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MAXIMUM MODULI OF UNIMODULAR POLYNOMIALS

Andreas Defant, Domingo García and Manuel Maestre

Abstract. Let $\sum_{|\alpha|=m} s_\alpha z^\alpha$, $z \in \mathbb{C}^n$ be a unimodular m-homogeneous polynomial in n variables (i.e. $|s_\alpha| = 1$ for all multi indices α), and let $R \subset \mathbb{C}^n$ be a (bounded complete) Reinhardt domain. We give lower bounds for the maximum modulus $\sup_{z\in R} |\sum_{|\alpha|=m} s_\alpha z^\alpha|$, and upper estimates for the average of these maximum moduli taken over all possible m-homogeneous Bernoulli polynomials (i.e. $s_\alpha = \pm 1$ for all multi indices α). Examples show that for a fixed degree m our estimates, for rather large classes of domains R, are asymptotically optimal in the dimension n.

0. Introduction

In his study [5] of Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ Harald Bohr considered vertical strips of uniform, but not absolute convergence of such a series. More precisely, he considered those nonnegative numbers A and B for which a given Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges conditionally (i.e., absolutely) in the half plane $\{s \in \mathbb{C} \ : \ \mathbf{Re}\, s > A\}$, but converges uniformly on the half planes $\{s \in \mathbb{C} \ : \ \mathbf{Re}\, s \geq B + \varepsilon\}$, $\varepsilon > 0$. Bohr asked for the maximal possible width $d := B - A$ of such a strip, and proved that $d \leq \frac{1}{2}$. Bohnenblust and Hille in [6] were able to show that $d = \frac{1}{2}$. In the context of nuclearity and absolute bases for spaces of holomorphic functions over infinite dimensional domains, Dineen and Timoney in [14] gave a new proof based on probabilistic methods. Finally, Boas in a beautiful paper [2] produced a proof in which one of the key ingredients is the Kahane-Salem-Zygmund Theorem (see

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[18, Theorem 4, Chap. 6, pp.70–71]). This probabilistic result, for each degree m and each dimension n, assures the existence of an m-homogenous Bernoulli polynomial $P(z) = \sum_{|\alpha|=m} s_\alpha z^\alpha$ on $\mathbb{C}^n$ for which $\sup\{|P(z)| : |z_k| < 1, k = 1, \ldots, n\} \leq C n^{\frac{m+1}{2}} \sqrt{\log m}$, where C depends neither on n nor on m. A similar argument was used by Boas and Khavinson [3, proof of Theorem 2] in their study of Bohr's power series theorem in several variables (see also [1], [9] and [13]).

The main goal of this paper is to give, in a systematic way, upper and lower estimates for the maximum modulus $\sup_{z\in R} |\sum_{|\alpha|=m} s_\alpha z^\alpha|$, where $\sum_{|\alpha|=m} s_\alpha z^\alpha$ is an arbitrarily given m-homogeneous Bernoulli polynomial in n variables (i.e., a polynomial of the form $\sum_{|\alpha|=m} s_\alpha z^\alpha$, where the coefficients s_α for all α are signs) and R some Reinhardt domain in $\mathbb{C}^n$. In section 2 we obtain for every unimodular m-homogeneous polynomial (i.e., every polynomial of the form $\sum_{|\alpha|=m} s_\alpha z^\alpha$, $z \in \mathbb{C}^n$, where $s_\alpha \in \mathbb{C}$ satisfies $|s_\alpha| = 1$ for all multi indices α) and every Reinhardt domain R lower bounds for the maximum modulus $\sup_{z\in R} |\sum_{|\alpha|=m} s_\alpha z^\alpha|$. In section 3 we give upper estimates for the average of such maximum moduli taken over all possible m-homogeneous Bernoulli polynomials. As a consequence we prove for certain classes of domains the existence of m-homogeneous Bernoulli polynomials which have maximum moduli as small as possible. Moreover, we apply our results to various concrete classes of domains R in $\mathbb{C}^n$.

1. Preliminaries

We shall use standard notation and notions from Banach space theory as presented e.g. in [22] or [28]. If X is a Banach space over the scalars $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then X^* is its topological dual and B_X is its open unit ball. We denote the Banach-Mazur distance of two Banach spaces X and Y by $d(X, Y)$.

For all needed background on polynomials defined on Banach spaces we refer to [12] and [16]. $\mathcal{P}(^m X)$ denotes the space of all m-homogeneous scalar valued polynomials P on a Banach space X, which together with the norm $\|P\| := \sup\{|P(x)| : \|x\| \leq 1\}$ forms a Banach space. A subset U of $\mathbb{C}^n$ is called circled if $\lambda x \in U$ for all $\lambda \in \mathbb{C}$, $|\lambda| = 1$. A complete bounded Reinhardt domain $R \subset \mathbb{C}^n$ is a bounded domain that is complete and n-circled, i.e. if $x = z_1 e_1 + \cdots + z_n e_n \in R$, then $\lambda(e^{\theta_1 i} z_1 e_1 + \cdots + e^{\theta_n i} z_n e_n) \in R$ for all $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$ and all $\theta_1, \ldots, \theta_n \in \mathbb{R}$. A basis (x_n) of a Banach space X is said to be unconditional if there is a constant $C \geq 1$ such that for all $\mu_1, \dots, \mu_n \in \mathbb{K}$ and all $s_1, \dots, s_n \in \mathbb{K}$ with $|s_k| \leq 1$, we have $\|\sum_{k=1}^n s_k \mu_k x_k\| \leq C \|\sum_{k=1}^n \mu_k x_k\|$; in this case the best constant is denoted by $\chi((x_n))$, the unconditional basis constant of (x_n). All unconditional bases considered in this paper are normalized. For $X = (\mathbb{K}^n, \|.\|)$ denote by $\chi_{mon}(\mathcal{P}(^m X))$ the unconditional basis constant of the monomials $\{z^\alpha : |\alpha| = m\}$.

We call a Banach space $X = (\mathbb{K}^n, \|.\|)$ symmetric if the canonical vectors e_k form a symmetric basis, i.e. $\|x\| = \|\sum_{k=1}^n s_k x_{\pi(k)} e_k\|$ for each $x \in X$, each permutation π of $\{1, \dots, n\}$ and each choice of scalars s_k with $|s_k| = 1$. A Banach space X for which $\ell_1 \subset X \subset c_0$ (with norm 1 inclusions) is said to be a Banach sequence space if the canonical sequence (e_k) forms a 1-unconditional basis; it is called symmetric if $\|x\| = \|\sum_{k=1}^\infty s_k x_{\pi(k)} e_k\|$ for each $x \in X$, each permutation π of $\mathbb{N}$ and each choice of scalars s_k with $|s_k| = 1$. Recall finally that a Banach lattice X is said to be 2-concave and 2-convex, respectively, if there is a constant $C > 0$ such that $\left(\sum_{k=1}^n \|x_k\|^2\right)^{1/2} \leq C \|\left(\sum_{k=1}^n |x_k|^2\right)^{1/2}\|$, and $\|\left(\sum_{k=1}^n |x_k|^2\right)^{1/2}\| \leq C \left(\sum_{k=1}^n \|x_k\|^2\right)^{1/2}$ for all $x_1, \dots, x_n \in X$, respectively. Here the best constant is denoted by $M_{(2)}(X)$ and $M^{(2)}(X)$, respectively (see [22]). For the notion of cotype q, $2 \leq q < \infty$, of a Banach space X (the cotype constant is denoted by $C_q(X)$) and its relation with convexity and concavity we also refer to [22].

2. Lower bounds for the maximum modulus of unimodular homogeneous polynomials on Reinhardt domains

Fix a degree m and a dimension n. We call an m-homogeneous polynomial

$$\sum_{|\alpha|=m} s_\alpha z^\alpha, \; z \in \mathbb{C}^n \tag{2.1}$$

unimodular whenever all coefficients $s_\alpha \in \mathbb{C}$ satisfy $|s_\alpha| = 1$. If all $s_\alpha \in \mathbb{C}$ are signs $+1$ and -1, then we speak of an m-homogeneous Bernoulli polynomial. Obviously,

$$\sup_{z \in B_{\ell_\infty^n}} \Big| \sum_{|\alpha|=m} s_\alpha z^\alpha \Big| \leq \sum_{|\alpha|=m} |s_\alpha| = \binom{n+m-1}{m}.$$

Conversely, since the monomials z^α, $|\alpha| = m$, form an orthogonal system of square integrable functions on the n-dimensional torus T_n in $\mathbb{C}^n$ (the

n-th cartesian product of $S_{\ell_2^1}$ endowed with the product measure of the normalized Lebesgue measure σ_1), we get

$$(2.2)\qquad \begin{aligned}\binom{n+m-1}{m}^{1/2} &= \Big(\int_{T_n} |\sum_{|\alpha|=m} s_\alpha z^\alpha|^2 dz\Big)^{1/2} \\ &\leq \sup_{z\in B_{\ell_\infty^n}} |\sum_{|\alpha|=m} s_\alpha z^\alpha| \leq \binom{n+m-1}{m}.\end{aligned}$$

Hence, the obvious estimate $\frac{1}{m!}n^m \leq \binom{n+m-1}{m} \leq n^m$ gives

$$(2.3)\qquad \frac{1}{\sqrt{m!}} n^{m/2} \leq \sup_{z\in B_{\ell_\infty^n}} |\sum_{|\alpha|=m} s_\alpha z^\alpha| \leq n^m.$$

Let us improve the lower bound. We want to show that there is a constant $c_m > 0$ such that for each unimodular m-homogeneous polynomial as in (2.1)

$$(2.4)\qquad c_m n^{\frac{m+1}{2}} \leq \sup_{z\in B_{\ell_\infty^n}} |\sum_{|\alpha|=m} s_\alpha z^\alpha|.$$

We are going to obtain this lower estimate as a consequence of the following more general result.

Proposition 2.1. *Let $X = (\mathbb{C}^n, \|.\|)$ be a Banach space such that the canonical basis $(e_k)_{k=1}^n$ is 1-unconditional. Then for each m*

$$(2.5)\qquad \frac{1}{m!}\frac{\big(\sup_{z\in B_X}\sum_{k=1}^n |z_k|\big)^m}{\chi_{mon}(\mathcal{P}(^mX))} \leq \sup_{z\in B_X} |\sum_{|\alpha|=m} s_\alpha z^\alpha|.$$

Proof. For each unimodular m-homogeneous polynomial

$$\sum_{|\alpha|=m} s_\alpha z^\alpha,\ z\in\mathbb{C}^n$$

we have

$$\begin{aligned}\Big(\sup_{z\in B_X}\sum_{k=1}^n |z_k|\Big)^m &= \Big(\sup_{z\in B_X} |\sum_{k=1}^n z_k|\Big)^m \\ &= \sup_{z\in B_X} |\sum_{|\alpha|=m} \frac{m!}{\alpha_1!\cdots\alpha_n!} z^\alpha|\end{aligned}$$

$$= \sup_{z\in B_X} |\sum_{|\alpha|=m} \frac{m!}{\alpha_1!\cdots\alpha_n! s_\alpha} s_\alpha z^\alpha|$$
$$\leq m!\chi_{mon}(\mathcal{P}(^mX)) \sup_{z\in B_X} |\sum_{|\alpha|=m} s_\alpha z^\alpha|.$$

□

By [8, Theorem 3] we know that for each m there is a constant $d_m > 0$ such that

$$\chi_{mon}(\mathcal{P}(^m\ell_\infty^n)) \leq d_m n^{\frac{m-1}{2}}. \tag{2.6}$$

As $n^m = \big(\sup_{z\in B_{\ell_\infty^n}} \sum_{k=1}^n |z_k|\big)^m$, inequality (2.6) together with Proposition 2.1 gives as desired (2.4) with $c_m := (m!d_m)^{-1}$. In contrast to (2.6) we know that $\chi_{mon}(\mathcal{P}(^m\ell_1^n))$ for fixed degree m is uniformly bounded from above in the dimension n (see [9, (4.6)]), more precisely

$$\chi_{mon}(\mathcal{P}(^m\ell_1^n)) \leq \frac{m^m}{m!}.$$

Hence by Proposition 2.1 we have

$$\frac{1}{m^m} \leq \sup_{z\in B_{\ell_1^n}} |\sum_{|\alpha|=m} s_\alpha z^\alpha|, \tag{2.7}$$

i.e., for each fixed degree m the maximum modulus of a unimodular m-homogeneous polynomial on $B_{\ell_1^n}$ is uniformly bounded from below in n.

More generally, we now prove a theorem which includes (2.4) and (2.7) as particular cases. We will need the following lemma which is related to [3, Theorem 3] and [13, Theorem 3.2].

LEMMA 2.2. *Given $r_1, \ldots, r_n > 0$, consider on $\mathbb{C}^n$ the norm $\|z\| := \sup\{|\frac{z_k}{r_k}| : k = 1, \ldots, n\}$ and denote the associated Banach space by $\ell_\infty^n(r_1, \ldots, r_n)$. Then*

$$\chi_{mon}(\mathcal{P}(^m\ell_\infty^n(r_1, \ldots, r_n))) \leq d_m n^{\frac{m-1}{2}},$$

where the constant $d_m > 0$ is the one from (2.6).

Proof. Clearly the open unit ball of $\ell_\infty^n(r_1, \ldots, r_n)$ is the polydisc $\{z \in \mathbb{C}^n : |z_k| < r_k,\ k = 1, \ldots, n\}$. Let $\sum_{|\alpha|=m} c_\alpha z^\alpha$ be an m-homogeneous polynomial on $\mathbb{C}^n$ such that

$$|\sum_{|\alpha|=m} c_\alpha z^\alpha| \leq 1, \text{ for all } z \in B_{\ell_\infty^n(r_1,\ldots,r_n)}.$$

Then

$$|\sum_{|\alpha|=m} c_\alpha (r_k z_k)^\alpha| \leq 1, \text{ for all } z \in B_{\ell_\infty^n},$$

which implies

$$|\sum_{|\alpha|=m} c_\alpha r^\alpha z^\alpha| \leq 1, \text{ for all } z \in B_{\ell_\infty^n}.$$

Hence, we obtain by (2.6) that

$$|\sum_{|\alpha|=m} |c_\alpha| r^\alpha z^\alpha| \leq d_m n^{\frac{m-1}{2}}, \text{ for all } z \in B_{\ell_\infty^n},$$

which finally gives as desired that

$$|\sum_{|\alpha|=m} |c_\alpha| z^\alpha| \leq d_m n^{\frac{m-1}{2}}, \text{ for all } z \in B_{\ell_\infty^n(r_1,\dots,r_n)}.$$

□

The following result is our main lower bound for maximum moduli of unimodular m-homogeneous polynomials.

THEOREM 2.3. *For each m there is a constant $c_m > 0$ such that for each unimodular m-homogeneous polynomial $\sum_{|\alpha|=m} s_\alpha z^\alpha$, $z \in \mathbb{C}^n$ the following estimates hold.*

(1) *For each Reinhardt domain $R \subset \mathbb{C}^n$*

$$c_m \frac{\left(\sup_{z\in R} \sum_{k=1}^n |z_k|\right)^m}{n^{\frac{m-1}{2}}} \leq \sup_{z\in R} |\sum_{|\alpha|=m} s_\alpha z^\alpha|.$$

(2) *For each Banach space $X := (\mathbb{C}^n, \|.\|)$ for which the e_k's form a 1-unconditional basis*

$$c_m \frac{\left(\sup_{z\in B_X} \sum_{k=1}^n |z_k|\right)^m}{d(X, \ell_1^n)^{m-1}} \leq \sup_{z\in B_X} |\sum_{|\alpha|=m} s_\alpha z^\alpha|.$$

(3) *If, additionally, $B_X \subset B_{\ell_2^n}$ we obtain*

$$c_m \sup_{z\in B_X} \sum_{k=1}^n |z_k| \leq \sup_{z\in B_X} |\sum_{|\alpha|=m} s_\alpha z^\alpha|.$$

Proof. Each Reinhardt domain R is, by definition, a union of open polydiscs. Given $u \in R$, we take $r_1, \dots, r_n > 0$ such that $u \in \{z \in \mathbb{C}^n :$

$|z_k| < r_k,\ k = 1, \dots, n\} \subset R$. By Proposition 2.1 and Lemma 2.2 there is a constant $c_m > 0$ for which

$$\begin{aligned} & c_m \frac{(\sum_{k=1}^n |u_k|)^m}{n^{\frac{m-1}{2}}} \\ & \le c_m \frac{\left(\sup_{z \in B_{\ell_\infty^n(r_1,\dots,r_n)}} \sum_{k=1}^n |z_k|\right)^m}{n^{\frac{m-1}{2}}} \\ & \le \sup_{z \in B_{\ell_\infty^n(r_1,\dots,r_n)}} |\sum_{|\alpha|=m} s_\alpha z^\alpha| \le \sup_{z \in R} |\sum_{|\alpha|=m} s_\alpha z^\alpha|, \end{aligned}$$

and hence (1) is proved. Statement (2) follows from Proposition 2.1 and the fact that by [9, Theorem 6.1] there exists a constant $d_m > 0$ such that

$$\chi_{mon}(\mathcal{P}(^m X)) \le d_m d(X, \ell_1^n)^{m-1}; \tag{2.8}$$

now it is enough to take $c_m := (m! d_m)^{-1}$. The proof of (3) is a consequence of the fact that by [26, Corollary 2] and [9, (5.2)]:

$$\frac{1}{\sqrt{2}} \sup_{z \in B_X} \sum_{k=1}^n |z_k| \le d(X, \ell_1^n) \le \sup_{z \in B_X} \sum_{k=1}^n |z_k|.$$

This completes the proof. □

Let us collect some concrete examples in order to illustrate our results.We start with estimates for mixed Minkowski spaces

$$\begin{aligned} \ell_p^u(\ell_q^v) &:= \{(x_k)_{k=1}^u : x_1, \dots, x_u \in \mathbb{C}^v\} \\ \|(x_k)_{k=1}^u\|_{p,q} &:= (\sum_{k=1}^u \|x_k\|_q^p)^{1/p}. \end{aligned}$$

EXAMPLE 2.4. Let $\sum_{|\alpha|=m} s_\alpha z^\alpha,\ z \in \mathbb{C}^u(\mathbb{C}^v)$ be a unimodular m-homogeneous polynomial in uv variables (for some order of these variables). Then

$$\sup_{z \in B_{\ell_p^u(\ell_q^v)}} |\sum_{|\alpha|=m} s_\alpha z^\alpha| \ge C \begin{cases} u^{\frac{m+1}{2} - \frac{m}{p}} v^{\frac{m+1}{2} - \frac{m}{q}} & \text{if } \ 2 \le p, q \le \infty, \\ u^{1-\frac{1}{p}} v^{1-\frac{1}{q}} & \text{if } \ 1 \le p, q \le 2, \\ u^{1-\frac{1}{p}} v^{\frac{m+1}{2} - \frac{m}{q}} & \text{if } \ 1 \le p \le 2 \le q \le \infty, \end{cases}$$

$C > 0$ some constant depending only on m, p and q.

Let us remark that, by taking $p = q$ and $v = 1$ in this example, we obtain lower estimates for every unimodular m-homogeneous polynomials on ℓ_p^n, $1 \leq p \leq \infty$:

$$\sup_{z \in B_{\ell_p^n}} |\sum_{|\alpha|=m} s_\alpha z^\alpha| \geq C \begin{cases} n^{\frac{m+1}{2} - \frac{m}{p}} & \text{if} \quad 2 \leq p \leq \infty, \\ n^{1-\frac{1}{p}} & \text{if} \quad 1 \leq p \leq 2. \end{cases} \tag{2.9}$$

The results of the next section will show in which sense these estimates are optimal. Our proof of Example 2.4 needs the following

LEMMA 2.5. *In the three cases $1 \leq p, q \leq 2$ or $2 \leq p, q \leq \infty$ or $1 \leq p \leq 2 \leq q \leq \infty$ we have*

$$d(\ell_p^u(\ell_q^v), \ell_1^{uv}) \leq d(\ell_p^u, \ell_1^u) d(\ell_q^v, \ell_1^v).$$

Proof. For $1 \leq p, q \leq 2$ we know from [26, Corollary 2] that

$$d(\ell_p^u(\ell_q^v), \ell_1^{uv}) \leq \| \sum_{k,l} e_k \otimes e_l \|_{\ell_{p'}^u(\ell_{q'}^v)} = u^{1-\frac{1}{p}} v^{1-\frac{1}{q}} = d(\ell_p^u, \ell_1^u) d(\ell_q^v, \ell_1^v);$$

see e.g. [28, 37.6]. If $2 \leq p, q \leq \infty$, then $M^{(2)}(\ell_p^u(\ell_q^v)) = 1$, and hence it follows from [28, Corollary 41.9] that as desired

$$d(\ell_p^u(\ell_q^v), \ell_1^{uv}) \leq C\sqrt{uv},$$

$C \geq 0$ some absolute constant. Finally, the remaining case: Note first that for every linear bijection $T : \ell_q^v \longrightarrow \ell_1^v$

$$d(\ell_1^u(\ell_q^v), \ell_1^{uv}) = d(\ell_1^u \otimes_\pi \ell_q^v, \ell_1^u \otimes_\pi \ell_1^v) \leq \|id \otimes T\| \|id \otimes T^{-1}\| = \|T\| \|T^{-1}\|,$$

hence $d(\ell_1^u(\ell_q^v), \ell_1^{uv}) \leq d(\ell_q^v, \ell_1^v)$; for $1 \leq p \leq 2 \leq q \leq \infty$ now by factorization

$$\begin{aligned} d(\ell_p^u(\ell_q^v), \ell_1^{uv}) &\leq d(\ell_p^u(\ell_q^v), \ell_1^u(\ell_q^v)) d(\ell_1^u(\ell_q^v), \ell_1^{uv}) \\ &\leq \|id : \ell_p^u(\ell_q^v) \longrightarrow \ell_1^u(\ell_q^v)\| d(\ell_q^v, \ell_1^v) \\ &= u^{1-1/p} d(\ell_q^v, \ell_1^v) \leq d(\ell_p^u, \ell_1^u) d(\ell_q^v, \ell_1^v). \end{aligned}$$

□

For the remaining case $1 \leq q \leq 2 \leq p \leq \infty$ the estimate from Lemma 2.5 is false; by a result of Kwapien and Schütt from [20, Corollary 3.3] we, e.g., see that $d(\ell_\infty^n(\ell_1^n)), \ell_1^{n^2}) \overset{n}{\asymp} n$ (and not, as one could expect from the estimate of the lemma, $\sqrt{n}$). We believe that at least in this special case the optimal lower bound for $\sup_{z \in B_{\ell_\infty^n(\ell_1^n))}} |\sum_{|\alpha|=m} s_\alpha z^\alpha|$ comes from Theorem 2.3(1), namely $n^{\frac{m+1}{2}}$.

Proof of the Example 2.4. All three cases are consequences of Theorem 2.3. We know that

$$\sup_{z\in B_{\ell_p^u(\ell_q^v)}} \sum_{k,l} | < z, e_k \otimes e_l > | = \|id : \ell_p^u(\ell_q^v) \longrightarrow \ell_1^{uv}\| = u^{1-1/p} v^{1-1/q}.$$

Hence the first case is a consequence of Theorem 2.3(1), the second one of 2.3(3), and the last of 2.3(2) combined with the preceding lemma. □

The next example deals with Orlicz spaces ℓ_φ, and is a considerable extension of (2.9) (take $\varphi(t) = t^p$).

EXAMPLE 2.6. Let φ be an Orlicz function satisfying the Δ_2-condition. Then for each m there is a constant $c_m > 0$ such that for each unimodular m-homogeneous polynomial $\sum_{|\alpha|=m} s_\alpha z^\alpha$ in n variables, the following estimates hold.

(1) $c_m n^{\frac{m+1}{2}} \varphi^{-1}(1/n)^m \le \sup_{z\in B_{\ell_\varphi}} |\sum_{|\alpha|=m} s_\alpha z^\alpha|$.

(2) $c_m n \varphi^{-1}(1/n) \le \sup_{z\in B_{\ell_\varphi}} |\sum_{|\alpha|=m} s_\alpha z^\alpha|$, provided that $t^2 \le K\varphi(t)$ for all t and some K.

Proof. The proof is again a simple consequence of Theorem 2.3, the observations (3.5) and (3.6) (anticipated from section 3), combined with the well-known equality $\|\sum_{k=1}^n e_k\|_{\ell_\varphi} = \frac{1}{\varphi^{-1}(1/n)}$ for the fundamental function; the condition in (2) assures that $\ell_\varphi \subset \ell_2$. □

We finish this section with an example for a non-convex domain.

EXAMPLE 2.7. Given $S > 1$ and $n \ge 2$ consider the Reinhardt domain $R := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1 \cdots z_n| < 1, |z_k| < S, k = 1, \ldots, n\}$. Then for each m there exists a constant $h_m > 0$ such that for all unimodular m-homogeneous polynomials $\sum_{|\alpha|=m} s_\alpha z^\alpha$ in n variables

$$h_m S^m n^{\frac{m+1}{2}} \le \sup_{z\in R} |\sum_{|\alpha|=m} s_\alpha z^\alpha|.$$

Proof. We have that

$$\sup_{z\in R} \sum_{k=1}^n |z_k| = (n-1)S + \frac{1}{S^{n-1}}; \tag{2.10}$$

indeed, if we take the compact set $K := \{x \in [0, S]^n : x_1 \cdots x_n \le r\}$ for $0 < r \le 1$, then the maximum of the function $f(x) := x_1 + \cdots + x_n$ on K is $(n-1)S + \frac{r}{S^{n-1}}$, attained at $(S, \ldots, S, \frac{r}{S^{n-1}})$ and at any point whose coordinates are a permutation of it. Let $x_0 \in K$ such that $f(x_0) = \sup_{x\in K} f(x)$. As f has no critical points, x_0 belongs to

∂K, the boundary of K. On the other hand x_0 does not belong to $\{x \in (0,S)^n : x_1 \cdots x_n = r\}$. In this case the Lagrange multiplier method gives only the critical point $(r^{1/n}, \ldots, r^{1/n})$. But the function $g : (0,\infty) \longrightarrow \mathbb{R}$ defined by $g(x) := (n-1)x + \frac{r}{x^{n-1}}$ attains a strict absolute minimum at $x = r^{1/n}$ and

$$\begin{aligned} f(r^{1/n}, \ldots, r^{1/n}) &= nr^{1/n} = g(r^{1/n}) \\ &< g(S) = (n-1)S + \frac{r}{S^{n-1}} = f(S, \ldots, S, \frac{r}{S^{n-1}}). \end{aligned}$$

If $x \in \partial K$ and for some k we have $x_k = 0$, then $f(x) \leq (n-1)S < f(S, \ldots, S, \frac{r}{S^{n-1}})$ and $x \neq x_0$. Finally, the only possibility left is that for some k we have $x_k = S$. Then, by the symmetry of the function f, we can assume $x_n = S$. By induction, we have

$$\sup\{(x_1, \ldots, x_{n-1}) \in [0,S]^{n-1} : x_1 \cdots x_{n-1} \leq rS^{-1}\} = (n-2)S + \frac{rS^{-1}}{S^{n-2}},$$

thus

$$\begin{aligned} & f(x_1, \ldots, x_{n-1}, S) \\ & \leq S + \sup\{(x_1, \ldots, x_{n-1}) \in [0,S]^{n-1} : x_1 \cdots x_{n-1} \leq rS^{-1}\} \\ & = (n-1)S + \frac{r}{S^{n-1}} = f(S, \ldots, S, \frac{r}{S^{n-1}}). \end{aligned}$$

Now, since $\sup_{z \in R} \sum_{k=1}^n |z_k|$ coincides with the maximum of f on $\{x \in [0,S]^n : x_1 \cdots x_n \leq 1\}$, we obtain the equality (2.10). Hence, by Theorem 2.3 (1) there exists a constant $c_m > 0$ such that

$$\begin{aligned} c_m \frac{1}{2^m} S^m n^{\frac{m+1}{2}} & \leq c_m \Big(\frac{n-1}{n}\Big)^m S^m n^{\frac{m+1}{2}} \leq c_m \frac{\big((n-1)S + \frac{1}{S^{n-1}}\big)^m}{n^{\frac{m-1}{2}}} \\ & \leq \sup_{z \in R} \Big| \sum_{|\alpha|=m} s_\alpha z^\alpha \Big|. \end{aligned}$$

The conclusion follows by taking $h_m = c_m \frac{1}{2^m}$. □

3. Expectations of the modulus maximum of homogeneous Bernoulli random polynomials on Reinhardt domains

The Kahane-Salem-Zygmund Theorem [18, Theorem 4, Chap. 6, pp.70–71] shows that for the polydisc typically the maximum modulus

$$\sup_{z \in B_{\ell_\infty^n}} \Big| \sum_{|\alpha|=m} s_\alpha z^\alpha \Big|$$

of an arbitrarily given unimodular m-homogeneous polynomial is as small as possible, namely $n^{\frac{m+1}{2}}$ (see (2.4)). To obtain polynomials on ℓ_p^n with "small" norms Boas in [1, Theorem 4] proves the existence of symmetric complex m-linear forms on $(\ell_p^n)^m$ with "small" norms. His proof, as explicitly stated, is made by a careful inspection of results due to Mantero-Tonge (see [23, Theorem 1.1] and also [24, Proposition 4]) which in turn were inspired by the Kahane-Salem-Zygmund Theorem. This section could be considered as an extension of the Mantero-Tonge results. It is worth to mention that [23, Theorem 1.1] was used by Dineen and Timoney in [13] and [14].

Let us clarify what we mean by "small" norm. Fix a degree m, a family $\varepsilon_\alpha : \Omega \to \{-1,1\}$, $|\alpha| = m$ of independent Bernoulli random variables on a probability space (Ω, μ) (each ε_α takes the values $+1$ and -1 with equal probability $1/2$), and $c_\alpha \in \mathbb{C}$. Then we call

$$\sum_{|\alpha|=m} c_\alpha \varepsilon_\alpha z^\alpha,\ z \in \mathbb{C}^n\,,$$

an m-homogeneous Bernoulli random polynomial. According to [18, Chapter 6, Theorem 3] there is an absolute constant $C > 0$ such that

$$\mu\Big(\sup_{z\in B_{\ell_\infty^n}} \Big|\sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha\Big| \geq C n^{\frac{m+1}{2}} (\log m)^{1/2}\Big) \leq \frac{1}{m^2 e^n}.$$

Hence with "high probability" the maximum modulus of every m-homogeneous Bernoulli polynomial $\sum_{|\alpha|=m} s_\alpha z^\alpha$, $z \in \mathbb{C}^n$, can be estimated from above by a constant times $(\log m)^{1/2} n^{\frac{m+1}{2}}$; in particular, there exists a set of signs s_α, $|\alpha| = m$ such that

$$\sup_{z\in B_{\ell_\infty^n}} \Big|\sum_{|\alpha|=m} s_\alpha z^\alpha\Big| \leq C(\log m)^{1/2} n^{\frac{m+1}{2}}.$$

To see from another point of view that (2.4) for fixed m is optimal in the dimension n, integrate (2.4) in order to obtain the following lower estimate for the expectation of the modulus maximum of an m-homogeneous Bernoulli random polynomial with respect to the polydisc $B_{\ell_\infty^n}$:

$$c_m n^{\frac{m+1}{2}} \leq \int_\Omega \sup_{z\in B_{\ell_\infty^n}} \Big|\sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha\Big| d\mu\,;$$

by [9, Corollary 6.5] we know that this result is optimal. More generally, given a Reinhardt domain R in $\mathbb{C}^n$ satisfying one of the assumptions of

Theorem 2.3, integration gives a lower bound for the averages

$$\int_\Omega \sup_{z\in R} |\sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha| d\mu\,,$$

and a modification of [9, Theorem 3.1] will show that for rich classes of domains R the lower bounds obtained in this way, are optimal.

Recall that a real valued random variable X on a probability space (Ω,μ) is said to be Gaussian whenever it has mean zero, is square integrable and its Fourier transform satisfies

$$\mathbb{E}e^{itX} = e^{\frac{-\|X\|_2^2}{2}t^2}, \text{ where } t\in\mathbb{R} \text{ and } \|X\|_2 = (\mathbb{E}X^2)^{\frac{1}{2}};$$

X is a standard Gaussian random variable if it is measurable and if for every Borel subset B of $\mathbb{R}$ we have

$$\mu\{\omega\in\Omega\ : g(\omega)\in B\} = \frac{1}{\sqrt{2\pi}}\int_B e^{-\frac{t^2}{2}}dt\,.$$

The following result is our main technical tool.

THEOREM 3.1. *Let U be a bounded circled set in $\mathbb{C}^n$, and $(g_\alpha)_{|\alpha|=m}$ and $(g_k)_{1\le k\le n}$ two families of independent standard Gaussian random variables on a probability space (Ω,μ). Then for each choice of scalars c_α, $|\alpha|=m$*

$$\int \sup_{z\in U} |\sum_{|\alpha|=m} c_\alpha g_\alpha z^\alpha| d\mu$$

$$\le C_m \sup_{|\alpha|=m}\left\{|c_\alpha|\sqrt{\frac{\alpha!}{m!}}\right\} \sup_{z\in U}(\sum_{k=1}^n |z_k|^2)^{\frac{m-1}{2}} \int \sup_{z\in U}|\sum_{k=1}^n g_k z_k| d\mu,$$

where $0\le C_m \le 2^{\frac{3}{2}m-\frac{1}{2}}m^{\frac{3}{2}}$.

Since the proof is a simple modification of [9, Theorem 3.1] we only sketch some relevant details. In fact, we prove a reformulation which needs some more notation. For natural numbers m,n we define $\mathcal{M}(m,n)$ $:= \{1,\dots,n\}^m$ and $\mathcal{J}(m,n) := \{\mathbf{j} = (j_1,\dots,j_m)\in\{1,\dots,n\}^m : j_1\le \cdots \le j_m\}$. Moreover, for $\mathbf{j}\in\mathcal{J}(m,n)$ let $|\mathbf{j}|$ be the cardinality of the set of all $\mathbf{i}\in\mathcal{M}(m,n)$ for which there is a permutation τ of $\{1,\dots,n\}$ such that $i_{\tau(k)} = j_k$ for $1\le k\le m$. Finally, if e_k^* denotes the kth coefficient functional on $\mathbb{C}^n$ and $\mathbf{j}\in\mathcal{J}(m,n)$, then we write $e_{\mathbf{j}}^*(z) := e_{j_1}^*(z)\cdots e_{j_m}^*(z)$, $z\in\mathbb{C}^n$. Now observe that if $\alpha\in(\mathbb{N}\cup\{0\})^n$ is a multi index such that $|\alpha|=m$, then $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n} = e_{\mathbf{j}}^*(z)$ for all $z\in\mathbb{C}^n$, where $\mathbf{j} = (\overset{\alpha_1-\text{times}}{1,\ldots\ldots,1},\ldots,\overset{\alpha_n-\text{times}}{n,\ldots\ldots,n})$, and that in this case $|\mathbf{j}| = \frac{m!}{\alpha!}$. Hence, the following inequality is a reformulation of Theorem 3.1:

$$\begin{aligned}
&\int \sup_{z\in U} \Big| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} g_{\mathbf{j}} e^*_{\mathbf{j}}(z)\Big| d\mu \\
&\leq C_m \sup_{\mathbf{j}\in\mathcal{J}(m,n)} \frac{|c_{\mathbf{j}}|}{\sqrt{|\mathbf{j}|}} \sup_{z\in U}\Big(\sum_{k=1}^{n}|z_k|^2\Big)^{\frac{m-1}{2}} \int \sup_{z\in U}\Big|\sum_{k=1}^{n} g_k z_k\Big| d\mu.
\end{aligned} \tag{3.1}$$

The crucial ingredient of the proof of Theorem 3.1 is Slepian's lemma. Given N real-valued random variables $X_k : \Omega \to \mathbb{R}$,

$$(X_1,\ldots,X_N) : \Omega \to \mathbb{R}^N$$

is said to be a Gaussian random vector provided that each real linear combination $\sum_{k=1}^{N} \alpha_k X_k$ is Gaussian. For example, if each X_k itself is a real linear combination of standard Gaussians, then they form a Gaussian random vector. The following comparison theorem originates in the work of Slepian [25] (see Fernique [15], and also [17, Remark 1.5] and [21, Corollary 3.14]): let $(X_1,\ldots,X_N)$ and $(Y_1,\ldots,Y_N)$ be Gaussian random vectors such that $\mathbb{E}|Y_i - Y_j|^2 \leq \mathbb{E}|X_i - X_j|^2$ for each pair (i,j). Then

$$\mathbb{E}\max_i Y_i \leq \mathbb{E}\max_i X_i. \tag{3.2}$$

Proof of inequality (3.1). Without loss of generality we may assume that all coefficients $c_{\mathbf{j}}$ are real, and $|c_{\mathbf{j}}| \leq \sqrt{|\mathbf{j}|}$ for all $\mathbf{j} \in \mathcal{J}(m,n)$. The aim is to estimate instead of the average

$$\int \sup_{z\in U} \Big| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} g_{\mathbf{j}} e^*_{\mathbf{j}}(z)\Big| d\mu\,,$$

for each finite set $D \subset U$, the average

$$\int \sup_{z\in D} \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} g_{\mathbf{j}} \mathbf{Re}\, e^*_{\mathbf{j}}(z) d\mu. \tag{3.3}$$

Indeed, if P is an m-homogeneous polynomial on $\mathbb{C}^n$, then it is very easy to prove, by the fact that U is circled and P is homogeneous, that $\|P\| = \sup_{z\in U} \mathbf{Re}\, P(z)$. Since, by hypothesis, $c_{\mathbf{j}} \in \mathbb{R}$ and $g_{\mathbf{j}} : \Omega \longrightarrow \mathbb{R}$ for all $\mathbf{j} \in \mathcal{J}(m,n)$, for each $w \in \Omega$

$$\sup_{z\in U} \Big| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} g_{\mathbf{j}}(w) e^*_{\mathbf{j}}(z)\Big| = \sup_{z\in U} \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} g_{\mathbf{j}}(w) \mathbf{Re}\, e^*_{\mathbf{j}}(z).$$

Moreover, by a compactness argument for each $\varepsilon > 0$ there is a finite subset $D \subset U$ such that

$$\begin{aligned} &\int \sup_{z\in U} \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} g_{\mathbf{j}}(w) \mathbf{Re}\, e^*_{\mathbf{j}}(z) d\mu \\ \leq &\int \sup_{z\in D} \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} g_{\mathbf{j}}(w) \mathbf{Re}\, e^*_{\mathbf{j}}(z) d\mu + \varepsilon. \end{aligned}$$

If now $\{\alpha^l\}_{l=1}^{2^{m-1}}$ is an ordering of $\{\alpha : \alpha = (\alpha_u)_{u=1}^m \in \{0,1\}^m$ such that $m - |\alpha|$ is even$\}$, then for every $\mathbf{j} \in \mathcal{J}(m,n)$ and $z \in \mathbb{C}^n$,

$$\mathbf{Re}\, e^*_{\mathbf{j}}(z) = \sum_{l=1}^{2^{m-1}} \varepsilon_l a^{l,1}_{j_1}(z) \cdots a^{l,m}_{j_m}(z),$$

where $a^{l,u}_k(z) := \mathbf{Re}\, e^*_k(z)^{\alpha^l_u} \mathbf{Im}\, e^*_k(z)^{1-\alpha^l_u}$ and $\varepsilon_l := (-1)^{(m-|\alpha^l|)/2}$, $1 \leq l \leq 2^{m-1}$, $1 \leq u \leq m$ and $1 \leq k \leq n$. Define the following two Gaussian random processes:

$$\begin{aligned} Y_z &:= \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} g_{\mathbf{j}}(w) \mathbf{Re}\, e^*_{\mathbf{j}}(z) \\ &= \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} g_{\mathbf{j}} \sum_{l=1}^{2^{m-1}} \varepsilon_l a^{l,1}_{j_1}(z) \cdots a^{l,m}_{j_m}(z),\ z \in \mathbb{C}^n, \\ X_z &:= \sum_{l=1}^{2^{m-1}} \sum_{u=1}^{m} \sum_{k=1}^{n} g_{l,u,k} a^{l,u}_k(z),\ z \in \mathbb{C}^n, \end{aligned}$$

where the $g_{\mathbf{j}}, g_{l,u,k}$ and g_k all stand for independent standard Gaussian random variables. Note that for fixed l and u, the term $\sum_{k=1}^n g_{l,u,k} a^{l,u}_k(z)$ coincides either with $\sum_{k=1}^n g_{l,u,k} \mathbf{Re}\, e^*_k(z)$ or with $\sum_{k=1}^n g_{l,u,k} \mathbf{Im}\, e^*_k(z)$ for all $z \in \mathbb{C}^n$. Fix now a finite set $D \subset U$, and show exactly as in the proof of [9, Theorem 3.1] that for each pair x, y in D

$$\begin{aligned} &\Big(\int |Y_x - Y_y|^2 d\mu\Big)^{1/2} \\ \leq\ &\sqrt{2^{m-1}}\sqrt{m} \sup_{z\in U}\Big(\sum_{k=1}^n |z_k|^2\Big)^{\frac{m-1}{2}} \Big(\int |X_x - X_y|^2 d\mu\Big)^{1/2}. \end{aligned}$$

By Slepian's lemma (3.2) we finally obtain

$$\begin{aligned}
&\int \sup_{z\in D} \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} g_{\mathbf{j}} \mathbf{Re}\, e^*_{\mathbf{j}}(z) d\mu \\
\leq\ & \sqrt{2^{m-1}}\sqrt{m}\, \sup_{z\in U}\Big(\sum_{k=1}^{n} |z_k|^2\Big)^{\frac{m-1}{2}} \int \sup_{z\in D} \sum_{l=1}^{2^{m-1}} \sum_{u=1}^{m} \sum_{k=1}^{n} g_{l,u,k} a_k^{l,u}(z) d\mu \\
\leq\ & \sqrt{2^{m-1}}\sqrt{m}\, \sup_{z\in U}\Big(\sum_{k=1}^{n} |z_k|^2\Big)^{\frac{m-1}{2}} \sum_{l=1}^{2^{m-1}} \sum_{u=1}^{m} \int \sup_{z\in D} \Big|\sum_{k=1}^{n} g_{l,u,k} z_k\Big| d\mu \\
=\ & \sqrt{2^{m-1}}\sqrt{m}\, 2^{m-1} m\, \sup_{z\in U}\Big(\sum_{k=1}^{n} |z_k|^2\Big)^{\frac{m-1}{2}} \int \sup_{z\in D} \Big|\sum_{k=1}^{n} g_k z_k\Big| d\mu.
\end{aligned}$$

Together with the consideration from (3.3) this completes the proof. □

Estimating Bernoulli by Gaussian averages, we add an explicit estimate of the expectation of the maximum modulus of an m-homogeneous Bernoulli random polynomial.

COROLLARY 3.2. *Let $(\varepsilon_\alpha)_{|\alpha|=m}$ and $(\varepsilon_k)_{1\leq k\leq n}$ be a families of independent standard Bernoulli random variables on a probability space (Ω,μ), and let c_α, $|\alpha|=m$ be scalars.*

(1) *For each bounded circled set U in $\mathbb{C}^n$ we have*

$$\begin{aligned}
&\int \sup_{z\in U} \Big|\sum_{|\alpha|=m} c_\alpha \varepsilon_\alpha z^\alpha\Big| d\mu \\
\leq\ & \sqrt{\log n}\, C_m \sup_{|\alpha|=m} \left\{ |c_\alpha| \sqrt{\frac{\alpha!}{m!}} \right\} \sup_{z\in U}\Big(\sum_{k=1}^{n} |z_k|^2\Big)^{\frac{m-1}{2}} \sup_{z\in U} \sum_{k=1}^{n} |z_k|.
\end{aligned}$$

(2) *Let $X=(\mathbb{C}^n,\|.\|)$ be a Banach space for which the e_k's form an 1-unconditional basis, and let $2\leq q<\infty$. Then we have*

$$\begin{aligned}
&\int \sup_{z\in B_X} \Big|\sum_{|\alpha|=m} c_\alpha \varepsilon_\alpha z^\alpha\Big| d\mu \\
\leq\ & c\sqrt{q}\, C_q(X^*)\, C_m \sup_{|\alpha|=m} \left\{ |c_\alpha| \sqrt{\frac{\alpha!}{m!}} \right\} \sup_{z\in B_X}\Big(\sum_{k=1}^{n} |z_k|^2\Big)^{\frac{m-1}{2}} \sup_{z\in B_X} \sum_{k=1}^{n} |z_k|.
\end{aligned}$$

Here $c>0$ is an absolute constant and C_m as above.

Proof. We use the following facts:

(i) Gauss averages in a Banach space always dominate Bernoulli averages (see e.g. [28, (4.2), p.15]),

(ii) given a Banach space Y and $y_1, \ldots, y_n \in Y$, then

$$\int \| \sum_{k=1}^{n} g_k y_k \|_Y d\mu \leq \sqrt{\log n} \int \| \sum_{k=1}^{n} \varepsilon_k y_k \|_Y d\mu$$

(see e.g. [28, (4.4) p.15]),

(iii) there is a constant $c > 0$ such that, given a Banach space Y of cotype q, for each choice of $y_1, \ldots, y_n \in Y$, we have

$$\int \| \sum_{k=1}^{n} g_k y_k \|_Y d\mu \leq c\sqrt{q} C_q(Y) \int \| \sum_{k=1}^{n} \varepsilon_k y_k \|_Y d\mu$$

(see e.g. [28, (4.3) p.15]), where (g_k), respectively (ϵ_k), is a family of independent standard Gaussian, respectively Bernoulli, random variables on a probability space. Now from (i), (ii) and Theorem 3.1 we obtain

$$\int \sup_{z \in U} | \sum_{|\alpha|=m} c_\alpha \varepsilon_\alpha z^\alpha | d\mu$$

$$\leq \sqrt{\log n} C_m \sup_{|\alpha|=m} \left\{ |c_\alpha| \sqrt{\frac{\alpha!}{m!}} \right\} \sup_{z \in U} (\sum_{k=1}^{n} |z_k|^2)^{\frac{m-1}{2}} \int \sup_{z \in U} | \sum_{k=1}^{n} \varepsilon_k z_k | d\mu.$$

But

$$\int \sup_{z \in U} | \sum_{k=1}^{n} \varepsilon_k z_k | d\mu \leq \int \sup_{z \in U} \sum_{k=1}^{n} |z_k| d\mu = \sup_{z \in U} \sum_{k=1}^{n} |z_k| , \tag{3.4}$$

which proves (3.2). (3.2) follows from (i), (iii), Theorem 3.1 and (3.4). □

Given $m \in \mathbb{N}$, if $(a_n(m))_{n=1}^{\infty}$ and $(b_n(m))_{n=1}^{\infty}$ are scalar sequences we write $a_n(m) \overset{m}{\asymp} b_n(m)$ whenever there are some $c_m, d_m > 0$ such that $c_m a_n(m) \leq b_n(m) \leq d_m a_n(m)$ for all n. Again we illustrate our results by several examples. The first example is the counterpart of Example 2.4.

EXAMPLE 3.3.

$$\int \sup_{z \in B_{\ell_p^u(\ell_q^v)}} | \sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha | d\mu \overset{m}{\asymp} \begin{cases} u^{\frac{m+1}{2} - \frac{m}{p}} v^{\frac{m+1}{2} - \frac{m}{q}} & \text{if } \quad 2 \leq p, q \leq \infty, \\ u^{1-\frac{1}{p}} v^{1-\frac{1}{q}} & \text{if } \quad 1 < p, q \leq 2, \\ u^{1-\frac{1}{p}} v^{\frac{m+1}{2} - \frac{m}{q}} & \text{if } \quad 1 < p \leq 2 \leq q \leq \infty. \end{cases}$$

For $v = 1$ and $p = q$ this result was proved in [9, Corollary 6.5]. If we allow p and q to equal 1, then in the second two cases the proof will show that we have to admit an additional *log* term. For $1 < q \leq 2 \leq p \leq \infty$ the upper estimate also holds, for the lower we don't know (see the remark after Lemma 2.5).

Proof of Example 3.3. The lower bound is immediate from Theorem 2.3. Upper bound: Since $\ell_{p'}(\ell_{q'})$ in all considered cases has finite cotype, by Corollary 3.2(2) and (3.5), we get

$$\begin{aligned} & \int \sup_{z \in B_{\ell_p^u(\ell_q^v)}} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha \Big| d\mu \\ \leq \;& D_m \| id : \ell_p^u(\ell_q^v) \longrightarrow \ell_2^{uv} \|^{m-1} \Big\| \sum_{k,l} e_k^* \otimes e_l^* \Big\|_{\ell_{p'}^u(\ell_{q'}^v)} \\ \leq \;& D_m \| id : \ell_p^u \longrightarrow \ell_2^u \|^{m-1} \| id : \ell_q^v \longrightarrow \ell_2^v \|^{m-1} u^{1-1/p} v^{1-1/q}, \end{aligned}$$

which by Hölder's inequality is the desired result. □

We go on with the counterpart of Example 2.7.

EXAMPLE 3.4. Given $S > 1$ and $n \geq 2$ consider the Reinhardt domain $R := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,:\, |z_1 \cdots z_n| < 1,\ |z_k| < S,\ k = 1, \ldots, n\}$. Then the following asymptotic estimate holds:

$$\int \sup_{z \in R} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha \Big| d\mu \overset{m}{\asymp} n^{\frac{m+1}{2}}.$$

Proof. The lower bound is a consequence of Example 2.7. The upper estimate follows from

$$\int \sup_{z \in R} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha \Big| d\mu \leq \int \sup_{\{z \,:\, |z_k| < S,\ k=1,\ldots,n\}} \Big| \sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha \Big| d\mu,$$

and Example 3.3 for the case ℓ_∞^n. □

Let us give some more abstract estimates for finite sections X_n of Banach sequence spaces X.

COROLLARY 3.5. *Let X be a Banach sequence space with non-trivial convexity, and $X_n := \operatorname{span}\{e_k : 1 \leq k \leq n\}$, $n \in \mathbb{N}$.*

(1) *For each m there is a constant $D_m > 0$ such that for every n*

$$\frac{1}{D_m}\frac{\left(\sup_{z\in B_{X_n}}\sum_{k=1}^n |z_k|\right)^m}{n^{\frac{m-1}{2}}}$$
$$\leq \int \sup_{z\in B_{X_n}} |\sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha| d\mu \leq D_m \sup_{z\in B_{X_n}} (\sum_{k=1}^n |z_k|^2)^{\frac{m-1}{2}} \sup_{z\in B_{X_n}} \sum_{k=1}^n |z_k|.$$

(2) *If X is symmetric and 2-convex, then*

$$\int \sup_{z\in B_{X_n}} |\sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha| d\mu \overset{m}{\asymp} \frac{\left(\sup_{z\in B_{X_n}}\sum_{k=1}^n |z_k|\right)^m}{n^{\frac{m-1}{2}}}.$$

(3) *If $X \subset \ell_2$, then*

$$\int \sup_{z\in B_{X_n}} |\sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha| d\mu \overset{m}{\asymp} \sup_{z\in B_{X_n}} \sum_{k=1}^n |z_k|.$$

This result is an improvement of [9, Corollary 6.5].

Proof. The lower bound in (1) is a consequence of Theorem 2.3 (1). Since X^* has non-trivial concavity, which implies that it has cotype q for some $2 \leq q < \infty$, the upper bound in (1) is a consequence of Corollary 3.2 (2). In (2) it remains to show the upper bound: We may assume that $M^{(2)}(X) = 1$, hence

$$\sup_{z\in B_{X_n}} (\sum_{k=1}^n |z_k|^2)^{1/2} = \frac{\sup_{z\in B_{X_n}}\sum_{k=1}^n |z_k|}{\sqrt{n}}$$

(see [27, Proposition 2.2] and [11, Proposition 3.5]). Hence, the conclusion in (2) is a consequence of the upper bound in (1). Finally, the upper bound in (3) follows from the upper bound in (1), and the lower bound in (3) by Theorem 2.3 (3). □

In the symmetric case the above three statements can be reformulated in terms of the fundamental function $\|\sum_{k=1}^n e_k\|_{X_n}$, $n \in \mathbb{N}$ of X:

REMARK 3.6. Let X be a symmetric Banach sequence space with non-trivial convexity, and $X_n := \operatorname{span}\{e_k : 1 \leq k \leq n\}$, $n \in \mathbb{N}$.

(1') For each m there is a constant D_m such that for every n

$$\frac{1}{D_m}\frac{n^{\frac{m+1}{2}}}{\|\sum_{k=1}^n e_k\|_{X_n}^m}$$
$$\leq \int \sup_{z\in B_{X_n}} |\sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha| d\mu \leq D_m \|id : X_n \longrightarrow \ell_2^n\|^{m-1} \frac{n}{\|\sum_{k=1}^n e_k\|_{X_n}}.$$

(2') If X is 2-convex, then

$$\int \sup_{z\in B_{X_n}} |\sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha| d\mu \overset{m}{\asymp} \frac{n^{\frac{m+1}{2}}}{\|\sum_{k=1}^n e_k\|_{X_n}^m}.$$

(3') If $X \subset \ell_2$, then

$$\int \sup_{z\in B_{X_n}} |\sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha| d\mu \overset{m}{\asymp} \frac{n}{\|\sum_{k=1}^n e_k\|_{X_n}}.$$

Proof. The proof easily follows from the following observations:

$$\begin{aligned} &\sup_{z\in B_{X_n}} (\sum_{k=1}^n |z_k|^2)^{1/2} = \|id : X_n \longrightarrow \ell_2^n\| \\ &\sup_{z\in B_{X_n}} \sum_{k=1}^n |z_k| = \|\sum_{k=1}^n e_k^*\|_{X_n^*} = \|id : X_n \longrightarrow \ell_1^n\|, \end{aligned} \tag{3.5}$$

and the well known fact that for symmetric X for all n

$$n = \|\sum_{k=1}^n e_k^*\|_{X_n^*} \|\sum_{k=1}^n e_k\|_{X_n} \tag{3.6}$$

(see [22, Proposition 3.a.6, p.118, vol. I]). □

In our Corollaries 3.2 and 3.5 and Examples 3.3 and 3.4 we have proved upper bounds for the average of the maximum moduli of m-homogenous Bernoulli random polynomials. This shows the existence of m-homogeneous Bernoulli polynomials which have norms being less or equal to the bounds given in these results. Moreover, in section 2 for some cases (see e.g. the Examples 2.4, 2.6 and 2.7) we have given lower bounds for m-homogeneous Bernoulli polynomials which coincide (up to a constant) with the upper bounds obtained for the expectations in 3.3 and 3.4. Hence, we have proved the existence of m-homogeneous Bernoulli polynomials which have norms as small as possible (up to a constant). Again we illustrate this by an example.

Example 3.7. Let φ be an Orlicz function satisfying the Δ_2-condition. Then for each m there is a constant $D_m > 0$ such that for every n there are signs s_α, $|\alpha| = m$ for which

(1) $\sup_{z\in B_{\ell_\varphi}} |\sum_{|\alpha|=m} s_\alpha z^\alpha| \le D_m n^{\frac{m+1}{2}} \varphi^{-1}(1/n)^m$, provided $\varphi(\lambda t) \le K\lambda^2\varphi(t)$ for all $0 \le \lambda, t \le 1$ and some $K > 1$,

(2) $\sup_{z\in B_{\ell_\varphi}} |\sum_{|\alpha|=m} s_\alpha z^\alpha| \le D_m n \varphi^{-1}(1/n)$, provided that $t^2 \le K\varphi(t)$ for all t and some K.

By Example 2.6 we know that the estimates in the statements (1) and (2) are the "smallest" possible.

As above this result easily follows from the fact that $\|\sum_{k=1}^{n} e_k\|_{\ell_\varphi} = \frac{1}{\varphi^{-1}(1/n)}$; the conditions in (1) and (2) make sure that ℓ_φ is 2-convex and contained in ℓ_2, respectively (see [19]).

Similar results can be obtained for various other types of Banach sequence spaces, e.g. Lorentz spaces.

References

[1] H. Boas, *Majorant series, Several complex vaiiables*, J. Korean Math. Soc. **37** (2000), no. 2, 321–337.

[2] ———, *The football player and the infinite series*, Notices Amer. Math. Soc. **44** (11) (1997), 1430–1435.

[3] H. P. Boas and D. Khavinson, *Bohr's power series theorem in several variables*, Proc. Amer. Math. Soc. **125**, 10 (1997), 2975–2979.

[4] H. Bohr, *A theorem concerning power series*, Proc. London Math. Soc. **13** (1914), no. 2, 1–5.

[5] ———, *Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen* $\sum a_n/n^s$, Nachrichten von der Königlichen Gesellschaft der Wissenschafen zu Göttingen, Mathematisch-Physikalische Klasse (1913), 441–488.

[6] H. F. Bohnenblust and E. Hille, *On the absolute convergence of Dirichlet series*, Ann. of Math. **32** (1931), 600–622.

[7] S. Chevet, *Séries de variable aléatories Gaussiens à valeur dans* $E \otimes F$, Application aux produits d'espaces de Wiener abstraits, Séminaire Maurey-Schwartz, Exp. 19. Ecole Polytechnique, Paliseau, (1977).

[8] A. Defant, J. C. Díaz, D. García and M. Maestre, *Unconditional basis and Gordon-Lewis constants for spaces of polynomials*, J. Funct. Anal. **181** (2001), 119–145.

[9] A. Defant, D. García and M. Maestre, *Bohr's power series theorem and local Banach space theory*, J. reine angew. Math. **557** (2003), 173–197.

[10] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North–Holland Math. Studies **176**, 1993.

[11] A. Defant, M. Mastyło and C. Michels, *Summing inclusion maps between symmetric sequence spaces, a survey*, Recent progress in functional analysis, (Valencia, 2000), 43–60, North-Holland Math. Stud., 189, North-Holland, Amsterdam, 2001.

[12] S. Dineen, *Complex Analysis on Infinite Dimensional Banach Spaces*, Springer-Verlag. Springer Monographs in Mathematics, Springer-Verlag, London, 1999.

[13] S. Dineen and R. M. Timoney, *Absolute bases, tensor products and a theorem of Bohr*, Studia Math. **94** (1989), 227–234.

[14] ———, *On a problem of H. Bohr*, Bull. Soc. Roy. Sci. Liège **60** (1991), no. 6, 401–404.

[15] X. Fernique, *Des resultats nouveaux sur les processus gaussiens*, C. R. Acad. Sci. Paris, Ser. A-B **278** (1974), 363–365.

[16] K. Floret, *Natural norms on symmetric tensor products of normed spaces*, Note Mat. **17** (1997), 153–188.

[17] Y. Gordon, *Some inequalities for Gaussians processes and applications*, Israel J. Math. **50** (1985), no. 4, 265–289.

[18] J. P. Kahane, *Some random series of functions*, second ed. Cambridge Stud. Adv. Math. **5**, Cambridge University Press, Cambridge, 1985.

[19] E. Katirtzoglou, *Type and cotype of Musielak-Orlicz sequence spaces*, Jour. Math. Anal. Appl. **226** (1998), 431–455.

[20] S. Kwapien and C. Schütt, *Combinatorial and probabilistic inequalities and their applications to Banach space theory II*, Studia Math. **95** (1989), 141–154.

[21] M. Ledoux and M. Talagrand, *Probability in Banach spaces. Isoperimetry and processes*, Springer-Verlag, Berlin, 1991.

[22] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I and II*, Springer-Verlag, 1977, 1979.

[23] A. M. Mantero and A. Tonge, *The Schur multiplication in tensor algebras*, Studia Math. **68** (1980), 1–24.

[24] ———, *Banach algebras and von Neumann's inequality*, Proc. London Math. Soc. **38** (1979), 309–334.

[25] D. Slepian, *The one-sided barrier problem for Gausssian noise*, Bell Syst. Tech. J. **41** (1962), 463–501.

[26] C. Schütt, *The projection constant of finite-dimesional spaces whose unconditional basis constant is 1*, Israel J. Math **30** (1978), 207–212.

[27] S. J. Szarek and N. Tomczak–Jaegermann, *On nearly Euclidean decompositions for some classes of Banach spaces*, Compositio Math. **40** (1980), 367–385.

[28] N. Tomczak–Jaegermann, *Banach–Mazur Distances and Finite–Dimensional Operators Ideals*, Longman Scientific & Technical, 1989.

Andreas Defant
Fachbereich Mathematik, Universitaet
D–26111, Oldenburg, Germany
E-mail: defant@mathematik.uni-oldenburg.de

Domingo García
Departamento de Análisis Matemático
Universidad de Valencia
Doctor Moliner 50
46100 Burjasot Valencia, Spain
E-mail: domingo.garcia@uv.es

Manuel Maestre
Departamento de Análisis Matemático
Universidad de Valencia
Doctor Moliner 50
46100 Burjasot Valencia, Spain
E-mail: manuel.maestre@uv.es