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A NOTE ON WEIGHTED COMPOSITION OPERATORS ON MEASURABLE FUNCTION SPACES

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Abstract. In this paper we will consider the weighted composition operators $W = uC_\tau$ between $L^p(X, \Sigma, \mu)$ spaces and Orlicz spaces $L^\varphi(X, \Sigma, \mu)$, generated by measurable and non-singular transformations τ from X into itself and measurable functions u on X. We characterize the functions u and transformations τ that induce weighted composition operators between L^p-spaces by using some properties of conditional expectation operator, pair (u, τ) and the measure space (X, Σ, μ). Also, some other properties of these types of operators will be investigated.

1. Preliminaries and notation

Let (X, Σ, μ) be a sigma finite measure space. By $L(X)$, we denote the linear space of all Σ-measurable functions on X. When we consider any subsigma algebra $\mathcal{A}$ of Σ, we assume they are completed; i.e., $\mu(A) = 0$ implies $B \in \mathcal{A}$ for any $B \subset A$. For any sigma finite algebra $\mathcal{A} \subseteq \Sigma$ and $1 \leq p \leq \infty$ we abbreviate the L^p-space $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ to $L^p(\mathcal{A})$, and denote its norm by $\|.\|_p$. We define the support of a measurable function f as $\sigma(f) = \{x \in X; f(x) \neq 0\}$. We understand $L^p(\mathcal{A})$ as a subspace of $L^p(\Sigma)$ and as a Banach space. Here functions which are equal μ-almost everywhere are identical. An atom of the measure μ is an element $B \in \Sigma$ with $\mu(B) > 0$ such that for each $F \in \Sigma$, if $F \subset B$ then either $\mu(F) = 0$ or $\mu(F) = \mu(B)$. A measure with no atoms is called non-atomic. We can easily check the following well known facts (see [12]):

(a) every sigma finite measure space (X, Σ, μ) can be decomposed into two disjoint sets B and Z, such that μ is a non-atomic over B and Z is a countable union of atoms of finite measure,

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(b) for each $f \in L^r(\Sigma)$, there exist two functions $f_1 \in L^p(\Sigma)$ and $f_2 \in L^q(\Sigma)$ such that $f = f_1 f_2$ and $\|f\|_r^r = \|f_1\|_p^p = \|f_2\|_q^q$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Associated with each sigma algebra $\mathcal{A} \subseteq \Sigma$, there exists an operator $E(\cdot|\mathcal{A}) = E^{\mathcal{A}}(\cdot)$, which is called *conditional expectation* operator, on the set of all non-negative measurable functions f or for each $f \in L^p$ for any p, $1 \leq p \leq \infty$, and is uniquely determined by the conditions

(i) $E^{\mathcal{A}}(f)$ is $\mathcal{A}$- measurable, and

(ii) if A is any $\mathcal{A}$- measurable set for which $\int_A f d\mu$ exists, we have $\int_A f d\mu = \int_A E^{\mathcal{A}}(f) d\mu$.

This operator is at the central idea of our work, and we list here some of its useful properties:

E1. $E^{\mathcal{A}}(f.g \circ T) = E^{\mathcal{A}}(f)(g \circ T)$,

E2. $E^{\mathcal{A}}(1) = 1$,

E3. $|E^{\mathcal{A}}(fg)|^2 \leq E^{\mathcal{A}}(|f|^2)E^{\mathcal{A}}(|g|^2)$,

E4. if $f > 0$ then $E^{\mathcal{A}}(f) > 0$,

E5. as an operator on $L^p(\Sigma)$, $E^{\mathcal{A}}$ is the orthogonal projection onto $L^p(\mathcal{A})$.

Properties E1 and E2 imply that $E^{\mathcal{A}}(\cdot)$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. Suppose that τ is a mapping from X into X which is measurable, (i.e., $\tau^{-1}(\Sigma) \subseteq \Sigma$) such that $\mu \circ \tau^{-1}$ is absolutely continuous with respect to μ (we write $\mu \circ \tau^{-1} \ll \mu$, as usual). Let h be the Radon-Nikodym derivative $h = \frac{d\mu \circ \tau^{-1}}{d\mu}$. If we put $\mathcal{A} = \tau^{-1}(\Sigma)$, it is easy to show that for each non-negative Σ-measurable function f or for each $f \in L^p(\Sigma)$ ($p \geq 1$), there exists a Σ-measurable function g such that $E^{\tau^{-1}(\Sigma)}(f) = g \circ \tau$. We can assume that the support of g lies in the support of h, and there exists only one g with this property. We then write $g = E^{\tau^{-1}(\Sigma)}(f) \circ \tau^{-1}$, though we make no assumptions regarding the invertibility of τ (see [2]). For a deeper study of the properties of E see the paper [8].

2. Some results on weighted composition operators between two L^p-spaces

Let $1 \leq q \leq p < \infty$ and we define $\mathcal{K}_{p,q}$ or $\mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$ as follows:

$$\mathcal{K}_{p,q} = \{u \in L(X) : uL^p(\mathcal{A}) \subseteq L^q(\Sigma)\}.$$

$\mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$ is a vector subspace of $L(X)$. Also note that if $1 \leq q = p < \infty$, then $L^\infty(\Sigma) \subseteq \mathcal{K}_{p,p}(\mathcal{A}, \Sigma)$ and $\mathcal{K}_{p,p}(\Sigma, \Sigma) = L^\infty(\Sigma)$ (see [3]; problem 64, 65).

For $u \in L(X)$, let M_u from $L^p(\mathcal{A})$ into $L(X)$ defined by $M_u f = u.f$ be the corresponding linear transformation. An easy consequence of the closed graph theorem and the result guaranteeing a pointwise convergent subsequence for each L^p convergent sequence assures us that for each $u \in \mathcal{K}_{p,q}(\mathcal{A}, \Sigma)$, the operator $M_u : L^p(\mathcal{A}) \to L^q(\Sigma)$ is a multiplication operator (bounded linear transformation).

We shall find the relationship between a sigma finite algebra $\mathcal{A} \subseteq \Sigma$ and the set of multiplication operators which map $L^p(\mathcal{A})$ into $L^q(\Sigma)$. Our first task is the description of the members of $\mathcal{K}_{p,q}$ in terms of the conditional expectation induced by $\mathcal{A}$.

THEOREM 2.1. *Suppose* $1 \leq q < p < \infty$ *and* $u \in L(X)$. *Then* $u \in \mathcal{K}_{p,q}$ *if and only if* $(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$, *where* $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$.

Proof. To prove the theorem, we adopt the method used by Axler [1]. Suppose $(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$, so $E^{\mathcal{A}}(|u|^q) \in L^{\frac{r}{q}}(\mathcal{A})$. For each $f \in L^p(\mathcal{A})$, we have $|f|^q \in L^{\frac{p}{q}}(\mathcal{A})$. Since $\frac{q}{p} + \frac{q}{r} = 1$, Hölder's inequality yields

$$\begin{aligned}
\|u.f\|_q &= \int |u|^q |f|^q d\mu \\
&= \int E^{\mathcal{A}}(|u|^q)|f|^q d\mu \\
&\leq \left\{ \left(\int E^{\mathcal{A}}(|u|^q)^{\frac{r}{q}} d\mu \right)^{\frac{q}{r}} \left(\int |f|^{q\frac{p}{q}} d\mu \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\
&= \|E^{\mathcal{A}}(|u|^q)\|_r \|f\|_p.
\end{aligned}$$

Hence $u \in \mathcal{K}_{p,q}$. Now suppose only that $u \in \mathcal{K}_{p,q}$. So the operator $M_u : L^p(\mathcal{A}) \to L^q(\Sigma)$ given by $M_u f = u.f$ is a bounded linear operator and $\|M_u\| < \infty$. Let r' and q' be conjugates to r and q respectively. We define a linear functional Φ on $L^{r'}(\mathcal{A})$ by

$$\Phi(f) = \int f(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}} d\mu \qquad (f \in L^{r'}(\mathcal{A})).$$

From $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$ we have $\frac{1}{p} + \frac{1}{q'} = \frac{1}{r'}$. So we can find two functions $f_1 \in L^p(\mathcal{A})$ and $f_2 \in L^{q'}(\mathcal{A})$ such that

$$f = f_1 f_2 \quad \text{and} \quad \|f\|_{r'}^{r'} = \|f_1\|_p^p = \|f_2\|_{q'}^{q'}.$$

By the following computations we show that Φ is a bounded linear functional.

$$\begin{aligned}|\Phi(f)| &= \left|\int f(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}} d\mu\right| \leq \int \left(|f|^q E^{\mathcal{A}}(|u|^q)\right)^{\frac{1}{q}} d\mu \leq \int |fu| d\mu \\ &= \int |M_u f_1||f_2| d\mu \leq \|M_u f_1\|_q \ \|f_2\|_{q'} \leq \|M_u\| \ \|f_1\|_p \ \|f_2\|_{q'} \\ &= \|M_u\| \ \|f\|_{r'}^{\frac{r'}{p}} \ \|f\|_{r'}^{\frac{r'}{q'}} \\ &= \|M_u\| \ \|f\|_{r'}.\end{aligned}$$

Now, since Φ is a bounded linear functional on $L^{r'}(\mathcal{A})$, by the Riesz representation theorem there exists $v \in L^r(\mathcal{A})$ such that

$$\Phi(f) = \int fv d\mu \quad \text{for all } f \in L^{r'}(\mathcal{A}) \quad \text{and} \quad \|\Phi\| = \|v\|_r.$$

Hence $\|(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}}\|_r = \|v\|_r = \|\Phi\| \leq \|M_u\| < \infty$ and so $(E^{\mathcal{A}}(|u|^q))^{\frac{1}{q}} \in L^r(\mathcal{A})$. □

COROLLARY 2.2. *Suppose* $1 \leq q < p < \infty$ *and* $u \in L(X)$. *Then* M_u *from* $L^p(\Sigma)$ *into* $L^q(\Sigma)$ *is bounded linear operator if and only if* $u \in L^{\frac{pq}{p-q}}(\Sigma)$. *In this case* $\|M_u\| = \|u\|_{\frac{pq}{p-q}}$.

Proof. Put $\mathcal{A} = \Sigma$ in the previous theorem. Then we will have $E^{\mathcal{A}} = I$ (identity operator). Then the proof holds. □

Let $u \in L(X)$ and $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. If $p = q$ then r must be ∞. So $M_u(L^p(\Sigma)) \subseteq L^p(\Sigma)$ if and only if $u \in L^\infty(\Sigma)$. In this case $\|M_u\| = \|u\|_\infty$. This fact is well-known. For the direct proof, see [3].

Take a function u in $L(X)$ and let $\tau : X \to X$ be a non-singular measurable transformation; i.e. $\mu(\tau^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. Then the pair (u, τ) induces a linear operator $W = uC_\tau$ from $L^p(\Sigma)$ into $L(X)$ defined by

$$W(f) = u.f \circ \tau \qquad (f \in L^p(\Sigma)).$$

Here, the non-singularity of τ guarantees that W is well defined as a mapping of equivalence classes of functions on support u. If W takes $L^p(\Sigma)$ into $L^q(\Sigma)$, then we call W a weighted composition operator $L^p(\Sigma)$ into $L^q(\Sigma)$ $(1 \leq q \leq \infty)$.

Boundedness of composition operators in $L^p(\Sigma)$ spaces $(1 \leq p \leq \infty)$ where measure spaces are sigma finite appeared already in Singh paper [10] and for two different $L^p(\Sigma)$ spaces in the paper [11]. Also boundedness of weighted composition operators on $L^p(\Sigma)$ spaces has already been studied in [4]. Namely, for a non-singular measurable transformation τ and complex valued measurable weight function u on X, W is bounded if and only if $hE^{\tau^{-1}(\Sigma)}(|u|^p) \circ \tau^{-1} \in L^\infty(\Sigma)$. In the following theorem we give a necessary and sufficient condition for boundedness of weighted composition operators from $L^p(\Sigma)$ into $L^q(\Sigma)$, where $p > q$ as follows:

THEOREM 2.3. *Suppose* $1 \leq q < p < \infty$ *and* $\frac{1}{p}+\frac{1}{r} = \frac{1}{q}$. *Let* $u \in L(X)$ *and* $\tau : X \to X$ *be a non-singular measurable transformation. Then the pair* (u, τ) *induces a weighted composition operator* W *from* $L^p(\Sigma)$ *into* $L^q(\Sigma)$ *if and only if* $J = hE^{\tau^{-1}(\Sigma)}(|u|^q) \circ \tau^{-1} \in L^{\frac{r}{q}}(\Sigma)$.

Proof. Let $f \in L^p(\Sigma)$. We will have

$$\begin{aligned}\|Wf\|_q^q &= \int |u.f \circ \tau|^q d\mu \\ &= \int hE^{\tau^{-1}(\Sigma)}(|u|^q) \circ \tau^{-1} |f|^q d\mu \\ &= \int |\sqrt[q]{J} f|^q d\mu \\ &= \|M_{\sqrt[q]{J}} f\|_q^q.\end{aligned}$$

So by Corollary 2.2, W is a weighted composition operator from $L^p(\Sigma)$ into $L^q(\Sigma)$ if and only if $\sqrt[q]{J} \in L^r(\Sigma)$ or equivalently $J \in L^{\frac{r}{q}}(\Sigma)$. □

COROLLARY 2.4. *Suppose* $1 \leq p \leq \infty$, $u \in L(X)$ *and* $\tau : X \to X$ *be a non-singular measurable transformation. Then the pair* (u, τ) *induces a weighted composition operator* W *from* $L^p(\Sigma)$ *into* $L^p(\Sigma)$ *if and only if* $hE^{\tau^{-1}(\Sigma)}(|u|^p) \circ \tau^{-1} \in L^\infty(\Sigma)$.

COROLLARY 2.5. *Under the same assumptions as in theorem 2.3,* τ *induces a composition operator* $C_\tau : L^p(\Sigma) \to L^q(\Sigma)$ *if and only if* $h \in L^{\frac{r}{q}}(\Sigma)$.

REMARK 2.6. One of the interesting features of a weighted composition operator is that the composition operator alone may not define a bounded operator between two $L^p(\Sigma)$ spaces. As an example, let X be $[0, 1]$, Σ the Borel sets, and μ Lebesgue measure. Let τ be the map $\tau(x) = x^3$ on $[0, 1]$. A simple computation shows that $h = 1/3x^{-2/3} \notin L^3(\Sigma)$. Then C_τ does not define a bounded operator from $L^3(\Sigma)$ into $L^2(\Sigma)$. However with $u(x) = x$, we have $\tau^{-1}(\Sigma) = \Sigma$ (so $E = I$) and $J = 1/3 \in L^3(\Sigma)$. Hence $W = M_u \circ C_\tau$ is bounded operator from $L^3(\Sigma)$ into $L^2(\Sigma)$.

The procedure which Axler has used for the case $p < q$ in [1], when X is the interval $[-\pi, \pi]$, can also be used here.

At this stage we investigate a necessary and sufficient condition for a multiplication operator to be compact. For a bounded linear operator A on a Banach space, we use the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ to denote the kernel and the range of A, respectively. We recall that A is said to be a Fredholm operator if $\mathcal{R}(A)$ is closed and if $\dim\mathcal{N}(A) < \infty$ and $\operatorname{codim}\mathcal{R}(A) < \infty$. Now we attempt to prove a theorem which is also likely to be found also elsewhere.

THEOREM 2.7. *Suppose that μ is a non-atomic measure on $L^2(\Sigma)$. Then the following conditions are equivalent:*

(a) *M_u is an invertible operator,*

(b) *M_u is a Fredholm operator,*

(c) *$\mathcal{R}(M_u)$ is closed and* $\operatorname{codim}\mathcal{R}(M_u) < \infty$,

(d) *$|u| \geq \delta$ almost everywhere on X for some $\delta > 0$.*

Proof. The implications (d) $\Longrightarrow$ (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) are obvious. We show (c) $\Longrightarrow$ (d).

Suppose that $\mathcal{R}(M_u)$ is closed and codim $\mathcal{R}(M_u) < \infty$. Then there exists a $\delta > 0$ such that $|u| \geq \delta$ on $\sigma(u)$. So it is enough to show that $\mu(\sigma(u)^c) = 0$. First of all we prove that M_u is onto. Let $0 \neq f_0 \in \mathcal{R}(M_u)^{\perp}$, therefore, for any $f \in L^2(\Sigma)$ we have $(M_u f, f_0) = 0$. Now we choose $t > 0$ such that the set

$$Z_t = \{s \in X : |f_0|^2(x) \geq t\}$$

is of positive measure. Since μ is a non-atomic measure we may choose a sequence of disjoint subsets Z_n of Z_t such that $0 < \mu(Z_n) < \infty$. Now let $g_n = \chi_{Z_n}.f_0$. It is clear that each g_n is non-zero element of $L^2(\Sigma)$, and for $n \neq m$, $(g_n, g_m) = 0$. Therefore, for $f \in L^2(\Sigma)$ we have

$$(f, M_u^* g_n) = (M_u f, \chi_{Z_n} f_0) = (M_u \chi_{Z_n} f, f_0) = 0.$$

So $g_n \in \mathcal{N}(M_u^*)$ for any n. Therefore, $\{g_n\}$ is a linearly independent subset of $\mathcal{N}(M_u^*)$, which is contradicts to $\dim \mathcal{N}(M_u^*) = \operatorname{codim} \mathcal{R}(M_u) < \infty$. If $\mu(\sigma(u)^c) > 0$, then there exists a set $Z \subset \sigma(u)^c$ such that $0 < \mu(Z) < \infty$, so we conclude that $\chi_Z \in L^2(\Sigma) \backslash \mathcal{R}(M_u)$, which contradicts the fact that M_u is onto. Therefore $\mu(\sigma(u)^c) = 0$. □

COROLLARY 2.8. *M_u is a Fredholm operator if and only if M_u^n $(= M_{u^n})$ is so.*

3. Weighted composition operators on Orlicz spaces

Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous convex function which satisfies the following conditions

(i) $\varphi(x) = 0$ if and only if $x = 0$,

(ii) $\lim_{x\to\infty} \varphi(x) = \infty$.

Such a function is called a Young function. Let (X, Σ, μ) be a sigma finite complete measure space. Let

$$L^{\varphi}(\mu) = \{f : X \to C \text{ measurable} : \int_X \varphi(\lambda|f|)d\mu < \infty \ \text{ for some } \ \lambda = \lambda(f) > 0\}.$$

The space $L^{\varphi}(\mu)$ is called an Orlicz space and is a Banach space with the norm defined by

$$\|f\|_{\varphi} = \inf\{\epsilon > 0 : \int_X \varphi(\frac{|f|}{\epsilon})d\mu \leq 1\}.$$

If $\varphi(x) = x^p$, $1 \leq p < \infty$, then $L^{\varphi}(\mu) = L^p(\mu)$, the well-known Banach space of p-integrable functions on X. Simple functions are not necessarily dense in $L^{\varphi}(\mu)$. But, if φ satisfies Δ_2-condition (i.e., $\varphi(2x) \leq k\varphi(x)$, $x > 0$, $k > 0$ constant), then simple functions are dense in $L^{\varphi}(\mu)$. For more literature concerning Orlicz spaces, we refer to Kufner, John and Fucik [6] and Rao [9].

It is well-known that if $\|f\|_{\varphi} \leq 1$ then $I_{\varphi}(f) := \int \varphi(|f|)d\mu \leq \|f\|_{\varphi}$. So $\|f_n - f\|_{\varphi} \to 0$ implies that $I_{\varphi}(f_n - f) \to 0$ for a sequence $\{f_n\}$ in $L^{\varphi}(\mu)$. If φ satisfy the Δ_2-condition, then the converse of the above fact is also true. From now on we assume that φ satisfies Δ_2-condition, φ is nonsingular and h is a finite valued function.

In this section we will present some results on boundedness, closed rangeness and compactness of weighted composition operators on Orlicz spaces. Some results on boundedness, closed rangeness and compactness of multiplication and composition operators on Orlicz space $L^{\varphi}(\mu)$ were obtained on [5] and [7].

THEOREM 3.1. *Let W be a weighted composition transformation from $L^{\varphi}(\mu)$ into $L(X)$. If $W(L^{\varphi}(\mu)) \subseteq L^{\varphi}(\mu)$ then W is bounded and so is a weighted composition operator on $L^{\varphi}(\mu)$.*

Proof. Let $\{f_n\}$ be a sequence in $L^{\varphi}(\mu)$ which converges to f and $\{Wf_n\}$ converges to some $g \in L^{\varphi}(\mu)$. Then

(1) $$\|f_n - f\|_{\varphi} \to 0 \quad \text{and} \quad \|Wf_n - g\|_{\varphi} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence we can choose some positive number n_0 such that $\|f_n - f\|_\varphi \leq 1$ and $\|Wf_n - g\|_\varphi \leq 1$ for all $n \geq n_0$. Consequently $I_\varphi(f_n - f) \leq \|f_n - f\|_\varphi$ and $I_\varphi(Wf_n - g) \leq \|Wf_n - g\|_\varphi$. Then we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\varphi(|f_{n_k} - f|) \to 0$ a.e. on X. Since τ is nonsingular, $\varphi(|f_{n_k} \circ \tau - f \circ \tau|) \to 0$ a.e. on X so $\varphi(|u.f_{n_k} \circ \tau - u.f \circ \tau|) \to 0$ a.e. on X. Again from (1) we can find a subsequence $\{f_{n_{k'}}\}$ of $\{f_{n_k}\}$ such that $\varphi(|u.f_{n_{k'}} \circ \tau - g|) \to 0$ and $\varphi(|u.f_{n_{k'}} \circ \tau - u.f \circ \tau|) \to 0$ a.e. on X. Hence $I_\varphi(Wf_{n_{k'}} - g) \to 0$ and $I_\varphi(Wf_{n_{k'}} - Wf) \to 0$. Since φ satisfies Δ_2-condition, $\|Wf_{n_{k'}} - g\|_\varphi \to 0$ and $\|Wf_{n_{k'}} - Wf\|_\varphi \to 0$ a.e. on X and so $Wf = g$. Hence W is bounded by the closed graph theorem. □

THEOREM 3.2. *Let u and h be in L^∞. Then W is a weighted composition operator on $L^\varphi(\mu)$.*

Proof. Let $f \in L^\varphi(\mu)$ and $h > 1$, then

$$\int \varphi\left(\frac{|Wf|}{\|f\|_\varphi \|u\|_\infty \|h\|_\infty}\right) d\mu \leq \int \varphi\left(\frac{|f \circ \tau|}{\|f\|_\varphi \|h\|_\infty}\right) d\mu$$
$$= \int \varphi\left(\frac{|f|}{\|f\|_\varphi \|h\|_\infty}\right) h d\mu \leq \int \varphi\left(\frac{|f|}{\|f\|_\varphi}\right) d\mu \leq 1.$$

Therefore $\|Wf\|_\varphi \leq \|f\|_\varphi \|u\|_\infty \|h\|_\infty$. Similarly when $h \leq 1$ we have

$$\int \varphi\left(\frac{|Wf|}{\|f\|_\varphi \|u\|_\infty}\right) d\mu \leq 1.$$

Therefore in this case $\|Wf\|_\varphi \leq \|f\|_\varphi \|u\|_\infty$. □

THEOREM 3.3. *Let $W : L^\varphi(\mu) \to L(X)$ be a weighted composition transformation. If $\max\{h, E(|u|) \circ \tau^{-1}\} \in L^\infty$, then W is a bounded linear operator (weighted composition operator) from $L^\varphi(\mu)$ into itself.*

Proof. Let $f \in L^\varphi(\mu)$ and $h > 1$. Put $J = E(|u|) \circ \tau^{-1}$. Then

$$\int \varphi\left(\frac{|Wf|}{\|hJ\|_\infty \|f\|_\varphi}\right) d\mu$$
$$\leq \int \varphi\left(\frac{E(|u.f \circ \tau|)}{\|hJ\|_\infty \|f\|_\varphi}\right) d\mu$$
$$\leq \int \varphi\left(\frac{E(|u|) \circ \tau^{-1}.|f|}{\|hJ\|_\infty \|f\|_\varphi}\right) d\mu \circ \tau^{-1}$$
$$= \int h\varphi\left(\frac{J.|f|}{\|hJ\|_\infty \|f\|_\varphi}\right) d\mu$$

$$\leq \int \varphi\left(\frac{hJ.|f|}{\|hJ\|_\infty \|f\|_\varphi}\right) d\mu$$
$$\leq \int \varphi\left(\frac{|f|}{\|f\|_\varphi}\right) d\mu$$
$$\leq 1.$$

Therefore $\|Wf\|_\varphi \leq \|hJ\|_\infty \|f\|_\varphi$. Now let $h \leq 1$. By similar computation we have $\|Wf\|_\varphi \leq \|J\|_\infty \|f\|_\varphi$. So in both cases W is a bounded linear operator from $L^\varphi(\mu)$ into itself. □

COROLLARY 3.4. *If $\mu \circ \tau^{-1}(A) \leq M\mu(A)$ for some $M \geq 1$ and for all $A \in \Sigma$ with $\mu(A) < \infty$, then the composition transformation C_τ is a bounded linear operator from $L^\varphi(\mu)$ into itself.*

DEFINITION 3.5. A Young function φ is said to satisfy the E-condition if $E(\varphi \circ f) = \varphi \circ E(f)$, for all non-negative measurable functions.

Next we consider the following operator M_J from $L^\varphi(\mu)$ into $L(X)$

$$M_J f = J.f, \qquad f \in L^\varphi(\mu).$$

Let φ satisfy the E-condition. The operator M_J is closely related to W by the quantity

$$\|Wf\|_{\varphi,\mu} = \|M_J f\|_{\varphi,\mu\circ\tau^{-1}}, \qquad f \in L^\varphi(\mu) \tag{2}$$

which is obtained through the computation

$$\|Wf\|_{\varphi,\mu} = \inf\{\epsilon > 0 : \int \varphi\left(\frac{|u.f \circ \tau|}{\epsilon}\right) d\mu \leq 1\}$$
$$= \inf\{\epsilon > 0 : \int \varphi\left(\frac{E(|u|) \circ \tau^{-1}|f|}{\epsilon}\right) d\mu \circ \tau^{-1} \leq 1\}$$
$$= \|M_J f\|_{\varphi,\mu\circ\tau^{-1}}.$$

It is well known that M_J is a multiplication operator from $L^\varphi(\mu)$ into the Orlicz space $L^\varphi(\mu\circ\tau^{-1})$, that is, a bounded linear operator on $L^\varphi(\mu)$ into $L^\varphi(\mu \circ \tau^{-1})$, if and only if J is essentially bounded with respect to the measure $\mu \circ \tau^{-1}$.

The compactness, closed rangeness and Fredholmness of multiplication operators on Orlicz space $L^\varphi(\mu)$ have been studied by Komal and Shally Gupta in [5] as follows:

THEOREM 3.6. *Let M_u be a bounded linear operator on $L^\varphi(\mu)$. Then*

(i) M_u is compact if and only if for any $\epsilon > 0$, the restriction of $L^\varphi(\mu)$ to the set $\{x \in X : u(x) \geq \epsilon \text{ a.e. } [\mu]\}$ is finite dimensional.

(ii) M_u *has closed range if and only if there exists* $\delta > 0$ *such that* $|u(x)| \geq \delta$ *for* μ- *almost all* $x \in \sigma(u)$.

(iii) M_u *is Fredholm if and only if there exists* $\delta > 0$ *such that* $|u(x)| \geq \delta$ *for* μ- *almost all* $x \in X$.

Here we extend their results for weighted composition operators as follows.

THEOREM 3.7. *Let* W *be a weighted composition operator on* $L^{\varphi}(\mu)$ *and* φ *satisfy the* E- *condition. Then*

(i) W *is compact if and only if for any* $\epsilon > 0$, *the restriction of* $L^{\varphi}(\mu)$ *to the set* $\{x \in X : J(x) \geq \epsilon \text{ a.e. } [\mu \circ \tau^{-1}]\}$ *is finite dimensional.*

(ii) W *has closed range if and only if there exists* $\delta > 0$ *such that* $|J(x)| \geq \delta$ *for* $\mu \circ \tau^{-1}$- *almost all* $x \in \sigma(J)$.

(iii) W *is Fredholm if and only if there exists* $\delta > 0$ *such that* $|J(x)| \geq \delta$ *for* $\mu \circ \tau^{-1}$- *almost all* $x \in X$.

Proof. By (2) and previous theorem the proof is clear. □

With each Young function φ we can associate another continuous convex function $\psi : R^+ \to R^+$ defined for all $y \in R^+$ as $\psi(y) = \sup\{x|y| - \varphi(x) : x \geq 0\}$. The function ψ is called the complementary function to φ.

Let φ, ψ be a pair of complementary Young functions. Then each $g \in L^{\psi}(\mu)$ defines a bounded linear functional F_g on $L^{\varphi}(\mu)$ by

$$F_g(f) = \int fg d\mu \qquad f \in L^{\varphi}(\mu).$$

Moreover, the mapping $g \to F_g$ is a isometry from $L^{\psi}(\mu)$ onto $(L^{\varphi})^*(\mu)$, so the norm dual of $L^{\varphi}(\mu)$ can be identified with $L^{\psi}(\mu)$ [9].

THEOREM 3.8. *Let* W *be a weighted composition operator on* $L^{\varphi}(\mu)$. *Then* $W^*g = hE(u.g) \circ \tau^{-1}$ *for all* $g \in L^{\psi}(\mu)$.

Proof. Take $A \in \Sigma$ such that $0 < \mu(A) < \infty$. For $g \in L^{\psi}(\mu)$,

$$\begin{aligned}(W^*F_g)(\chi_A) &= F_g(W\chi_A) = \int (W\chi_A) g d\mu \\ &= \int u.\chi_A \circ \tau g d\mu = \int hE(u.g) \circ \tau^{-1} \chi_A d\mu \\ &= F_{hE(u.g)\circ\tau^{-1}} \chi_A.\end{aligned}$$

Hence, $W^*F_g = F_{hE(u.g)\circ\tau^{-1}}$. After identifying $(L^{\varphi})^*(\mu)$ with $L^{\psi}(\mu)$ and g with F_g, we can write $W^*g = hE(u.g) \circ \tau^{-1}$ for all $g \in L^{\psi}(\mu)$.

THEOREM 3.9. *Let W be a weighted composition operator on $L^{\psi}(\mu)$. If X is a non-atomic measure space, then the nullity of W is either zero or infinite.*

Proof. Replacing g by Wf in theorem 3.8. we get

$$W^*(Wf) = hE(u^2.f \circ \tau) \circ \tau^{-1} = hfE(u^2) \circ \tau^{-1} = Jf = M_J f,$$

where $J = hE(u^2) \circ \tau^{-1}$ and $f \in L^{\psi}(\mu)$. Let $X_o = \{x \in X : J(x) = 0\}$, since $KerW = KerW^*W = KerM_J = L^{\psi}(X_o)$ and X is non-atomic it follows that $\dim L^{\psi}(X_o)$ is zero if $\mu(X_o) = 0$ and infinite if $\mu(X_o) \neq 0$. □

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