}^{-1}(x) \in B_{p+m_n}$$

for all $x \in B_{p+m_n}$. We affirm that the sequence $(\phi_n)_{n=1}^{\infty}$ satisfies the following conditions:

- (i) $\phi_n \in A_b(B_{p+m_n})$ and $\|\phi_n\| \leq 1$ for all $n \in \mathbb{N}$,
- (ii) $|\phi_n(x)| < 1 - \delta$, for all $x \in S(v_n) \cap B_{p+m_n}$,
- (iii) $\lim_{n \rightarrow \infty} |\phi_n(0)| = 1$.

Indeed: (i) follows from $\|f\| = 1$ and from the linearity and continuity of the mapping

$$\pi_{\text{supp}(y)}^{-1} : G_{p+m_n} \longrightarrow G_{p+m_n}.$$

Given any $x \in S(v_n)$, by definition 1.2 we have $x = \pi_{\text{supp}(v_n)}(z)$ where $v_n + z \in S$ and $\text{supp}(v_n) \cap \text{supp}(z) = \emptyset$ and (ii) is true as $z = \pi_{\text{supp}(v_n)}^{-1}(x)$ and $|\phi_n(x)| = |f(v_n + z)| < 1 - \delta$. Finally,

$$\lim_{n \rightarrow \infty} |\phi_n(0)| = \lim_{n \rightarrow \infty} |f(v_n)| = 1$$

gives (iii). Now (i) and (ii) imply

$$\begin{aligned} C_{m_n}(S(v_n), \delta) &= \sup\{|g(0)| : g \in A_b(B_{p+m_n}), \|g\| \leq 1, |g(x)| < 1 - \delta, \\ &\quad \forall x \in S(v_n) \cap B_{p+m_n}\} \\ &\geq |\phi_n(0)|. \end{aligned}$$

Consequently $\sup_{k \in \mathbb{N}} C_k(S(v_n), \delta) \geq |\phi_n(0)|$, and by (iii)

$$\sup_{v \in V} \sup_{k \in \mathbb{N}} C_k(S(v), \delta) \geq \sup\{|\phi_n(0)| : n \in \mathbb{N}\} = 1.$$

This contradicts the fact that $\{S(v)\}_{v \in V}$ is a uniform family of 0-boundaries for $A_b(B_p)$. □

LEMMA 1.3. *Let $\{S_\gamma\}_{\gamma \in \Gamma}$ be a family of subsets of B_p . If there exists $r \in \mathbb{R}$ satisfying $0 \leq r < 1$ and $S_\gamma \cap rB_{p+k} \neq \emptyset$ for all $\gamma \in \Gamma$ for all $k \in \mathbb{N}$, then $\{S_\gamma\}_{\gamma \in \Gamma}$ is a uniform family of 0-boundaries for $A_b(B_p)$.*

Proof. In case $r = 0$ is clear as in this case $0 \in S_\gamma$ for all $\gamma \in \Gamma$. Suppose $0 < r < 1$ and take $\delta = \frac{1}{r} - 1$. Fix $k \in \mathbb{N}$ arbitrary. Given $y \in S_\gamma \cap rB_{p+k}$, let $v \in B_{p+k}$ such that $y = rv$. By lemma 1.4 of [5] there exists $C(\delta) < \infty$ (independent of k) such that every $g \in A_b(B_{p+k})$, $\|g\| \leq 1$, satisfies

$$|g(0) - g(y)| \leq C(\delta)[1 - |g(0)|]$$

and so

$$|g(0)| \leq \frac{C(\delta)}{1 + C(\delta)} + \frac{|g(y)|}{1 + C(\delta)}.$$

If $|g(x)| < 1 - \epsilon$ for all $x \in S_\gamma \cap B_{p+k}$, we have

$$|g(0)| = 1 - \frac{\epsilon}{1 + C(\delta)}$$

and this implies

$$C_k(S_\gamma, \epsilon) \leq 1 - \frac{\epsilon}{1 + C(\delta)} \text{ for all } k \in \mathbb{N}, \text{ for all } \gamma \in \Gamma$$

where $C(\delta)$ is independent of $k \in \mathbb{N}$ and $\gamma \in \Gamma$. So,

$$\sup_{\gamma \in \Gamma} \sup_{k \in \mathbb{N}} C_k(S_\gamma, \epsilon) < 1$$

for all $\epsilon > 0$. □

LEMMA 1.4. *Let $\theta_0 > 0$ and let $\{S_\gamma\}_{\gamma \in \Gamma}$ be a family of subsets of B_p such that for each $\gamma \in \Gamma$ and each $k \in \mathbb{N}$ there exists $x_\gamma \in S_\gamma$ satisfying $e^{i\theta} x_\gamma \in S_\gamma \cap B_{p+k}$ for all $\theta \in \mathbb{R}$, $|\theta| \leq \theta_0$. Then $\{S_\gamma\}_{\gamma \in \Gamma}$ is a uniform family of 0-boundaries for $A_b(B_p)$.*

Proof. Take $k \in \mathbb{N}$, $0 < \delta < 1$ and $\gamma \in \Gamma$ arbitrarily. If $\phi \in A_b(B_{p+k})$ is such that $\|\phi\| \leq 1$ and $|\phi(x)| < 1 - \delta$ for all $x \in S_\gamma \cap B_{p+k}$ by hypothesis there exists $x_\gamma \in S_\gamma$ satisfying

$$|\phi(e^{i\theta} x_\gamma)| < 1 - \delta \text{ for all } |\theta| \leq \theta_0.$$

Now, if we define $f : \bar{\Delta} \rightarrow \mathcal{C}$ by $f(z) = \phi(z.x_\gamma)$, the mean value property for analytic functions gives

$$|\phi(0)| = |f(0)| < 1 - \frac{\delta\theta_0}{\pi}$$

since

$$|f(e^{i\theta})| = |\phi(e^{i\theta} x_\gamma)| < 1 - \delta \text{ for all } |\theta| \leq \theta_0.$$

Therefore

$$\sup_{\gamma \in \Gamma} \sup_{k \in \mathbb{N}} C_k(S_\gamma, \delta) \leq 1 - \frac{\delta\theta_0}{\pi} < 1.$$

□

COROLLARY 1.5. *Let V be a set of finite vectors of B_p which is dense in some boundary of $A_b(B_p)$. Assume that $S \subset B_p$ and assume that there is some $0 \leq r < 1$ such that given any $v \in V$ and $k \in \mathbb{N}$ there exists some $y \in G_p$ satisfying $\|y\|_{p+k} \leq r$, $\text{supp}(v) \cap \text{supp}(y) = \emptyset$ and $v + y \in S$. Then S is a boundary for $A_b(B_p)$.*

Proof. Use Theorem 1.2 and Lemma 1.3. □

COROLLARY 1.6. *Let V be a set of finite vectors of B_p which is dense in some boundary of $A_b(B_p)$. Assume that $S \subset B_p$ and assume that there is some $\theta_0 > 0$ such that given any $v \in V$ and $k \in \mathbb{N}$ there exists some $y \in B_{p+k}$ satisfying $\text{supp}(v) \cap \text{supp}(y) = \emptyset$ and $v + e^{i\theta} y \in S$ for all $|\theta| \leq \theta_0$. Then S is a boundary for $A_b(B_p)$.*

Proof. Use Theorem 1.2 and Lemma 1.4. □

2. Examples

The results proved in Section 1 allow us to give examples of boundaries for $A_b(B_p)$ different of that given by Proposition 1.1.

EXAMPLE 2.1. Let $\theta_0 > 0$. The set

$$H = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \{x = (x_t)_{t=1}^{\infty} \in B_p : x_n = \frac{e^{i\theta}}{p+j-1}, |\theta| \leq \theta_0\}$$

is a boundary for $A_b(B_p)$.

Proof. We are going to use corollary 1.6. For each $j \in \mathbb{N}$ and $1 \leq n \leq j$ let

$$J_{n,j} = \{k \in \mathbb{N} : k \neq n \text{ and } 1 \leq k \leq j\}$$

and let $T_p^{n,j}$ denote the torus in G_p associated to a $J_{n,j}$. By theorem 1.1 we get that $V = \cup_{j=1}^{\infty} \cup_{n=1}^j T_p^{n,j}$ is a boundary for $A_b(B_p)$ and it is clear that V is a set of finite vectors. For each $v \in V$ we have $v \in T_p^{n,j}$ for some $j \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $1 \leq n \leq j$. For each $k \in \mathbb{N}$, let $y_k = (y_s^k)_{s=1}^{\infty}$ where $y_s^k = 0$ for all $s \neq n$ and $y_n^k = \frac{1}{p+k}$. It is clear that $y_k \in B_{p+k}$, $\text{supp}(v) \cap \text{supp}(y_k) = \emptyset$ and $v + e^{i\theta}y_k \in H$ for all $|\theta| \leq \theta_0$. Therefore by Corollary 1.6 we have that H is a boundary for $A_b(B_p)$. \square

EXAMPLE 2.2. Let $0 \leq r < 1$. The set

$$H = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{x \in B_p : x_n = \frac{r}{p+k-1}\}$$

is a boundary for $A_b(B_p)$.

Proof. Let V and $T_p^{n,j}$ as in the proof of the Example 2.1. For each $j \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $1 \leq n \leq j$, let $v \in T_p^{n,j}$. Given $k \in \mathbb{N}$, let $y = (y_s)_{s=1}^{\infty}$ where $y_s = 0$ for all $s \neq n$ and $y_n = \frac{r}{p+k}$. It is clear that $\|y\|_{p+k} \leq r$, $\text{supp}(v) \cap \text{supp}(y) = \emptyset$ and $v + y \in H$. By Corollary 1.5 we have that H is a boundary for $A_b(B_p)$. \square

EXAMPLE 2.3. For each $\theta_0 > 0$, and $m \in \mathbb{N} \cup \{0\}$ fixed, the set

$$H_m = \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \{x = (x_j)_{j=1}^{\infty} \in B_p : (x_j)_{j=1}^{n-1} \in T_p^{n-1},$$

$$x_n = \frac{e^{i\theta}}{p+m+k-1}, \quad |\theta| \leq \theta_0, x_j = 0 \text{ for all } j > n\},$$

where $T_p^{n-1} = T_p^{\{1, \dots, n-1\}}$

is a boundary for $A_b(B_p)$.

Proof. Let $V = \cup_{n=1}^{\infty} T_p^n$. By Proposition 1.1, V is a boundary for $A_b(B_p)$ and it is clear that V is a set of finite vectors. If $v \in V$, then $v \in T_p^n$ for some $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, we take $y = (y_s)_{s=1}^{\infty}$ where $y_s = 0$ for all $s \neq n+1$ and $y_{n+1} = \frac{1}{p+m+k+n}$. It is clear that

$$\|y\|_{p+k} \leq 1, \text{supp}(v) \cap \text{supp}(y) = \emptyset \text{ and } v + e^{i\theta}y \in H_m$$

for all $|\theta| \leq \theta_0$. By corollary 1.6 we have that H_m is a boundary for $A_b(B_p)$. \square

EXAMPLE 2.4. For each $0 \leq r < 1$ and $m \in \mathbb{N}$ fixed, the set

$$H_m = \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \{x = (x_j)_{j=1}^{\infty} \in B_p : (x_j)_{j=1}^{n-1} \in T_p^{n-1}, \\ x_n = \frac{r}{p+m+k-1}, x_j = 0 \text{ for all } j > n\}$$

is a boundary for $A_b(B_p)$.

Proof. Let $y = (y_s)_{s=1}^{\infty}$ be defined by $y_s = 0$ for all $s \neq n+1$ and $y_{n+1} = \frac{r}{p+m+k+n}$. Apply Corollary 1.5 as in Example 2.3. \square

EXAMPLE 2.5. For each $\theta_0 > 0$ and $m \in \mathbb{N} \cup \{0\}$ fixed, the set

$$H_m = \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \{x = (x_j)_{j=1}^{\infty} \in B_p : (x_j)_{j=1}^{n-1} \in T_p^{n-1}, \\ x_{n+m} = \frac{e^{i\theta}}{p+m+k-1} \quad |\theta| \leq \theta_0, x_j = 0 \\ \text{for all } j > n, j \neq n+m\}$$

is a boundary for $A_b(B_p)$.

Proof. Let V as in the proof of Example 2.3. If $v \in T_p^n$ and $k \in \mathbb{N}$, take $y = (y_s)_{s=1}^{\infty}$ where $y_s = 0$ for all $s \neq n+m$ and $y_{n+m} = \frac{1}{p+m+k+n}$. It is clear that

$\|y\|_{p+k} \leq 1$, $\text{supp}(v) \cap \text{supp}(y) = \emptyset$ and $v + e^{i\theta}y \in H_m$ for all $|\theta| \leq \theta_0$. Therefore by Corollary 1.6 H_m is a boundary for $A_b(B_p)$. \square

EXAMPLE 2.6. For each $0 \leq r < 1$ and $m \in \mathbb{N}$ fixed, the set

$$H_m = \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \{x = (x_j)_{j=1}^{\infty} \in B_p : (x_j)_{j=1}^{n-1} \in T_p^{n-1}, \\ x_{n+m} = \frac{r}{p+m+k-1}, x_j = 0 \text{ for all } j > n, j \neq n+m\}$$

is a boundary for $A_b(B_p)$.

Proof. Let V as in the proof of Example 2.3. If $v \in T_p^n$ and $k \in \mathbb{N}$, take $y = (y_s)_{s=1}^{\infty}$, where $y_s = 0$ for all $s \neq n+m$ and $y_{n+m} = \frac{r}{p+m+k+n}$. It is clear that

$$\|y\|_{p+k} \leq r, \quad \text{supp}(v) \cap \text{supp}(y) = \emptyset \quad \text{and} \quad x + y \in H_m.$$

By corollary 1.5 H_m is a boundary for $A_b(B_p)$. □

REMARK 2.1. For each $m \in \mathbb{N}$, let U_m be a countable dense subset of T_p^m . We can replace T_p^{n-1} by U_{n-1} for each $n \in \mathbb{N}$ in Examples 2.3 to 2.6 and get examples of countable boundaries for $A_b(B_p)$.

REMARK 2.2. In Example 2.3 (equivalently Example 2.4, 2.5 and 2.6), $\{H_m\}_{m \in \mathbb{N}}$ gives a disjoint family of closed boundaries for $A_b(B_p)$ and therefore the Shilov boundary for this algebra doesn't exist.

Next, we give an example of a set that is a null set for some $f \in A_b(B_p)$ and is a boundary for $A_u(B_p)$.

EXAMPLE 2.7. The set

$$S = \bigcup_{n=1}^{\infty} \left\{ x = (x_n)_{n=1}^{\infty} \in B_p : x_{n+1} = \frac{1}{p+n} \right\}$$

is a boundary for $A_u(B_p)$ and is a null set for some non-zero $f \in A_b(B_p)$.

Proof. By Proposition 3.5 of [8] S is a boundary for $A_u(B_p)$. Choose an increasing sequence $(k_n)_{n=1}^{\infty}$ so that

$$(1) \quad |1 - (1 - z)^{\frac{1}{k_n}}| \quad (|z| < \frac{1}{2}, n \in \mathbb{N})$$

and

$$(2) \quad \frac{\log 2}{k_n} \leq \frac{1}{2^n}.$$

Define, for each $x = (x_n)_{n=1}^{\infty} \in B_p$,

$$(3) \quad f(x) = \prod_{n=2}^{\infty} \left(1 - \frac{x_n}{p+n-1} \right)^{\frac{1}{k_n}}.$$

It is easy to check that the factors in the above product belong to $A_u(B_p)$. Let $x \in B_p$ and consider $U(x) = \{y \in B_p : \|x - y\|_p < \frac{1}{4}\}$. There exists some $n_0 \in \mathbb{N}$ such that $|x_n| < \frac{1}{4}$ for all $n \geq n_0$. So, given any $y = (y_n)_{n=1}^{\infty} \in U(x)$, we have

$$\frac{|y_n|}{p+n-1} \leq \frac{|y_n - x_n|}{p} + |x_n| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad \text{for all } n \geq n_0.$$

So, by Theorem 15.4 of [10] it follows that the product (3) converges uniformly on $U(x)$ and $f \neq 0$. Since x is arbitrary it follows that f is continuous on B_p and analytic in the interior of B_p . Further, given $x = (x_n)_{n=1}^{\infty} \in B_p$ we have

$$\frac{|x_n|}{p+n-1} \leq \frac{1}{2}$$

for all $n \geq 2$. This follows from $|x_n/n| \leq 1/n$ in case where $p = 1$ and from

$$|x_n/(p + n - 1)| \leq |x_n| \leq 1/p$$

in case where $p \geq 2$. Consequently,

$$\frac{1}{2} \leq 1 - \frac{|x_n|}{p + n - 1} \leq |1 - \frac{x_n}{p + n - 1}| \leq 1 + \frac{1}{2} \leq 2,$$

and this implies $\log |1 - \frac{x_n}{p+n-1}| \leq \log 2$ for all $n \geq 2$ (where \log denotes the principal branch of logarithm). Now, by (2) we get

$$\begin{aligned} & \left| \prod_{n=2}^m \left(1 - \frac{x_n}{p + n - 1}\right)^{\frac{1}{k_n}} \right| = \prod_{n=2}^m \left| \exp \left[\log \left(1 - \frac{x_n}{p + n - 1}\right)^{\frac{1}{k_n}} \right] \right| \\ &= \prod_{n=2}^m \exp \left\{ \operatorname{Re} \left[\frac{1}{k_n} \log \left(1 - \frac{x_n}{p + n - 1}\right) \right] \right\} \\ &= \prod_{n=2}^m \exp \left(\frac{1}{k_n} \log \left|1 - \frac{x_n}{p + n - 1}\right| \right) \\ &\leq \prod_{n=2}^m \exp \left(\frac{1}{k_n} \log 2 \right) \leq \prod_{n=2}^m \exp \frac{1}{2^n} \leq \exp \sum_{n=2}^m \frac{1}{2^n} \end{aligned}$$

for all $m \geq 2$ and for all $x \in B_p$. So, $f \in A_b(B_p)$. □

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