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A HAHN-BANACH EXTENSION THEOREM FOR ENTIRE FUNCTIONS OF NUCLEAR TYPE

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ABSTRACT. Let E and F be locally convex spaces over $\boldsymbol{C}$. We assume that E is a nuclear space and F is a Banach space. Let f be a holomorphic mapping from E into F. Then we show that f is of uniformly bounded type if and only if, for an arbitrary locally convex space G containing E as a closed subspace, f can be extended to a holomorphic mapping from G into F.

1. Introduction

Let E, F and G be locally convex spaces over $\boldsymbol{C}$. We assume that E is a closed subspace of G. Then we consider the problem to ask whether a holomorphic mapping from E into F can be extended to a holomorphic mapping from G into F. This problem has been mainly investigated in case E is a Banach space or a nuclear space. When we consider this problem, we should remark that we need the bounded condition of a given hololomorphic function because it is well-known that a holomorphic function f on c_0 has a holomorphic extension to l_∞ if and only if f is of bounded type. Aron and Berner [1] proved various extension Theorems with the bounded condition of holomorphic functions between Banach spaces. Boland [2] showed that for every closed subspace E of an arbitrary DFN-space G every entire function on E can be extended to an entire function on G. Meise and Vogt [7] generalized the result of Boland to a nuclear space as follows:

THEOREM 1.1. *Let f be a holomorphic function on a nuclear locally convex space E. Then the following statements are equivalent:*

(a) *for an arbitrary locally convex space G of Hilbert type containing E as a closed subspace f can be extended to a holomorphic*

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function on G where a locally convex space of Hilbert type means that it has a fundamental system of seminorms consisting of semi-inner-products,

(b) *for an arbitrary locally convex space G of Hilbert type containing E as a closed subspace f can be extended to a holomorphic function of uniformly bounded type on G,*

(c) *f is of uniformly bounded type.*

The equivalence between (b) *and* (c) *is true for holomorphic mappings from E into F where E is a nuclear subspace of a locally convex space of Hilbert type G and F is any locally convex space.*

Nishihara [9] showed that the condition that G is Hilbert type can be removed in the above Meise-Vogt Theorem by using the fact that the projective tensor product topology is equal to the injective tensor product topology in a nuclear space and the fact that the injective tensor product topology inherits the induced topology. The aim of this paper is to extend the results of Meise and Vogt [7] and Nishihara [9] to the case of a holomorphic mapping with range of values in a Banach space as follows:

THEOREM 1.2 (Main Theorem). *Let E and F be locally convex spaces over $\mathbf{C}$. We assume that E is a nuclear space and F is a Banach space. Let f be a holomorphic mapping from E into F. Then the following statements are equivalent:*

(a) *for an arbitrary locally convex space G containing E as a closed subspace, f can be extended to a holomorphic mapping from G into F,*

(b) *for an arbitrary locally convex space G containing E as a closed subspace, f can be extended to a holomorphic mapping of uniformly bounded type from G into F,*

(c) *for an arbitrary locally convex space G of Hilbert type containing E as a closed subspace f can be extended to a holomorphic mapping from G into F,*

(d) *for an arbitrary locally convex space G of Hilbert type containing E as a closed subspace f can be extended to a holomorphic mapping of uniformly bounded type from G into F,*

(e) *f is of uniformly bounded type,*

(f) *f is of nuclear uniformly bounded type.*

We shall first show that a holomorphic mapping f from a locally convex space E into a Banach space F can be extended if f is of nuclear uniformly bounded type, that is, of uniformly bounded type with respect to the nuclear norm. Next we shall show that a holomorphic mapping with range of values in a Banach space on a nuclear locally convex space is of nuclear type and of nuclear uniformly bounded type if and only if f is of uniformly bounded type. By using these results we shall prove the Main Theorem.

2. Notations and preliminaries

In this section we collect some notation, definitions and basic properties of locally convex spaces. Throughout this paper we assume that all locally convex spaces are complex and Hausdorff. Let E and F be locally convex spaces. We denote by $cs(E)$ the set of all continuous seminorms on E. We denote by $P(^nE\,;F)$ the space of all continuous n-homogeneous polynomials from E into F. When $E = \boldsymbol{C}$, we use the notation $P(^nE)$ in place of $P(^nE\,;\boldsymbol{C})$. Let f be a mapping from a set S into a locally convex space F. For $\alpha \in cs(F)$ we set

$$\|f\|_{\alpha,S} = \sup\{\alpha\,(f(x))\ ; \quad \text{for every} \quad x \in S\}.$$

If α is clearly understood from the context, then we just write $\|f\|_S$. For every $\alpha \in cs(E)$ we set

$$B_\alpha(r) = \{x \in E\,;\, \alpha(x) < r\}.$$

Let B be a bounded convex balanced subset of a locally convex space E. Let E_B be a complex vector subspace of E defined by

$$E_B = \bigcup_{r>0} r\,B$$

where we denote by $\lambda\,A$ the set $\{\lambda\,x\,;\, x \in A\}$ for a subset A of a vector space and a scalar λ. Let μ_B be the Minkovski functional of B in E_B. Then the complex vector space E_B is a normed space with the norm μ_B. Let E and F be locally convex spaces and let $H(E\,;F)$ be the vector space of all holomorphic mappings from E into F. A mapping $f \in H(E\,;F)$ is said to be of *uniformly bounded type* if there exist a convex balanced neighborhood V of 0 in E and a convex balanced bounded subset B of F such that $f(E) \subset F_B$ and $\|f\|_{\mu_B, rV} < +\infty$ for every positive number r. We denote by $H_{ub}(E\,;F)$ the space of all holomorphic mappings from E into F which are of uniformly bounded type.

LEMMA 2.1. *Let F be a Banach space and let f be a holomorphic mappings from E into F. Then the following statements are equivalent.*

(a) $f \in H_{ub}(E; F)$.

(b) *Let* $f(x) = \sum_{n=0}^{\infty} p_n(x)$ *be the Taylor expansion of* f *at 0. Then there exists a balanced convex open neighborhood* V *of 0 in* E *such that*

$$\sum_{n=0}^{\infty} \|p_n\|_V r^n < \infty \tag{2.1}$$

for every $r > 0$.

(c) $\limsup_{n\to 0} \sqrt[n]{\|p_n\|_V} = 0$.

Proof. The equivalence between (b) and (c) is just the Cauchy-Hadamard's root test.

We show that (a) implies (b). Since $f \in H_{ub}(E)$, there exists a balanced convex open neighborhood V of 0 in E such that

$$\|f\|_{rV} < +\infty \quad \text{for every} \quad r > 0.$$

We set $M(r) = \|f\|_{2rV}$ for every $r > 0$. Then by Cauchy Inequalities we have $\|p_n\|_{rV} \leq \frac{1}{2^n} M(r)$. Therefore we have

$$\sum_{n=0}^{\infty} \|p_n\|_{rV} \leq 2M(r) < +\infty.$$

Thus

$$\sum_{n=0}^{\infty} \|p_n\|_V r^n = \sum_{n=0}^{\infty} \|p_n\|_{rV} \leq 2M(r) < +\infty$$

and we have proved that (a) implies (b).

It follows from the following inequality that (b) implies (a)

$$\|f\|_{rV} \leq \sum_{n=0}^{\infty} \|p_n\|_{rV} = \sum_{n=0}^{\infty} \|p_n\|_V r^n < +\infty$$

for every $r > 0$. This completes the proof. □

3. Holomorphic mappings of nuclear type

Let E and F be locally convex spaces. An n-homogeneous polynomial mapping $p \in P(^nE; F)$ is called a *nuclear n-homogeneous polynomial* from E to F if for each $\alpha \in cs(F)$ there exist an equicontinuous sequence $(\phi_j)_{j=1}^{\infty}$ in E', $(\lambda_j) \in l^1$ and $(y_j)_j$, α-bounded sequence in $\hat{F}$ (the completion of F) such that

$$p(x) = \sum_{j=1}^{\infty} \lambda_j \phi_j^n(x) y_j$$

for all $x \in E$. We denote by $P_N({}^nE\,;F)$ the space of all nuclear n-homogeneous polynomial mappings from E into F. If A is a subset of E and $\alpha \in cs(F)$, then we set

$$\pi_{N,A,\alpha}(p) = \|p\|_{N,A,\alpha} = \inf\left\{\sum_{j=1}^{\infty} |\lambda_j| \|\phi_j\|_A^n \alpha(y_i)\,;\, p = \sum_{j=1}^{\infty} \lambda_j \phi_j^n y_j\right\}$$

for $p \in P_N({}^nE\,;F)$. In this section after this we assume that the space F is a Banach space, and we use the notation $\pi_{N,A}$ in place of $\pi_{N,A,\alpha}$. Let f be a holomorphic mapping from E into F. We denote by $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}$ the Taylor series expansion of f at 0 in E. An $f \in H(E\,;F)$ is said to be of *nuclear type* if $\hat{d}^n f(0) \in P_N({}^nE\,;F)$ for all n. The set of all holomorphic mappings of nuclear type from E into F is denoted by $H_N(E\,;F)$. For an $f \in H_N(E\,;F)$ and an balanced convex subset A of E we set

$$\pi_{N,A}(f) = \sum_{n=0}^{\infty} \pi_{N,A}\left(\frac{\hat{d}^n f(0)}{n!}\right).$$

Let V be a convex balanced subset of a locally convex space E. We set

$$H_N^{\infty}(V\,;F) = \{f \in H(V\,;F)\,;\, \hat{d}^n f(0) \in P_N({}^nE\,;F) \text{ for all } n \text{ and } \pi_{N,V}(f) < \infty\}.$$

If $\mathcal{V} = (V_n)_{n=1}^{\infty}$ is an increasing countable balanced open cover of E, let

$$H_{N,\mathcal{V}}(E\,;F) = \{f \in H(E\,;F)\,;\, f|V_n \in H_N^{\infty}(V_n\,;F) \quad \text{for all } n\}.$$

We have

$$H_N(E\,;F) \supset \bigcup_{\mathcal{V}} H_{N,\mathcal{V}}(E\,;F)$$

where $\mathcal{V}$ ranges over all increasing countable convex balanced open covers of E. For every convex balanced open neighborhood U, let

$$\mathcal{V}(U) = \{nU\,;\, n \geq 1\}.$$

We set

$$H_{N,ub}(E\,;F) = \bigcup_{U} H_{N,\mathcal{V}(U)}(E\,;F)$$

where U ranges over all convex balanced open neighborhood of 0 in E. An $f \in H(E;F)$ is said to be of *nuclear uniformly bounded type* if $f \in H_{N,ub}(E;F)$. An $f \in H_N(E;F)$ is of *nuclear uniformly bounded type* if and only if there exists a convex balanced open subset V 0f E such that

$$\limsup_{n\to\infty} \sqrt[n]{\pi_{N,V}\left(\frac{\hat{d}^n f(0)}{n!}\right)} = 0.$$

We have the following relation:

$$H_{N,ub}(E;F) \subset H_N(E;F) \quad \text{and} \quad H_{N,ub}(E;F) \subset H_{ub}(E;F).$$

LEMMA 3.1. *Let E be a closed subspace of a locally convex space G and F be a Banach space. If $p \in P_N(^nE;F)$ and $\pi_{N,W}(p) < \infty$ then for any neighborhood V of 0 in G such that $V \cap E \subset W$ there is $\tilde{p} \in P_N(^nG;F)$ such that*

$$\pi_{N,V}(\tilde{p}) = \pi_{N,V\cap E}(p)$$

and

$$\tilde{p}|_E = p.$$

Proof. Let V an arbitrary balanced convex open neighborhood of 0 in G such that $V \cap E \subset W$ and let α be the Minkovski functional of V. By hypothesis there exists $(\phi_j)_j \subset E'$ satisfying $\|\phi_j\|_{V\cap E} = 1$ for every j, $(\lambda_j)_j \in l^1$ and a bounded sequence $(y_j)_j$ in F such that

$$p = \sum_{j=1}^{\infty} \lambda_j \phi_j^n y_j.$$

Then we have

$$\|\phi_j(x)\| \le \alpha(x) \quad \text{for every } x \in E.$$

By the Hahn-Banach Theorem there exists $(\tilde{\phi}_j)_j \subset G'$ such that

$$\tilde{\phi}_j|_E = \phi_j$$

and

$$\|\tilde{\phi}_j(x)\| \le \alpha(x) \quad \text{for every } x \in G.$$

Thus we have $\|\tilde{\phi}_j\|_V = 1$. If we set

$$\tilde{p} = \sum_{j=1}^{\infty} \lambda_j \tilde{\phi}_j^n y_j,$$

then $\tilde{p} \in P(^nG;F)$, $\tilde{p}|_E = p$ and $\pi_{N,V}(\tilde{p}) = \pi_{N,V\cap E}(p)$. This completes the proof. □

THEOREM 3.1. *Let E, F and G be locally convex space. We assume that E is a closed subspace of G and F is a Banach space. Let f be a holomorphic mapping from E into F which is of nuclear uniformly bounded type. Then there exists a holomorphic mapping $\tilde{f}$ of nuclear uniformly bounded type from G into F which is of nuclear uniformly bounded type such that $\tilde{f}|_E = f$.*

Proof. There exists a sequence (p_n) of continuous n-homogeneous nuclear polynomials from E into F and a convex balanced open neighborhood V of 0 in G such that $f(x) = \sum_{n=0}^{\infty} p_n(x)$ for every $x \in E$ and $\limsup_{n\to\infty} \sqrt[n]{\pi_{N,V\cap E}(p_n)} = 0$. By Lemma 3.1, there exists $\tilde{p_n} \in P_N({}^nG\,;\,F)$ for every n such that $\tilde{p}_n|_E = p_n$ and $\pi_{N,V\cap E}(p_n) = \pi_{N,V}(\tilde{p}_n)$ for every n. Thus we have

$$\limsup_{n\to\infty} \sqrt[n]{\pi_{N,V}(\tilde{p}_n)} = 0.$$

Therefore if we define a holomorphic mapping $\tilde{f} : G \to F$ by $\tilde{f} = \sum_{n=0}^{\infty} \tilde{p}_n$, then $\tilde{f} \in H_{N,ub}(G\,;\,F)$ and $\tilde{f}|_E = f$. This completes the proof. □

A locally convex space E is said to be *nuclear* if for each $\gamma \in cs(E)$ there exist $\alpha \in cs(E)$ with $\alpha \geq \gamma$, $(\lambda_j) \in l^1$, $(\phi_j)_j \subset E'$ bounded in $(E_\alpha)'$and $(y_j)_j$ bounded in E_γ such that

$$x = \sum_{j=1}^{\infty} \lambda_j\, \phi_j(x)\, y_j$$

for every $x \in E$ where the series converges in E_γ. The following Proposition can be proved by following the proof in Proposition 2.12 of Dineen [4].

PROPOSITION 3.1. *Let E be a nuclear locally convex space. For every $\gamma \in cs(E)$ there exist $\alpha \in cs(E)$ and a positive constant $C(\gamma,\alpha)$ such that*

$$\|p\|_{N,B_\alpha(1)} \leq C(\gamma,\alpha)^n \|p\|_{B_\gamma(1)}$$

for every $p \in P({}^nE\,;\,F)$.

Proof. Since E is nuclear, there exist $\alpha \in cs(E)$ with $\alpha \geq \gamma$, $(\lambda_j) \in l^1$, $(\phi_j)_j \subset E'$ bounded in $(E_\alpha)'$and $(y_j)_j$ bounded in E_γ such that

$$x = \sum_{j=1}^{\infty} \lambda_j\, \phi_j(x)\, y_j$$

for every $x \in E$ where the series converges in E_γ. Let $p \in P({}^nE\,;\,F)$ with $\|p\|_{B_\gamma(1)} < \infty$. There exists a unique continuous symmetric n-linear mapping $\check{p}$ from the product space E^n into F such that $p(x) = \check{p}(x,\ldots,x)$ for every $x \in E$. We remark that

$$\begin{aligned}&\check{p}(\sum_{j=1}^{m}\lambda_j\,\phi_j(x)\,y_j,\ldots,\sum_{j=1}^{m}\lambda_j\,\phi_j(x)\,y_j)\\ = &\sum_{1\le k_1,\ldots,k_n\le m}\lambda_{k_1}\cdots\lambda_{k_n}\phi_{k_1}(x)\cdots\phi_{k_n}(x)\check{p}(y_{k_1},\ldots,y_{k_n}).\end{aligned}$$

We denote by $\boldsymbol{N}$ the set of all positive integers. We can estimate $\|\check{p}((y_{k_j}))\| = \|\check{p}(y_{k_1},\ldots,y_{k_n})\|$ and $\sum_{(k_1,\ldots,k_n)\in\boldsymbol{N}^n}\|\lambda_{k_1}\cdots\lambda_{k_n}\phi_{k_1}(x)\cdots\phi_{k_n}(x)\check{p}(y_{k_1},\ldots,y_{k_n})\|$ as follows:

$$\begin{aligned}\sup_{(k_1,k_2,\ldots,k_n)\in\boldsymbol{N}^n}\|\check{p}((y_{k_j}))\| &\le \frac{n^n}{n!}\|p\|_{B_\gamma(1)}\left(\sup_k\gamma(y_k)\right)^n\\ &\le e^n\|p\|_{B_\gamma(1)}\left(\sup_k\gamma(y_k)\right)^n\\ &= \left(e\sup_k\gamma(y_k)\right)^n\|p\|_{B_\gamma(1)}.\end{aligned}$$

$$\begin{aligned}&\sum_{(k_1,\ldots,k_n)\in\boldsymbol{N}^n}\|\lambda_{k_1}\cdots\lambda_{k_n}\phi_{k_1}(x)\cdots\phi_{k_n}(x)\check{p}(y_{k_1},\ldots,y_{k_n})\|\\ \le &\sum_{(k_1,\ldots,k_n)\in\boldsymbol{N}^n}|\lambda_{k_1}\cdots\lambda_{k_n}|\left(\sup_{k\in\boldsymbol{N}}\|\phi_k\|_{B_\alpha(1)}\right)^n\\ &\cdot\left(e\sup_k\gamma(y_k)\right)^n\|p\|_{B_\gamma(1)}\\ = &(\sum_{j=1}^{\infty}|\lambda_j|)^n\left(\sup_{k\in\boldsymbol{N}}\|\phi_k\|_{B_\alpha(1)}\right)^n\\ &\cdot\left(e\sup_k\gamma(y_k)\right)^n\|p\|_{B_\gamma(1)}\quad\text{for every } x\in B_\alpha(1).\end{aligned}$$

Thus $\sum_{(k_1,\dots,k_n)\in \boldsymbol{N}^n} \lambda_{k_1}\cdots\lambda_{k_n}\phi_{k_1}(x)\cdots\phi_{k_n}(x)\check{p}(y_{k_1},\dots,y_{k_n})$ is absolutely convergent and

$$p(x) = \check{p}(\sum_{j=1}^{\infty}\lambda_j\,\phi_j(x)\,y_j,\dots,\sum_{j=1}^{\infty}\lambda_j\,\phi_j(x)\,y_j).$$

By the polarization formula for the polynomial $\zeta\to\zeta^n$ $(\zeta\in \boldsymbol{C})$ it holds that

$$\phi_{k_1}(x)\cdots\phi_{k_n}(x) = \frac{1}{2^n n!}\sum_{(\epsilon_1,\dots,\epsilon_n)\in\{-1,1\}^n}\epsilon_1\cdots\epsilon_n\left(\sum_{j=1}^{n}\epsilon_j\phi_{k_j}(x)\right)^n.$$

Therefore we have

$$\begin{aligned}
&\sum_{(k_1,\dots,k_n)\in \boldsymbol{N}^n}\lambda_{k_1}\cdots\lambda_{k_n}\phi_{k_1}(x)\cdots\phi_{k_n}(x)\check{p}(y_{k_1},\dots,y_{k_n})\\
=&\sum_{(k_1,\dots,k_n)\in \boldsymbol{N}^n}\lambda_{k_1}\cdots\lambda_{k_n}\\
&\cdot\left(\frac{1}{2^n n!}\sum_{(\epsilon_1,\dots,\epsilon_n)\in\{-1,1\}^n}\epsilon_1\cdots\epsilon_n\left(\sum_{j=1}^{n}\epsilon_j\phi_{k_j}(x)\right)^n\right)\check{p}(y_{k_1},\dots,y_{k_n})\\
=&\sum_{(k_1,\dots,k_n)\in \boldsymbol{N}^n}\ \sum_{(\epsilon_1,\dots,\epsilon_n)\in\{-1,1\}^n}\frac{1}{2^n}\lambda_{k_1}\cdots\lambda_{k_n}\epsilon_1\cdots\epsilon_n\\
&\cdot\left(\frac{1}{\sqrt[n]{n!}}\sum_{j=1}^{n}\epsilon_j\phi_{k_j}(x)\right)^n\check{p}(y_{k_1},\dots,y_{k_n}).
\end{aligned}$$

For every $\boldsymbol{k}=(k_l)_{l=1}^n\in \boldsymbol{N}^n$ and $\boldsymbol{\epsilon}=(\epsilon_j)_{j=1}^n\in\{-1,\,1\}^n$ we set

$$\delta_{\boldsymbol{k},\boldsymbol{\epsilon}} = \frac{1}{2^n}\lambda_{k_1}\cdots\lambda_{k_n}\epsilon_1\cdots\epsilon_n,$$

$$\psi_{\boldsymbol{k},\boldsymbol{\epsilon}} = \sum_{j=1}^{n}\frac{1}{\sqrt[n]{n!}}\epsilon_j\phi_j$$

$$z_{\boldsymbol{k},\boldsymbol{\epsilon}} = (y_{k_1},\dots,y_{k_n}).$$

Then we have

$$\begin{aligned}
&\sum_{(k_1,\dots,k_n)\in \boldsymbol{N}^n} \lambda_{k_1}\cdots\lambda_{k_n}\phi_{k_1}(x)\cdots\phi_{k_n}(x)\check{p}(y_{k_1},\dots,y_{k_n}) \\
=& \sum_{(\boldsymbol{k},\boldsymbol{\epsilon})\in \boldsymbol{N}^n\times\{-1,1\}^n} \delta_{\boldsymbol{k},\boldsymbol{\epsilon}}\psi^n_{\boldsymbol{k},\boldsymbol{\epsilon}}(x)\check{p}(z_{\boldsymbol{k},\boldsymbol{\epsilon}})
\end{aligned}$$

for every $x \in E$. Moreover it is valid that

$$\begin{aligned}
\sum_{(\boldsymbol{k},\boldsymbol{\epsilon})\in \boldsymbol{N}^n\times\{-1,1\}^n} \left|\delta_{\boldsymbol{k},\boldsymbol{\epsilon}}\right| &= \sum_{(\boldsymbol{k},\boldsymbol{\epsilon})\in \boldsymbol{N}^n\times\{-1,1\}^n} \frac{1}{2^n}\left|\lambda_{k_1}\cdots\lambda_{k_n}\right| \\
&= \sum_{\boldsymbol{k}\in\boldsymbol{N}^n} \left|\lambda_{k_1}\cdots\lambda_{k_n}\right| \\
&= \left(\sum_{j=1}^{\infty}|\lambda_j|\right)^n < +\infty,
\end{aligned}$$

$$\begin{aligned}
\|\psi^n_{\boldsymbol{k},\boldsymbol{\epsilon}}\|_{B_\alpha(1)} &= \left\|\frac{1}{n!}\left(\sum_{j=1}^{n}\epsilon_j\psi_j\right)^n\right\|_{B_\alpha(1)} \\
&\le \frac{1}{n!}\left(\sum_{j=1}^{n}\|\phi_j\|_{B_\alpha(1)}\right)^n \\
&\le \frac{n^n}{n!}\left(\sup_k\|\phi_k\|_{B_\alpha(1)}\right)^n \\
&\le e^n\left(\sup_k\|\phi_k\|_{B_\alpha(1)}\right)^n.
\end{aligned}$$

Thus we have

$$\|p\|_{N,\alpha} \le \left(\sum_{j=1}^{\infty}|\lambda_j|\right)^n \left(e\sup_k\|\phi_k\|_{B_\alpha(1)}\right)^n \left(e\sup_j\gamma(y_j)\right)^n \|p\|_{B_\gamma(1)}.$$

Here if we set

$$C = C(\alpha,\gamma) = e^2\left(\sum_{j=1}^{\infty}|\lambda_j|\right)\left(\sup_k\|\phi_k\|_{B_\alpha(1)}\right)\left(\sup_j\gamma(y_j)\right),$$

then

$$\|p\|_{N,B_\alpha(1)} \le C^n\|p\|_{B_\gamma(1)} \quad \text{for every } p\in P({}^nE\,;\,F).$$

This completes the proof. □

We obtain the following Theorem from Proposition 3.1.

THEOREM 3.2. *Let E be a nuclear locally convex space and let F be a Banach space. Then $f \in H(E\,;F)$ is of uniformly bounded type if and only if f is of nuclear uniformly bounded type.*

Proof. We assume that f is of uniformly bounded type. Then there exists $\gamma \in cs(E)$ such that

$$\limsup_{n\to\infty} \sqrt[n]{\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{B_\gamma(1)}} = 0.$$

By Proposition 3.1 there exist $\alpha \in cs(E)$ and a positive constant $C(\gamma,\alpha)$ such that

$$\sqrt[n]{\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{N,B_\alpha(1)}} \leq C(\gamma,\alpha)\sqrt[n]{\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{B_\gamma(1)}} \quad \text{for every } n.$$

This implies

$$\limsup_{n\to\infty} \sqrt[n]{\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{N,B_\alpha(1)}} = 0.$$

This implies that f is of nuclear uniformly bounded type. The converse can be proved from the inequality

$$\|p\|_{B_\alpha(1)} \leq \|p\|_{N,B_\alpha(1)}$$

for every $n \in \boldsymbol{N}$, $\alpha \in cs(E)$ and $p \in P(^nE\,;F)$. □

LEMMA 3.2. *Let E be a locally convex space and let F be a Banach space. Let I be an indexing set. Then for every $f \in H(E^I\,;F)$ there exist a finite subset J of I and $f_J \in H(E^J\,;F)$ such that $f = f_J \circ \delta_J$ where δ_J means the canonical projection of E^I onto E^J.*

Proof. There exists a balanced convex neighborhood W such that $\|f\|_W < \infty$. By the definition of the product topology there exist a finite set J and a convex balanced neighborhood V such that $V \times E^{I\setminus J} \subset W$. By Liouville Theorem and the identity Theorem f is constant with respect to the direction $E^{I\setminus J}$. Thus we can define $f_J \in H(E^J\,;F)$ by

$$f_J(x) = f \circ \delta_J^{-1}(x) \quad \text{for every } x \in E^J.$$

Then f_J satisfies all required conditions. This completes the proof. □

THEOREM 3.3 (Meise-Vogt [7]). *Let E, F and G be locally convex spaces over $\boldsymbol{C}$. We assume that E is a nuclear space and F is a Banach space. Let f be a holomorphic mapping from E into F.Then if for an arbitrary locally convex space of Hilbert type G containing E as a closed subspace f can be extended to a holomorphic mapping from G into F, then f is of uniformly bounded type.*

Proof. Meise-Vogt [7] proved that the nuclear space E can be realized as a subspace of L^I where I is some indexing set and L is a DFS-space of Hilbert type. By the assumption, there exists $g \in H(L^I\,;\,F)$ such that $g|_E = f$. By Lemma 3.2, there exist a finite subset J of I and $g_J \in H(L^J\,;\,F)$ such that $g = g_J \circ \delta_J$ where δ_J means the canonical projection of L^I onto L^J. Since L^J is a DFS-space, $H(L^J\,;\,F) = H_{ub}(L^J\,;\,F)$. Thus there exists a convex balanced open neighborhood U of 0 in L^J such that $\|g_J\|_{r\,U} < \infty$ for every $r > 0$. Since $\|g\|_{\delta_J^{-1}(r\,U)} = \|g_J\|_{r\,U} < \infty$ for every $r > 0$, $\|f\|_{r\,(\delta_J^{-1}(U)\cap E)} = \|f\|_{\delta_J^{-1}(r\,U)\cap E} = \|g\|_{\delta_J^{-1}(r\,U)\cap E} < \infty$. This implies that f is of uniformly bounded type. This completes the proof. □

We can complete the proof of Main Theorem by Theorems 3.1, 3.2 and 3.3.

Proof of Main Theorem. (b)⇒(a), (a)⇒(c), (b)⇒(d) and (d)⇒(c) are trivial. The equivalence (f) $\Longleftrightarrow$ (e) follows from Theorem 3.2. It follows from Theorem 3.1 that (f) implies (b). It follows from Theorem 3.3 that (c) implies (e). Thus all statements are equivalent each other. This completes the proof. □

COROLLARY 3.1. *Let E and F be locally convex spaces over $\boldsymbol{C}$. We assume that E is a nuclear space and F is a Banach space. Let f be a holomorphic mapping from E into F. Then the following statements are equivalent:*

(a) *for an arbitrary locally convex space G containing E as a closed subspace, the restriction mapping $H(G\,;\,F) \to H(E\,;\,F)$ is surjective,*
(b) $H(E\,;\,F) = H_{ub}(E\,;\,F)$.

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