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HOLOMORPHIC MAPPINGS INTO SOME DOMAIN IN A COMPLEX NORMED SPACE

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ABSTRACT. Let D_1, D_2 be convex domains in complex normed spaces E_1, E_2 respectively. When a mapping $f : D_1 \to D_2$ is holomorphic with $f(0) = 0$, we obtain some results like the Schwarz lemma. Furthermore, we discuss a condition whereby f is linear or injective or isometry.

1. Introduction

Let $\Delta = \{z \in \mathbf{C}; |z| < 1\}$ be the unit disc in $\mathbf{C}$. The classical Schwarz lemma in one complex variable is as follows:

THE CLASSICAL SCHWARZ LEMMA. *Let $f : \Delta \to \Delta$ be a holomorphic mapping with $f(0) = 0$. Then the following statements hold:*
(i) $|f(z)| \leq |z|$ *for any* $z \in \Delta$,
(ii) *if there exists* $z_0 \in \Delta\backslash\{0\}$ *such that* $|f(z_0)| = |z_0|$, *or if* $|f'(0)| = 1$, *then there exists a complex number λ of modulus 1 such that $f(z) = \lambda z$ and f is an automorphism of Δ.*

It is natural to consider an extension of the above results to more general domains or higher dimensional spaces. However, condition (ii) in above no longer holds even for the bidisc $\Delta \times \Delta$. In fact, one can easily construct a holomorphic mapping $f : \Delta \times \Delta \to \Delta \times \Delta$ such that $f(0) = 0$ and $\|f(z)\| = \|z\|$ for z in an open subset of $\Delta \times \Delta$, but f is not an isometry (cf. J. P. Vigué [18]). Nevertheless, E. Vesentini [15], [16] showed that if $\|f(w)\| = \|w\|$ holds on B_1 and if every boundary point of the unit ball B_2 is a complex extreme point, then $f : B_1 \to B_2$ is a linear isometry, where B_1, B_2 are the open unit balls in normed spaces E_1, E_2 over $\mathbf{C}$ respectively. J. P. Vigué [18], [19] proved that if every boundary point of the unit ball B for some norm in $\mathbf{C}^n$ is a complex extreme

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point of $\overline{B}$ and if $\|f(w)\| = \|w\|$ holds on an open subset U of B, then $f : B \to B$ is a linear automorphism of $\mathbf{C}^n$. H. Hamada [6] generalized the above classical Schwarz lemma to the case where $\|f(w)\| = \|w\|$ holds on some local complex submanifold of codimension 1. The author [10], [11] generalized to the case where $\|f(w)\| = \|w\|$ holds on a non-pluripolar subset. H. Hamada and the author [8] generalized to the case where $\|f(w)\| = \|w\|$ holds on a totally real submanifold.

In this paper, we consider some condition whereby a holomorphic mapping is linear or injective or isometric.

2. Notation and preliminaries

All topologies considered throughout this paper shall be Hausdorff. A vector space E over $\mathbf{C}$ is said to be *locally convex* if E is equipped with the Hausdorff topology defined by some family Π of seminorms such that $\sup_{\alpha\in\Pi} \alpha(x) > 0$ for all $x \in E \setminus \{0\}$, that is, a fundamental system of neighborhoods of x in this topology is made up of finite intersections of sets $x + \alpha^{-1}([0, a])$, $\alpha \in \Pi$, $0 < a < \infty$. Then all seminorms in Π are continuous, but the family $cs(E)$ of all continuous semi-norm on E is generally larger than Π. A sequence $\{z_n\}_{n\in\mathbf{N}}$ on a locally convex space E is a Cauchy sequence in E if for each $\alpha \in \Pi$ and each $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $\alpha(z_n - z_m) < \varepsilon$ for all $m, n \geq n_0$. A locally convex space E is said to be *sequentially complete* if any Cauchy sequence converges.

Let F be a locally convex space, let E be a sequentially complete locally convex space. Let U be an open subset in F, and let $f : U \to E$ be a holomorphic mapping. For $a \in U$, there uniquely exists a sequence of n-homogeneous polynomials $P_n : F \to E$ such that the expansion

$$f(a+z) = f(a) + \sum_{n=1}^{\infty} P_n(z)$$

holds for all z in the largest balanced subset of $U - a$. This series is called *the Taylor expansion of f by n-homogeneous polynomials P_n at a.*

Let $\Delta = \{z \in \mathbf{C}; |z| < 1\}$ be the unit disc in the complex plane $\mathbf{C}$, and let $\gamma(\lambda) = 1/(1 - |\lambda|^2)$. The Poincaré distance ρ on Δ is defined for $z, w \in \Delta$ as follows:

$$\rho(z, w) = \frac{1}{2} \log \frac{1 + |(z-w)/(1-z\bar{w})|}{1 - |(z-w)/(1-z\bar{w})|}.$$

Let D be a domain in a sequentially complete locally convex space E. The gauge N_D of D is defined for $z \in E$ as follows:

$$N_D(z) = \inf\{\alpha > 0; z \in \alpha D\}.$$

The Carathéodory pseudodistance C_D on D is defined for $p, q \in D$ as follows:

$$C_D(p,q) = \sup\{\rho(f(p), f(q)); f \in \mathrm{H}(D, \Delta)\}.$$

The Lempert function δ_D of D is defined for $p, q \in D$ as follows:

$$\begin{aligned}\delta_D(p,q) \quad = \quad & \inf\{\rho(\xi,\eta); \xi, \eta \in \Delta, {}^{\exists}\varphi \in \mathrm{H}(\Delta, D) \\ & \text{such that } \varphi(\xi) = p, \varphi(\eta) = q\}.\end{aligned}$$

The Kobayashi pseudodistance K_D on D is defined for $p, q \in D$ as follows:

$$K_D(p,q) = \inf\left\{\sum_{k=1}^{m} \delta_D(x_k, x_{k+1}); m \in \mathbf{N}, \{p = x_1, x_2, \ldots, x_{m+1} = q\} \subset D\right\}.$$

Then we have established between the various pseudodistances on a domain D:

$$C_D \leq K_D \leq \delta_D \text{ on } D \times D.$$

The infinitesimal Carathéodory pseudometric c_D for D is defined for $z \in D$, $v \in E$ as follows:

$$c_D(z,v) = \sup\{|d\psi(z)(v)|; \psi \in \mathrm{H}(D, \Delta)\}.$$

The infinitesimal Kobayashi pseudometric κ_D for D is defined for $z \in D$, $v \in E$ as follows:

$$\begin{aligned}\kappa_D(z,v) \quad = \quad & \inf\{\gamma(\lambda)|\alpha|; {}^{\exists}\varphi \in \mathrm{H}(\Delta, D), {}^{\exists}\lambda \in \Delta \\ & \text{such that } \varphi(\lambda) = z, \alpha\varphi'(\lambda) = v\}.\end{aligned} \tag{2.1}$$

Then holomorphic mappings $\varphi \in \mathrm{H}(\Delta, D)$ as in (2.1) certainly exist. In fact, if R is the radius of the open disc $\{\lambda \in \mathbf{C}; \lambda v \in U(z)\}$, where $U(z)$ is a neighborhood of z, we may take the mapping

$$\varphi(\lambda) = z + \frac{\lambda}{\zeta} v$$

for $|\zeta| \geq 1/R$. Hence $\kappa_D(z,v) \leq 1/R$.

Moreover, for any $\psi \in \mathrm{H}(D, \Delta)$ with $\psi(z) = 0$, we have $(\psi \circ \varphi)'(0) = d\psi(z)(\varphi'(0))$. It follows from this that

$$c_D \leq \kappa_D \text{ on } D \times E.$$

We use convexity to obtain the relationship among the pseudodistances or pseudometrics (S. Dineen [3], T. Franzoni and E. Vesentini [5], M. Hervé [9] etc).

PROPOSITION 2.1. *If D is a balanced convex domain in a sequentially complete locally convex space E, then*
(i) $C_D(0,x) = K_D(0,x) = \delta_D(0,x) = \rho(0, N_D(x))$ *for any* $x \in D$,
(ii) $c_D(0,v) = \kappa_D(0,v) = N_D(v)$ *for any* $v \in E$.

Let D be a balanced pseudoconvex domain in a sequentially complete locally convex space E. Then we have the following proposition as the gauge N_D is plurisubharmonic on E.

PROPOSITION 2.2. *If D is a balanced pseudoconvex domain in a sequentially complete locally convex space E, then*
(i) $K_D(0,x) = \delta_D(0,x) = \rho(0, N_D(x))$ *for any* $x \in D$,
(ii) $\kappa_D(0,v) = N_D(v)$ *for any* $v \in E$.

Using the above proposition, we obtain the following generalization of part (i) of the Schwarz lemma to balanced pseudoconvex domains in sequentially complete locally convex spaces.

PROPOSITION 2.3. *Let E_j be a sequentially complete locally convex space and let D_j be a balanced pseudoconvex domain in E_j for $j = 1, 2$. Let $f : D_1 \to D_2$ be a holomorphic mapping with $f(0) = 0$. Then*

$$N_{D_2} \circ f(z) \leq N_{D_1}(z).$$

Proof. By Proposition 2.2 (i), we have

$$\rho(0, N_{D_1}(z)) = \delta_{D_1}(0,z) \geq \delta_{D_2}(0, f(z)) = \rho(0, N_{D_2} \circ f(z)).$$

Since $\rho(0,r)$ is strictly increasing for $0 \leq r < 1$, we obtain this proposition. □

The following definition of a complex geodesic due to E. Vesentini [15, 16, 17].

DEFINITION 2.4. *Let D be a domain in a sequentially complete locally convex space E endowed with a pseudodistance d_D. A holomorphic mapping $\varphi : \Delta \to D$ is said to be a complex d_D-geodesic for (x,y) if*

$$d_D(x,y) = \rho(\xi, \eta)$$

for any points $\xi, \eta \in \Delta$ such that $\varphi(\xi) = x$ and $\varphi(\eta) = y$.

A holomorphic mapping $\varphi : \Delta \to D$ is said to be *a complex c_D-geodesic for (z,v)* if $c_D(z,v) = \gamma(\lambda)|\alpha|$ holds for any $\lambda \in \Delta$ and any $\alpha \in \mathbf{C}$ such that $\varphi(\lambda) = z$ and $\alpha\varphi'(\lambda) = v$.

A holomorphic mapping $\varphi : \Delta \to D$ is said to be *a complex κ_D-geodesic for* (z, v) if $\kappa_D(z, v) = \gamma(\lambda)|\alpha|$ holds for any $\lambda \in \Delta$ and $\alpha \in \mathbf{C}$ such that $\varphi(\lambda) = z$ and $\alpha\varphi'(\lambda) = v$.

The following results about a complex geodesic are well-known (cf. S. Dineen [3], T. Franzoni and E. Vesentini [5], M. Hervé [9] etc).

PROPOSITION 2.5. *Let D be a domain in a sequentially complete locally convex space E endowed with a pseudodistance d_D or a pseudometric μ_D. Then the following statements hold:*

(i) *a holomorphic mapping $\varphi : \Delta \to D$ is a complex d_D-geodesic for (x, y) if and only if there exists only one pair $(\xi, \eta) \in \Delta^2$ with $(\xi \neq \eta)$ such that $\varphi(\xi) = x$, $\varphi(\eta) = y$ and*

$$d_D(x, y) = \rho(\xi, \eta),$$

(ii) *a holomorphic mapping $\varphi : \Delta \to D$ is a complex μ_D-geodesic for (z, v) if and only if there exists only one point $\lambda \in \Delta$ such that $\varphi(\lambda) = z$, $\alpha\varphi'(\lambda) = v$ and*

$$\mu_D(\varphi(\lambda), \varphi'(\lambda)) = |\alpha|\gamma(\lambda).$$

A point x of the closure $\overline{D}$ of D is said to be *a complex extreme point of* $\overline{D}$ if $y = 0$ is the only vector in E such that the function $: \zeta \longmapsto x + \zeta y$ maps Δ into D. For example, C^2-smooth strictly convex boundary points are complex extreme points.

For a bounded balanced pseudoconvex domain D, the holomorphic mapping $\varphi(\zeta) = \zeta a/N_D(a)$ is a complex δ_D-geodesic and κ_D-geodesic for $(0, a)$ for any $a \in D$ with $N_D(a) > 0$. In fact, M. Hervé [9] has given the following characterization of the uniqueness of complex geodesics (see e.g. E. Vesentini [15], [16], [17]).

PROPOSITION 2.6. *Let D be a balanced convex domain in a sequentially complete locally convex space E. Let $a \in D$ be such that $N_D(a) > 0$, and let $\varphi : \Delta \to D$ be the holomorphic mapping defined by $\varphi(\zeta) = \zeta a/N_D(a)$. Then the following conditions are equivalent :*

(i) *the point $b = a/N_D(a)$ is a complex extreme point of $\overline{D}$;*
(ii) *φ is the unique (modulo* $\mathrm{Aut}(\Delta)$*) complex C_D-geodesic for $(0, a)$;*
(iii) *φ is the unique (modulo* $\mathrm{Aut}(\Delta)$*) complex K_D-geodesic for $(0, a)$;*
(iv) *φ is the unique (modulo* $\mathrm{Aut}(\Delta)$*) complex δ_D-geodesic for $(0, a)$;*
(v) *φ is the unique (modulo* $\mathrm{Aut}(\Delta)$*) complex c_D-geodesic for $(0, a)$;*
(vi) *φ is the unique (modulo* $\mathrm{Aut}(\Delta)$*) complex κ_D-geodesic for $(0, a)$.*

Using the uniqueness of complex geodesics, we obtain the linearity of complex geodesics as in the following proposition.

PROPOSITION 2.7. *Let D_j be a bounded balanced convex domain in complex normed spaces E_j for $j = 1, 2$, and let $f : D_1 \to D_2$ be a holomorphic mapping with $f(0) = 0$. Let $x \in D_1 \setminus \{0\}$ and let $\varphi(\zeta) = \zeta x / N_{D_1}(x)$. We assume that $f(x)/N_{D_2} \circ f(x)$ is a complex extreme point of $\overline{D_2}$. If one of the following conditions is satisfied, then $f \circ \varphi$ is a linear complex δ_{D_2}-geodesic.*
(i) $N_{D_2} \circ f(x) = N_{D_1}(x)$.
(ii) $\delta_{D_2}(f(0), f(x)) = \delta_{D_1}(0, x)$.
(iii) $K_{D_2}(f(0), f(x)) = K_{D_1}(0, x)$.
(iv) $C_{D_2}(f(0), f(x)) = C_{D_1}(0, x)$.

Proof. By Proposition 2.1 (i), the conditions (i), (ii), (iii) and (iv) are equivalent. Suppose that (i) is satisfied. By Proposition 2.1 (i),

$$\begin{aligned} \delta_{D_2}(f \circ \varphi(0), f \circ \varphi \circ N_{D_1}(x)) &= \delta_{D_2}(0, f(x)) \\ &= \delta_{D_1}(0, x) \\ &= \rho(0, N_{D_1}(x)). \end{aligned}$$

By Proposition 2.5 (i), $f \circ \varphi$ is a complex δ_{D_2}-geodesic for $(0, f(x))$. By Proposition 2.6, we have

$$f \circ \varphi(\zeta) = \zeta e^{i\theta} \frac{f(x)}{N_{D_2} \circ f(x)}$$

for some $\theta \in \mathbf{R}$. □

3. Special versions of the Schwarz Lemma

Now we introduce the projective space $\mathbf{P}(E)$. Let E be a locally convex space. Let z and z' be points in $E \setminus \{0\}$. z and z' are said to be *equivalent* if there exists $\lambda \in \mathbf{C}^*$ such that $z = \lambda z'$. We denote by $\mathbf{P}(E)$ the quotient space of $E \setminus \{0\}$ by this equivalence relation. Then $\mathbf{P}(E)$ is a Hausdorff space. The Hausdorff space $\mathbf{P}(E)$ is called the *complex projective space induced by E*. We denote by Q the quotient map from $E \setminus \{0\}$ to $\mathbf{P}(E)$ (see M. Nishihara [14]).

THEOREM 3.8. *Let E_j be a complex normed space, let D_j be a bounded balanced convex domain in E_j for $j = 1, 2$ and let $f : D_1 \to D_2$ be a holomorphic mapping with $f(0) = 0$. Let X be a non-empty subset* *of D_1 such that X is mapped homeomorphically onto an open subset Ω in the complex projective space $\mathbf{P}(E_1)$ by the quotient map Q from $E_1 \setminus \{0\}$ onto $\mathbf{P}(E_1)$. We assume that $f(x)/N_{D_2}(f(x))$ is a complex extreme point of $\overline{D_2}$ for any $x \in X$ and that there exists $w_0 \in X$ such that $w_0/N_{D_1}(w_0)$ is a complex extreme point of $\overline{D_1}$. If one of the following conditions is satisfied, then f is linear and injective.*

(i) $N_{D_2}(f(x)) = N_{D_1}(x)$ *for any* $x \in X$.

(ii) $C_{D_2}(f(0), f(x)) = C_{D_1}(0, x)$ *for any* $x \in X$.

(iii) $K_{D_2}(f(0), f(x)) = K_{D_1}(0, x)$ *for any* $x \in X$.

(iv) $\delta_{D_2}(f(0), f(x)) = \delta_{D_1}(0, x)$ *for any* $x \in X$.

Proof. By Proposition 2.1 (i), the conditions (i), (ii), (iii) and (iv) are equivalent. Suppose that (i) is satisfied. We take a point $w \in X \setminus \{0\}$ and set $\varphi(\zeta) = \zeta w/N_{D_1}(w)$ for $\zeta \in \Delta$. Then φ is a complex δ_{D_1}-geodesic. We have

$$\delta_{D_2}(f \circ \varphi(0), f \circ \varphi(N_{D_1}(w))) = \rho(0, N_{D_1}(w)).$$

By Proposition 2.7, $f \circ \varphi$ is a complex δ_{D_2}-geodesic. It follows from this that there exists a point $y \in D_2 \setminus \{0\}$ such that

$$f \circ \varphi(\zeta) = \zeta \frac{y}{N_{D_2}(y)}. \tag{3.1}$$

On the other hand, let $f(x) = \sum_{n=1}^{\infty} P_n(x)$ be the Taylor expansion of f by n-homogeneous polynomials P_n in a neighborhood V of 0 in E_1. Then we have

$$f \circ \varphi(\zeta) = \sum_{n=1}^{\infty} P_n(\zeta \frac{w}{N_{D_1}(w)}) = \sum_{n=1}^{\infty} \left(\frac{\zeta}{N_{D_1}(w)}\right)^n P_n(w) \tag{3.2}$$

in a neighborhood of 0 in Δ. By (3.1) and (3.2), we obtain

$$P_n(w) = 0 \quad \text{for } w \in X, n \geq 2.$$

We take any point $y \in \mathbf{C}^* X = \{tx; t \in \mathbf{C}^*, x \in X\}$. Then there exist $t \in \mathbf{C}^*$ and $x \in X$ such that $y = tx$. Hence

$$\begin{aligned} P_n(y) &= P_n(tx) \\ &= t^n P_n(x) \\ &= 0, \end{aligned}$$

that is, $P_n \equiv 0$ on $\mathbf{C}^* X \subset E_1$ for every $n \geq 2$. Since Q is continuous, the set $\mathbf{C}^* X = Q^{-1}(\Omega)$ contains an open subset U of E. By the identity theorem,

$$P_n \equiv 0 \quad \text{on } E_1 \text{ for every } n \geq 2.$$

Therefore $f = P_1$, that is, f is linear.

Next we show that f is injective. Let z be a point of E_1 with $f(z) = 0$. Since f is linear, we have

$$\begin{aligned} N_{D_2} \circ f(tx) &= N_{D_2}(tf(x)) \\ &= |t| N_{D_2} \circ f(x) \\ &= |t| N_{D_1}(x) \\ &= N_{D_1}(tx) \end{aligned}$$

for every $t \in \mathbf{C}^*, x \in X$. It follows from this that

$$N_{D_2} \circ f(y) = N_{D_1}(y) \quad \text{for all } y \in \mathbf{C}^* X.$$

Since $\mathbf{C}^* X$ is open, there exists a positive number $r > 0$ such that $w_0 + \zeta z \in \mathbf{C}^* X$ for $\zeta \in \mathbf{C}, |\zeta| < r$. Then we have

$$N_{D_2} \circ f(w_0 + \zeta z) = N_{D_1}(w_0 + \zeta z). \tag{3.3}$$

On the other hand,

$$\begin{aligned} N_{D_2} \circ f(w_0 + \zeta z) &= N_{D_2}(f(w_0) + \zeta f(z)) \\ &= N_{D_2} \circ f(w_0) \\ &= N_{D_1}(w_0). \end{aligned} \tag{3.4}$$

By (3.3) and (3.4), we have

$$N_{D_1}(w_0 + \zeta z) = N_{D_1}(w_0).$$

Hence

$$N_{D_1}\left(\frac{w_0}{N_{D_1}(w_0)} + \frac{\zeta}{N_{D_1}(w_0)} z\right) = 1 \quad \text{for } |\zeta| < r.$$

Since $w_0/N_{D_1}(w_0)$ is a complex extreme point of $\overline{D_1}$, we have

$$z = 0.$$

Therefore f is injective. □

Since complex Hilbert spaces are endowed with the norm which is induced from its inner products, we have the following corollary.

COROLLARY 3.9. *Let H_j be a complex Hilbert space with the inner product $< \cdot, \cdot >_j$, let B_j be the open unit ball of H_j for the norm $\| \cdot \|_j = < \cdot, \cdot >_j^{\frac{1}{2}}$ for $j = 1, 2$. Let $f : B_1 \to B_2$ be a holomorphic map with $f(0) = 0$. Let X be a non-empty subset of B_1 such that X is mapped onto an open subset Ω in the projective space $\mathbf{P}(H_1)$ by the quotient map Q. If $\|w\|_1 = \|f(w)\|_2$ holds for every $w \in X$, then f is a linear isometry.*

If $H_1 = H_2 = \mathbf{C}^n$ with the Euclidean unit ball B, then f is a linear automorphism of B.

Proof. Since every point of the boundary $\partial B_j = \{z \in H_j; \|z\|_j - 1 = 0\}$ of B_j is a complex extreme point of the closure $\bar{B}_j$ of B_j for $j = 1, 2$, by Theorem 3.8, f is linear and injective.

We consider a function

$$g(z) = \|z\|_1^2 - \|f(z)\|_2^2$$

for $z \in H_1$. By Proposition 2.3, we have $g \geq 0$ on H_1.

Since $\partial\bar{\partial}g \geq 0$, the non-negative valued function g is plurisubharmonic on H_1. Hence $\log g$ is plurisubharmonic on H_1. Since $\|w\|_1 = \|f(w)\|_2$ for every $w \in X$,

$$\log g \equiv -\infty$$

on an open subset $\mathbf{C}^* X = Q^{-1}(\Omega)$. Therefore f is a linear isometry. □

4. Infinitesimal pseudometrics

PROPOSITION 4.10. *Let D_j be a bounded balanced convex domain in a complex normed space E_j for $j = 1, 2$, and let $f : D_1 \to D_2$ be a holomorphic mapping with $f(0) = 0$. Let $x \in D \setminus \{0\}$ and let $\varphi(\zeta) = \zeta x / N_{D_1}(x)$. We assume that $df(0)x/N_{D_2}(df(0)x)$ is a complex extreme point of $\overline{D_2}$. If one of the following conditions is satisfied, then $f \circ \varphi$ is a linear complex κ_{D_2}-geodesic.*

(i) $N_{D_2} \circ f(x) = N_{D_1}(x)$.

(ii) $c_{D_2}(f(0), f(x)) = c_{D_1}(0, x)$.

(iii) $\kappa_{D_2}(f(0), f(x)) = \kappa_{D_1}(0, x)$.

Proof. By Proposition 2.1 (ii),

$$\begin{aligned} \kappa_{D_2}(0, df(0)x) &= \kappa_{D_1}(0, x) \\ &= N_{D_1}(x). \end{aligned}$$

Since $N_{D_1}(x)(f \circ \varphi)'(0) = df(0)x$, $f \circ \varphi$ is a complex κ_{D_2}-geodesic for $(0, df(0)x)$. By Proposition 2.6, we have

$$f \circ \varphi(\zeta) = \zeta e^{i\theta} \frac{df(0)x}{N_{D_2}(df(0)x)}$$

for some $\theta \in \mathbf{R}$. □

THEOREM 4.11. *Let E_j be a complex normed space, let D_j be a bounded balanced convex domain in E_j for $j = 1, 2$, and let $f : D_1 \to D_2$ be a holomorphic mapping. Let V be a connected open neighborhood of the origin in D_1. We assume that $\kappa_{D_2}(0, df(0)x) = \kappa_{D_1}(0, x)$ for $x \in V$. If $f(0) = 0$ and $df(0)x/N_{D_2}(df(0)x)$ is a complex extreme point of $\overline{D_2}$ for any $x \in V \setminus \{0\}$, and if there exists $w \in V \setminus \{0\}$ such that $w/N_{D_1}(w)$ is a complex extreme point of $\overline{D_1}$, then f is linear and injective.*

Proof. Let $f(z) = \sum_{n=1}^{\infty} P_n(z)$ be the expansion of f by n-homogeneous polynomials P_n in a neighborhood of 0 in E_1. Since $\kappa_{D_2}(f(0), df(0)v) = \kappa_{D_1}(0, v)$ for any $v \in V$, by Proposition 4.10, $f(\zeta x/N_{D_1}(x))$ is the restriction of a linear map for any $x \in V$. Then we have

$$P_n(x) = 0 \quad \text{on } V \text{ for } n \geq 2$$

as in the proof of Theorem 3.8. By the analytic continuation theorem, we have P_n is identically 0 for $n \geq 2$. Therefore f is the restriction of a linear map.

Let $\varphi(\zeta) = \zeta w/N_{D_1}(w)$. By Proposition 2.6, $f \circ \varphi$ is a complex δ_{D_2}-geodesic for $(0, f(v))$. By Proposition 2.1,

$$\rho(0, N_{D_2}(f(v))) = \delta_{D_2}(0, f(v)) = \rho(0, N_{D_1}(v)).$$

This implies that $N_{D_2}(f(v)) = N_{D_1}(v)$. The rest of the proof is same as Theorem 3.8. □

We note that the map f is not necessarily a linear isometry under the assumption of the above theorem (cf. J. P. Vigué [18]). The following theorem was obtained by H.Cartan for bounded domain in $\mathbf{C}^2$ (see T. Franzoni and E. Vesentini [5] etc).

THEOREM 4.12. *Let D be a bounded domain in a complex normed space E, and let $f : D \to D$ be a holomorphic mapping. If there exists $x_0 \in D$ such that $f(x_0) = x_0$ and $df(x_0)$ is an identity, f is the identity map.*

Using the above theorem of Cartan, we obtain the following theorem.

THEOREM 4.13. *Let E_j be a complex normed space, let D_j be a bounded balanced convex domain in E_j for $j = 1, 2$, and let $f : D_1 \to D_2$ be a holomorphic mapping. Let V be a connected open neighborhood of the origin in D_1. We assume that $\kappa_{D_2}(0, df(0)x) = \kappa_{D_1}(0, x)$ for $x \in V$. If the inverse $df(0)^{-1}$ exists, then $f(0) = 0$ and f is the restriction of $df(0)$ to D_1.*

Proof. First we will show $f(0) = 0$. We assume that $a = f(0) \neq 0$. Since $a \in D$, there exists a point $v \in E_1$ such that $N_{D_1}(v) = 1$ and $df(0)v = a/N_{D_2}(a)$. Then we have

$$\begin{aligned} \kappa_{D_2}(a, a/N_{D_2}(a)) &= \kappa_{D_2}(f(0), df(0)v) \\ &\leq \kappa_{D_1}(0, v) \\ &= N_{D_1}(v) \\ &= 1. \end{aligned}$$

Therefore

$$\kappa_{D_2}(a, a) \leq N_{D_2}(a). \tag{4.1}$$

On the other hand, we set $\varphi(\zeta) = \zeta a/N_{D_2}(a)$ for $\zeta \in \Delta$. Then φ is a complex κ_{D_2}-geodesic for $(0, a)$. So we have

$$\begin{aligned} \kappa_{D_2}(a, a) &= \kappa_{D_2}(\varphi(N_{D_2}(a)), N_{D_2}(a)\varphi'(N_{D_2}(a))) \\ &= \kappa_\Delta(N_{D_2}(a), N_{D_2}(a)) \\ &= \frac{N_{D_2}(a)}{1 - \{N_{D_2}(a)\}^2}. \end{aligned}$$

Therefore $\kappa_{D_2}(a, a) > N_{D_2}(a)$. This contradicts with (4.1). We obtain $f(0) = 0$.

By the assumptions, we have $N_{D_1}(df(0)^{-1}(w)) < 1$ for $w \in D_2$. Now we consider a holomorphic mapping $g = df(0)^{-1} \circ f$. Then g is a holomorphic mapping from D_1 to D_1 such that $g(0) = 0$ and $dg(0)$ is identity. By Theorem 4.12, g is identity. □

In the Hilbert space case, since every boundary point of the unit ball is a complex extreme point, by the proofs of Corollary 3.9, Theorem 4.11 and Theorem 4.13, we obtain the following corollary.

COROLLARY 4.14. *Let H_j be a complex Hilbert space with the inner product $< \cdot, \cdot >_j$, let B_j be the open unit ball of H_j for the norm $\| \cdot \|_j = < \cdot, \cdot >_j^{\frac{1}{2}}$ for $j = 1, 2$. Let $f : B_1 \to B_2$ be a holomorphic map. Let V be a connected open neighborhood of the origin in B_1. We assume that $\kappa_{B_2}(0, df(0)x) = \kappa_{B_1}(0, x)$ for $x \in V$. Then $f(0) = 0$ and f is a linear isometry.*

If $H_1 = H_2 = \mathbf{C}^n$ with the Euclidean unit ball B, then f is a linear automorphism of B.

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