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# ZEROS OF REAL POLYNOMIALS ON BANACH SPACES

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ABSTRACT. This paper is an expository presentation of a large part of the results, about zeros of real polynomials on Banach spaces, that have been obtained in recent years. Also new results, for orthogonally additive polynomials on $L_p$ spaces, are given.

## 1. Introduction

The study of the zeros of a complex polynomial has a long history, with results coming via complex analysis, algebraic geometry, and functional analysis. (See, e.g. [10], [11], [16]). Similar studies for real polynomials are somewhat less common. In this paper we survey a large part of the results on zeros of real polynomials that have been obtained in recent years. If $P$ and $P_n$ are $k$-homogeneous polynomials and $\{P_n\} \to P$ pointwise or uniformly on bounded sets, what about the convergence of the zero sets $Z(P_n)$? Section 1 is dedicated to this problem. We give results and examples to show the diference between the complex and the real case.

If $P$ is a homogeneous polynomial, what about the dimension of subspaces included in $Z(P) = P^{-1}(0)$? Section 2 deals with this question. Specifically treated is the case where the real Banach space does not admit a positive definite 2-homogeneous polynomial.

Finally section 3 is devoted to approximation by zeros of orthogonally additive polynomials on real $l_p$ and $L_p$ spaces. Results about the number of zeros that are involved in the decomposition of each $e_j$ in $l_p$ or each characteristic function in $L_p[0,1]$ are given.

Throughout, $X$ will be a Banach space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). $\mathcal{P}(^kX)$ will denote the Banach space of $k$-homogeneous continuous polynomials on

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$X$ under the norm:

$$\| P \| = \text{Sup}\{| P(x) | : x \in X, \| x \| \leq 1\}$$

(see [6], [14] for a general reference on polynomials on Banach spaces).

We will denote the closure of a set $A$ in a topological space by $cl(A)$.

## Section 1

Definition 1.1. ([7]). We say that a sequence of nonempty closed subsets of a Banach space $X$, $\{A_n\}$, converges in the Mosco sense to a closed subset $A$, [$(A_n \xrightarrow{M} A)$ for short ], whenever the following two conditions hold:

(i) for every $x \in A$ there exists a sequence $\{x_n\}$, norm convergent to $x$, such that $x_n \in A_n$ for every $n$,

(ii) given $J \subseteq \mathbb{Z}^+$ cofinal, for every sequence $\{x_{n_j}\}_{j \in J}$ weakly convergent to $x$, the condition $x_{n_j} \in A_{n_j}$ for every $j$, implies $x \in A$.

-*Kuratowski convergence* is defined in the same way changing weak convergence by norm convergence in (ii).

Notation: $A_n \xrightarrow{K} A$

Condition (i) is equivalent to $A \subset L_i A_n$ and (ii), in Kuratowski convergence, is equivalent to $L_s A_n \subset A$.

Recall that $x \in L_i A_n \Leftrightarrow \exists (x_n) : x = \lim\limits_n x_n$ and $x_n \in A_n \quad \forall n \in \mathbb{N}$, and that $x \in L_s A_n \Leftrightarrow \exists (x_{k_n}), k_1 < k_2 < \cdots$ such that $x = \lim\limits_n x_{k_n}$ and $x_{k_n} \in A_{k_n}, \quad \forall n \in \mathbb{N}$.

-*We say that a sequence of nonempty closed subsets of a Banach space $X$, $\{A_n\}$, converges in strong (respectively, Wijsman) sense to a closed set $A$, provided that the sequence $\{\lambda_n\}$ converges to $\lambda$ uniformly (respectively, pointwise) on bounded sets, where $\lambda_n$ and $\lambda$ are defined as*:

$$\lambda(x) = d(x, A),\ \lambda_n(x) = d(x, A_n).$$

Notation: $A_n \xrightarrow{r} A$, $A_n \xrightarrow{W} A$.

Remark 1.1. Mosco convergence implies *Kuratowski convergence. If $X$ is a Schur space both convergences agree. If $X$ is a finite dimensional Banach space then $M, K, W$ and $r$-convergence agree. $W$-convergence* is weaker than *$r$-convergence* but stronger than *$K$-convergence* ([4]).

It is assumed usually that the sets in the definition of Mosco convergence are convex, and consequently weakly closed. We do not, but let us observe that without that condition a constant sequence may be non-convergent.

In [5] it is proved that if $X$ is a Banach space then $r$-convergence is equivalent to the following condition: $\forall r > 0$, $\forall \varepsilon > 0$, there exists $n_0$ such that: (i) $A + rB \subset A_n + \varepsilon B$, (ii) $A_n + rB \subset A + \varepsilon B$, $\forall n \geq n_0$, where $B$ is the unit ball of the Banach space.

Given $P \in \mathcal{P}(^kX)$ and $\alpha \in \mathbb{K}$, we will denote $\{x \in X : P(x) = \alpha\}$ by $V(P - \alpha)$. When $\alpha = 0, V(P - 0) = Z(P)$.

DEFINITION 1.2. ([7], [8]). If $\{P_n\}$ and $P$ are $k$-homogeneous polynomials on $X$, we say that $\{P_n\} \xrightarrow{M} P$ (Mosco convergent), $\{P_n\} \xrightarrow{K} P$ (Kuratowski convergent), $\{P_n\} \xrightarrow{W} P$ (Wijsman convergent), $\{P_n\} \xrightarrow{r} P$ ($r$-convergent) if $V(P_n - \alpha)$ is $M$-convergent, $K$-convergent, $W$-convergent, $r$-convergent to $V(P - \alpha)$ for all $\alpha \in \mathbb{K}$, $\alpha \neq 0$, respectively.

In [7] it is shown that $K$-convergence is equivalent to uniform convergence on compacts sets and strong convergence is equivalent to norm convergence. Also $\{P_n\} \xrightarrow{W} P \iff \lim_n cl(P_n(B)) = cl(P(B))$, for every ball $B \subseteq X$.

In [8] it is shown that $\{P_n\} \xrightarrow{M} P \iff \{P_n\} \longrightarrow P$ uniformly on weakly compact sets and $P \in \mathcal{P}(^kX)$ is weakly sequentially continuous.

What about the convergence of $Z(P_n)$? The different behavior of the case $\alpha = 0$ is related to the fact that 0 is the unique critical value of a homogeneous polynomial, and consequently it is possible to have a change of the topology of $V(P - \alpha)$ near $\alpha = 0$.

PROPOSITION 1.1. ([7]). *Let $X$ be a complex Banach space, $P$ and $P_n$ nonzero $k$-homogeneous polynomials on $X$. If $\{P_n\} \to P$ pointwise, then $Z(P_n) \xrightarrow{K} Z(P)$.*

*Proof.* We have that $L_s Z(P_n) \subset Z(P)$, because if $P_{n_k}(x_{n_k}) = 0$ and $\{x_{n_k}\}$ converges, then $P\left(\lim_k x_{n_k}\right) = 0$.

Let's prove that $Z(P) \subset L_i Z(P_n)$. Let $x$ be such that $P(x) = 0$. If there does not exists a sequence $\{x_n\}$ converging to $x$ such that $P_n(x_n) = 0$, then we may assume that there exists an $\varepsilon$ such that $Z(P_n) \cap B(x, \varepsilon) = \phi$ $\forall n$. Now we consider a complex line $L = \{x + \lambda z : \lambda \in \mathbb{C}\}$ such that $P$ is not identically 0 on $L$. Let $h_0$ and $h_n$ denote the restriction to $L$ of $P$ and $P_n$ respectively, and $\Omega = B(x, \varepsilon) \cap L$. Then $h_n, h_0 : \Omega \to \mathbb{C}$ are 1-dimensional holomorphic functions, and $\{h_n\}$ converges uniformly to $h_0$ on $\Omega$ because $\{P_n\}$ converges uniformly to $P$ on the compact $cl(\Omega)$.

The fact that $h_n$ does not have zeros in $\Omega$ give us the following alternative (Hurwitz's Theorem): either $h_0$ is identically 0 or it does not have zeros in $\Omega$. Both are impossible by the choice of $L$ and the fact that $h_0(x) = 0$. □

The following easy example shows us that in the real case things are worse.

EXAMPLE 1.1. ([7]). $X = \mathbb{R}^2$, $k = 2$, $P_n(x, y) = x^2 + \frac{1}{n}y^2$, $P(x, y) = x^2$. We have $\| P_n \| = \| P \| = 1$ and $\lim_n P_n(x, y) = P(x, y)$ for all $(x, y) \in X$. But $\{Z(P_n)\}$ does not converge to $Z(P)$ in the Kuratowski sense, because $Z(P_n) = \{(0, 0)\}$ for all $n$ and $Z(P) = \{(0, y) : y \in \mathbb{R}\}$.

However we have:

PROPOSITION 1.2. ([7]). *Let $X$ be a real Banach space, $P$ and $P_n$ nonzero $k$-homogeneous polynomials on $X$ such that $dP(x) \neq 0, \forall x \neq 0$. If $\{P_n\} \to P$ pointwise, then $Z(P_n) \xrightarrow{K} Z(P)$.*

*Proof.* If $(P_n) \to P$ pointwise, it is clear that $\{P_n(x_n)\} \to P(x)$ for every sequence $\{x_n\}$ converging to $x \in X$. Then it is clear that $L_s Z(P_n) \subset Z(P)$. To prove that $Z(P) \subset L_i Z(P_n)$, we consider a point $x \in X$ such that $P(x) = 0$. If $x = 0$, then clearly $0 \in L_i Z(P_n)$; otherwise we choose $y$ such that $dP(x)(y) \neq 0$ and define $Q_n(r) = P_n(x + ry)$ and $Q(r) = P(x + ry)$; $Q$ satisfies that $Q'(0) \neq 0$. But $Q_n$ and $Q$ are polynomials of degree $k$ over $\mathbb{R}$ such that $\{Q_n\} \to Q$ pointwise and consequently uniformly on bounded sets.

Now the fact that 0 is a root of $Q$ and $Q'(0) \neq 0$ enables us to claim that there exists a sequence $\{\lambda_n\}$ of roots of the $Q_n$ converging to 0. So $(x + \lambda_n y) \in Z(P_n)$ and $\lim_n (x + \lambda_n y) = x$, and the proof is finished. □

PROPOSITION 1.3. ([7]). *Let $X$ be a complex Banach space, $P$ and $P_n$ nonzero $k$-homogeneous polynomials on $X$. If $\lim_n cl(P_n(B)) = cl(P(B))$ for every open ball $B \subsetneq X$, then $Z(P_n) \xrightarrow{W} Z(P)$.*

*Proof.* We consider $\lambda_n(x) = d(x, Z(P_n))$ and $\lambda(x) = d(x, Z(P))$. Since $\{P_n\} \to P$ pointwise, by [Th.1.17 [7]] $P_n \xrightarrow{K} P$ and then it is obvious that $\limsup \lambda_n(x) \leq \lambda(x)$, and consequently if the result does not hold then there exists $x$ such that $\liminf \lambda_n(x) < \lambda(x) = \lambda$. Passing to a subsequence if necessary, we may assume that $\lim_n \lambda_n(x) = \lambda^* < \lambda' < \lambda$. Let $x_0 \in B(0, 1)$ be such that $P(x_0) \neq 0$. Then $0 \in P_n(B(x, \lambda'))$ for every $n$, and therefore $0 \in cl\,(P(B(x,\lambda')))$. This allows us to choose a sequence $\{y_n\}$ in $B(x,\lambda')$ such that $\lim_n P(y_n) = 0$.

Let $\varepsilon = \lambda - \lambda'$ and define $\phi_n : D(0:\varepsilon) \to \mathbb{C}$ as $\phi_n(\omega) = P(y_n + \omega x_0)$. A subsequence of $\{\phi_n\}$ converges uniformly to a 1-dimensional polynomial $\phi$ such that $\phi(0) = \lim_n \phi_n(0) = \lim_n P(y_n) = 0$. On the other hand, $\phi_n$ does not vanish, because $P$ does not have zeros on $B(x,\lambda)$ by the definition of $\lambda$. Therefore using Hurwitz's Theorem we conclude that $\phi$ must be identically 0, contradicting the fact that $P(x_0) \neq 0$. □

Proposition 1.3 is false in the real case even if we have stronger hypothesis:

EXAMPLE 1.2. ([7]). Let $P$, $P_n : c_0 \to \mathbb{R}$ defined as:

$$P(x) = \sum_{k=1}^{\infty} \frac{1}{k^2}\,(x_k - x_{k+1})^2$$

$$P_n(x) = \sum_{k=1}^{n} \frac{1}{k^2}\,(x_k - x_{k+1})^2.$$

We have $\{P_n\} \to P$ uniformly on bounded sets and $P(x) = 0 \iff x = 0$; $Z(P) = \{0\}$, $Z(P_n) = \{x \in c_0 : x_1 = \cdots = x_{n+1}\}$. Therefore we have that $d(e_1, Z(P)) = 1$ and $d(e_1,\, Z(P_n)) = 1/2$, and consequently $\lim_n d(e_1,\, Z(P_n)) \neq d(e_1,\, Z(P))$. Obviously $\lim_n cl(P_n(B)) = cl(P(B))$, since $\{P_n\} \to P$ uniformly on bounded sets.

REMARK 1.2. ([7]). The Example 1.2 shows that the following relation, which is true in the complex case, does not hold in the real case:

$''B = B(x_0;\, r),\, 0 \in cl(P(B)),\, \eta > 0 \Longrightarrow Z(P) \cap B(x_0;\, r+\eta) \neq \phi''$.

It is enough to consider $x_0 = e_1$, $r = 1/2$, $\eta = 1/4$. Then it is clear that $0 \notin P\,(B(e_1;\, 1/2 + 1/4))$ but $0 \in cl(P(B))$ since

$$z_n = (1/2 + 1/n)\,e_1 + \sum_{j=2}^{n} (1/2 - 1/n)\,e_j \in B$$

and

$$0 \leq P(z_n) \leq 5/n^2 \to 0.$$

REMARK 1.3. ([7]) In Example 1.2, $dP(x) = 0 \iff x = 0$. Then by Proposition 1.2 we have that $Z(P_n) \xrightarrow{K} Z(P)$.

Stronger conditions on $P$ give us a similar result in the real case:

Proposition 1.4. ([7]). *Let $X$ be a real Banach space, $P$ and $P_n$ $k$-homogeneous polynomials on $X$ such that $P$ satisfies the following condition*:

$$(*) \qquad \lim_n \| dP(y_n) \| = 0 \Longrightarrow 0 \in \overline{co}(\{y_n\}).$$

*If* $\lim_n cl(P_n(B)) = cl(P(B))$ *for every open ball* $B \subseteq X$, *then* $Z(P_n) \xrightarrow{W} Z(P)$.

*Proof.* From the condition on the polynomial $P$ we deduce that $dP(x) = 0$ if and only if $x = 0$, and therefore, by Proposition 1.2, $Z(P_n) \xrightarrow{K} Z(P)$. As in Proposition 1.3, if $\lim_n \lambda_n(x) \neq \lambda(x)$ we would have $\{y_n\} \subset B(x, \lambda')$ such that $\lim_n P(y_n) = 0$. Now we define

$$\phi_n : [-\varepsilon, \varepsilon] \longrightarrow \mathbb{R}, \ \phi_n(t) = P(y_n + tz_n)$$

where $z_n$ are points in the unit ball such that $|dP(y_n)(z_n)| > 1/2 \| dP(y_n) \|$. These polynomials never vanish and converge to a polynomial $\phi$ such that $\phi(0) = 0$. Then necessarily $\phi'(0) = 0$ and $\lim_n dP(y_n)(z_n) = \lim_n \phi'_n(0) = 0$, so we conclude that $\lim_n \| dP(y_n) \| = 0$, which is not possible because $0 \notin B(x, \lambda)$ and $\overline{c_o}(\{y_n\}) \subset B(x, \lambda)$. □

Remark 1.4. ([7]). Condition $(*)$, in Proposition 1.4, on the polynomial $P$ is weaker than the property of being a separating polynomial.

Proposition 1.5. ([7]). *If $X$ is a complex Banach space and $P$, $P_n$ are nonzero $k$-homogeneous polynomials on $X$ such that $\{P_n\} \to P$ uniformly on bounded sets, then* $Z(P_n) \xrightarrow{r} Z(P)$.

*Proof.* Let $B$ denote the unit ball of $X$. First we will prove that $\forall r, \varepsilon > 0$ there exists $n_0$ such that $Z(P_n) \cap rB \subset Z(P) + \varepsilon B$, $\forall n \geq n_0$. Indeed, otherwise there would exists $r, \varepsilon > 0$ and a sequence $\{x_n\}$ contained in $rB$ such that $P_n(x_n) = 0$ but $B(x_n, \varepsilon) \cap Z(P) = \phi$. Now uniform convergence on $rB$ of $\{P_n\}$ gives us $\lim_n P(x_n) = 0$.

Let's choose $z_0 \in B$ such that $P(z_0) \neq 0$ and define $\phi_n : D(0, \varepsilon) \to \mathbb{C}$ as $\phi_n(\omega) = P(x_n + \omega z_0)$. Then $\phi_n(x) = P(x_n) + \binom{k}{1} A(x_n, \ldots, x_n, z_0)\omega + \cdots + \binom{k}{k-1} A(x_n, z_0, \ldots, z_0)\omega^{k-1} + P(z_0)\omega^k$, where $A$ denotes the $k$-linear form associated to $P$. A subsequence of $\{\phi_n\}$ converges to a polynomial $\phi$ such that $\phi^{(k)}(0) = k!P(z_0) \neq 0$. But $\phi(0) = \lim_n \phi_n(0) = \lim_n P(x_n) = 0$, and this, together with the fact that $\phi_n$ does nor vanishs, gives us (Hurwitz's Theorem) that $\phi = 0$, which is a contradiction.

We proceed with the other inclusion: $\forall r,\ \varepsilon > 0$ there exists $n_0$ such that

$$Z(P) \cap rB \subset Z(P_n) + \varepsilon B \quad \forall n \geq n_0.$$

If this is not the case, then there exist $r > 0$ and $\varepsilon > 0$ and $\{x_n\}$ be such that $x_n \in rB$, $P(x_n) = 0$ and $Z(P_n) \cap B(x_n, \varepsilon) = \phi$. Let $z_0 \in B$ be such that $P(z_0) \neq 0$, and define $\phi_n, \widetilde{\phi}_n : D(0;\varepsilon) \to \mathbb{C}$ as $\phi_n(\omega) = P(x_n + \omega z_0)$, $\widetilde{\phi}_n(\omega) = P_n(x_n + \omega z_0)$. $\widetilde{\phi}_n$ never vanishes. We choose now a subsequence of $\{\phi_n\}$ converging to a polynomial $\phi$ identically zero, such that $\phi(0) = 0$; it is clear that the corresponding subsequence of $\left\{\widetilde{\phi}_n\right\}$ converges to $\phi$ too. Hurwitz's Theorem again give us the contradiction. □

The Example 1.2 shows that in the real case uniformly convergence, on bounded sets, of $\{P_n\}$ does not imply even $Z(P_n) \xrightarrow{W} Z(P)$.

The following example shows us that , even assuming $W$-convergence, $r$-convergence of $Z(P_n)$ to $Z(P)$ does not follow from uniform convergence on bounded sets (in the real case) of the sequence $\{P_n\}$.

Example 1.3. ([7]). Let $X = c_0$, $P_n = \sum_{k=1}^{n} \frac{1}{k^2} x_k^2$, $P(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} x_k^2$. Then $\{P_n\} \to P$ uniformly and $Z(P) = \{0\}$, $Z(P_n) = \{x \in c_0 : x_1 = \cdots = x_n = 0\}$. $Z(P_n) \xrightarrow{W} Z(P)$ since $d(x, Z(P)) = \| x \|$ and $d(x, Z(P_n)) = \| x \|$ too when $n > n_0$, where $n_0$ is such that $| x_n | < 1/2 \| x_n \|$, $\forall n \geq n_0$. However $Z(P_n) \cap 2B$ is never included in $Z(P) + \frac{1}{2}B$, since $e_{n+1} \in Z(P_n) \cap 2B$ but $e_{n+1} \notin \frac{1}{2}B, \forall n$. Therefore $\{Z(P_n)\}$ does not converge to $Z(P)$ in the strong sense.

Remark 1.5. ([7]). In Example 1.3, $dP(x) = 2\sum_{k=1}^{\infty} \frac{1}{k^2} x_k e_k^*$, and therefore $\| dP(x) \|_1 = 2\sum_{k=1}^{\infty} \frac{1}{k^2} | x_k |$. Hence $\inf\{\| dP(x) \|_1 \colon x \in S\} = 0$. This fact suggests the following result.

Proposition 1.6. ([7]). *Let $X$ be a real Banach space, $P$ and $P_n$ $k$-homogeneous polynomials such that* $\inf\{\| dP(x) \| : \| x \| = 1\} > 0$. *If $\{P_n\} \to P$ uniformly on bounded sets, then $Z(P_n) \xrightarrow{r} Z(P)$.*

*Proof.* First we will prove that, if $B$ is the unit ball of $X$, then $\forall r, \varepsilon > 0$ there exists $n_0$ such that

$$Z(P_n) \cap rB \subset Z(P) + \varepsilon B \quad \forall n \geq n_0.$$

Otherwise there exists $r, \varepsilon > 0$ and a sequence $\{x_n\}$ contained in $rB$ such that $P_n(x_n) = 0$ but $B(x_n, \varepsilon) \cap Z(P) = \phi$. Now uniform convergence on $rB$ of $\{P_n\}$ gives us $\lim_n P(x_n) = 0$. Since $x_n \in rB - \varepsilon B \quad \forall n$, we have a positive constant $c$ such that $\| dP(x_n) \| > c \quad \forall n$, and we may choose $y_n$ in the unit sphere such that $| dP(x_n)(y_n) | > c \quad \forall n$. Let's define $\varphi_n : (-\varepsilon, \varepsilon) \to \mathbb{R}$ as $\varphi_n(r) = P(x_n + ry_n)$. A subsequence of $\{\varphi_n\}$ converges to a polynomial $\varphi$ such that $\varphi(0) = \lim_n \varphi_n(0) = \lim_n P(x_n) = 0$, and the fact that $\varphi_n$ does nor vanish gives us that $\varphi'(0) = 0$ or equivalently that $\lim_n dP(x_n)(y_n) = 0$ which is a contradiction.

We are now going to prove the other inclusion: $\forall r, \varepsilon > 0$ there exists $n_0$ such that

$$Z(P) \cap rB \subset Z(P_n) + \varepsilon B, \quad \forall n \geq n_0.$$

If this is not so, then there exist $r > 0$ and $\varepsilon > 0$ and $\{x_n\}$ such that $x_n \in rB$, $P(x_n) = 0$ and $Z(P_n) \cap B(x_n, \varepsilon) = \phi$. Considering $\{y_n\}$ as in the other inclusion, we may define

$$\phi, \phi_n : (-\varepsilon, \varepsilon) \to \mathbb{R} \text{ as } :$$

$\phi(r) = \lim_n P(x_n + ry_n)$, $\phi_n(r) = P_n(x_n + ry_n)$ passing to a subsequence if necessary; $\phi_n$ never vanishes, and $\{\phi_n\} \to \phi$ because of the uniform convergence of the sequence $\{P_n\}$ to $P$ on $(r + \varepsilon)B$; but $\phi(0) = 0$ and consequently $\phi'(0) = 0$ too, and so is clear that $\lim_n dP(x_n)(y_n) = 0$, which gives us the contradiction. □

REMARK 1.6. ([7]). The condition $\inf\{\| dP(x) \| : \| x \| = 1\} > 0$ is weaker than the property of being a separating polynomial but strictly stronger than property (*) in Proposition 1.4, as the polynomial $P$ in Example 1.3 proves.

Finally in the real case we cannot infer $Z(P_n) \xrightarrow{M} Z(P)$ even under very strong conditions as the following example shows:

EXAMPLE 1.4. ([8]). Let us suppose that $k$ is odd (the even case is easier). Let us take $\varphi_1$, $\varphi_2 \in X^*$ linearly independent, (we are only assuming that $\dim X > 1$). Let us define $P$ and $P_n$ as $\varphi_1^{k-1}(\varphi_1 + \varphi_2)$ and $\left(\varphi_1^{k-1} + \frac{1}{n}\varphi_2^{k-1}\right)(\varphi_1 + \varphi_2)$ respectively. $P \in \mathcal{P}(^kX)$ is weakly sequentially continuous, $P = \lim_n P_n$ in norm but $Z(P) = \ker \varphi_1 \cup \ker(\varphi_1 + \varphi_2)$, $Z(P_n) = \ker(\varphi_1 + \varphi_2)$ and consequently the sequence $\{Z(P_n)\}$ does not converge to $Z(P)$ even in the Kuratowski sense.

However, in the real case, we have the following:

PROPOSITION 1.7. ([8]). *If $P \in \mathcal{P}(^kX)$ is weakly sequentially continuous and $dP(x) = 0$, for all $x \neq 0$, then the uniform convergence on weakly compacts sets of the sequence $\{P_n\}$ to $P$ implies $Z(P_n) \xrightarrow{M} Z(P)$.*

*Proof.* Since $P$ is weakly sequentially continuous, the proof of the second condition is trivial. To establish the first condition we have to prove that for every $x \in Z(P)$, there exists a norm convergence to $x$ sequence $(x_n)$ such that $P_n(x_n) = 0$. If $x = 0$ the constant sequence $x_n = 0$ works. Hence we may assume $x \neq 0$; let us consider $z \in S_E$ such that $dP(x)(z) \neq 0$. The following one-dimensional polynomials:

$$g_n(t) = P_n(x + tz),\ g(t) = P(x + tz)$$

verifies that $\{g_n\}$ converges to $g$ uniformly on the compact interval $[-1, 1]$, $g(0) = 0$ and $g'(0) = dP(x)(z) \neq 0$. Consequently, there exists a sequence $\{t_n\}$ such that $\lim_n t_n = 0$ and $g_n(t_n) = 0$ eventually. If we define $x_n = x + t_n z$, the sequence $\{x_n\}$ fulfils the required condition. □

## Section 2

Plichko and Zagorodnyuk [16] have prove that for any positive integers $n$ and $d$, there is a positive integer $m = m(n, d)$ such that for any complex polynomial $P : \mathbb{C}^m \to \mathbb{C}$ of degree $d$, there is a vector subspace $X_p \subset \mathbb{C}^m$ of dimension $n$ such that $P|_{X_p} = P(0)$. For further background on related problems see [15]. In [3] the problem of finding a good bound on $m = m(n, d)$ is treated. Estimates for degrees 2, 3 and 4 are obtained. For instance, it is not difficult to see that $m(n, 2) = 2n+1$, and for example the polynomial $P(z_1, \ldots, z_{2n}) = \sum_{j=1}^{2n} z_j^2$ vanishes on the $n$-dimensional subspace generated by $\{e_1 + ie_2, \ldots, e_{2n-1} + ie_{2n}\}$.

Now in the real case we study four special situations. The first is a general result for real, symmetric, homogeneous polynomials of odd degree. We next study real 2-homogeneous polynomials $P$, relating the size of the subspace of zeros of $P$ with the signature of the associated matrix. We show a general result concerning the zeros of real 3-homogeneous polynomials and finally we present several dichotomy results related with existence of positive definite 2-homogeneous polynomials.

PROPOSITION 2.1. ([2]). *Let $P$ be a symmetric homogeneous polynomial $P : \mathbb{R}^m \to \mathbb{R}$ of odd degree. Then there is an $[m/2]$-dimensional subspace contained in $P^{-1}(0)$.*

*Proof.* Let $P$ be a $d$-homogeneous polynomial on $\mathbb{R}^m$ where $d$ is an odd integer. It is known that the polynomials $\{\sum_{i=1}^m x_i^r \mid r \in \mathbb{N}\}$ form an algebraic basic for the symmetric polynomials on $\mathbb{R}^m$ (see, e.g. [18], p. 79). Therefore there is a polynomial $Q$ such that

$$P\left(\sum_{i=1}^m x_i e_i\right) = Q\left(\sum_{i=1}^m x_i, \dots, \sum_{i=1}^m x_i^d\right).$$

Now,

$$P\left(\sum_{i=1}^m x_i e_i\right) = \sum_{\substack{i_1,\dots,i_s;\\ i_1k_1+i_2k_2+\cdots,i_sk_s=d}} \alpha_{i_1,\dots,i_s}^{k_1,\dots,k_s} \left(\sum_{i=1}^m x_i^{i_1}\right)^{k_1} \cdots \left(\sum_{i=1}^m x_i^{i_s}\right)^{k_s}.$$

Note that in each expression $i_1k_1 + \cdots + i_sk_s = d$, some of the $i_j$'s must be odd. Consider the $[m/2]$-dimensional subspace

$$H = \left[e_1 - e_2, \dots, e_{2[m/2]-1} - e_{2[m/2]}\right];$$

clearly, if $x \in H$ then $P(x) = 0$. So, $H \subset P^{-1}(0)$. □

The same works in the infinite dimensional case:

PROPOSITION 2.2. ([2]). *Let $X$ be a real Banach space with a symmetric basis and let $P$ be a homogeneous symmetric polynomial of odd degree. Then $P^{-1}(0)$ contains an infinite dimensional subspace.*

*Proof.* Let $P$ be a $d$-homogeneous polynomial on $X$ where $d$ is an odd integer, and let $\{e_n\}$ be the symmetric basic of $X$. By the representation of symmetric polynomials given in [9], either $P = 0$ or there exists an integer $N \geq 1$ such that the set of polynomials $\{\sum_{i=1}^\infty x_i^r \mid r \geq N\}$ is an algebraic basis for the space of symmetric polynomials on $X$. Therefore, if $d < N$ we have $P = 0$; otherwise, there is a polynomial $Q : \mathbb{R}^{d-N+1} \to \mathbb{R}$ such that

$$P\left(\sum_{i=1}^\infty x_i e_i\right) = Q\left(\sum_{i=1}^\infty x_i^N, \dots, \sum_{i=1}^\infty x_i^d\right).$$

Then,

$$P\left(\sum_{i=1}^\infty x_i e_i\right) = \sum_{\substack{i_1,\dots,i_s;\\ i_1k_1+i_2k_2+\cdots+i_sk_s=d}} \alpha_{i_1,\dots i_s}^{k_1,\dots,k_s} \left(\sum_{i=1}^\infty x_i^{i_1}\right)^{k_1} \cdots \left(\sum_{i=1}^\infty x_i^{i_s}\right)^{k_s}.$$

Consider now the infinite dimensional subspace:

$$H = [e_1 - e_2, e_3 - e_4, \ldots, e_{2n-1} - e_{2n}, \ldots, ].$$

Clearly, $H$ is an infinite dimensional subspace contained in $P^{-1}(0)$, as we required. □

We turn now to searching for subspaces contained in the zero set of 2-homogeneous real polynomials. We are able to obtain a simple, general result, which depends only on the signs of the eigenvalues of the quadratic form associated to the polynomial.

We recall that a quadratic form $Q$ on $\mathbb{R}^k$ (or, equivalently, the associated symmetric matrix or bilinear form $A$) is said to be *positive definite* if $Q(x) > 0$ (equivalently, $A(x,x) > 0$) for all $x \in \mathbb{R}^k$, $x \neq 0$. In the same way, $Q$ will be *negative definite* whenever $-Q$ is positive definite. We denote by $p(Q)$ (resp. $n(Q), z(Q)$) the number of positive (resp. negative, zero) eigenvalues with their multiplicity.

PROPOSITION 2.3. ([2]). *Let $Q \in \mathcal{P}(^2\mathbb{R}^k)$. Then, if $r = \min\{p(Q), n(Q)\} + z(Q)$ there is an $r$-dimensional subspace $Y$ such that $Y \subset Q^{-1}(0)$.*

*Proof.* Consider a basis $\{w_1, \ldots, w_k\}$ with respect to which $Q$ is diagonal. Then,

$$Q\left(\sum_{i=1}^{k} y_i w_i\right) = \sum_{i=1}^{k} \mu_i y_i^2,$$

where without loss of generality, we may assume that $\mu_i = \pm 1$ or 0. We may also assume that these eigenvalues are written so that $\mu_1 = 1$, $\mu_2 = -1$, $\mu_3 = 1, \ldots, \mu_{2s-1} = 1$, $\mu_{2s} = -1$, where $s = \min\{p(Q), n(Q)\}$, and that $\mu_{k-z(Q)+1} = \cdots = \mu_k = 0$. It is easy to verify that the $r$-dimensional subspace

$$Y = \left[w_1 + w_2, \ldots, w_{2s-1} + w_{2s}, w_{k-z(Q)+1}, \ldots, w_k\right]$$

is contained in $Q^{-1}(0)$. □

Next result shows that every 3-homogeneous polynomial $P : \mathbb{R}^k \to \mathbb{R}$ vanishes on a subspace whose dimension $n$ depends only on $k$, where $n \to \infty$ as $k \to \infty$.

THEOREM 2.1. ([2]). *Let $P : \mathbb{R}^k \to \mathbb{R}$ be a 3-homogeneous polynomial and*

$$k \geq \frac{3^n(6n+5) - 1}{4}$$

*for some $n > 0$. Then, there is an $[(n/2)]$-dimensional subspace contained in $P^{-1}(0)$.*

It is easy to see that every 3-homogeneous polynomial in at least $k = 2$ variables vanishes on an one dimensional subspace. The estimate provided by the theorem in this case is for $k$ to be not smaller than 38. If a 2-dimensional subspace is required, so that $n = 4$, then the theorem gives the estimate that any 3-homogeneous polynomial in $k = 587$ variables has such a subspace. We remark that more, and better, information seems to be known for the analogous problem for complex polynomials. For example, every homogeneous complex polynomial(of any degree) in two or more variables vanishes on a complex line. In fact, one can show that any 3-homogeneous complex polynomial in $2^{n-1}(n+1)$ variables vanishes on a $n$-dimensional subspace. ([3]).

An $k$-homogeneous polynomial $P : X \longrightarrow \mathbb{K}$ is said to be *positive definite* if $P(x) \geq 0$ for every $x$ and $P(x) = 0$ implies that $x = 0$.

PROPOSITION 2.4. ([1]). *A Banach space $X$ admits a positive definite 2-homogeneous polynomial if and only if there is a 2-homogeneous polynomial $P$ on $X$ whose set of zeros is contained in a finite dimensional subspace of $X$ .*

REMARK 2.1. ([1]). Any separable space and $\mathcal{C}(K)$ spaces, when $K$ is compact and separable, admit a positive definite 2-homogeneous polynomial. On the other hand, $X = c_0(\Gamma)$ and $X = l_p(\Gamma)$, where $\Gamma$ is an uncountable index set and $p > 2$, do not admit positive definite 2-homogeneous polynomials.

As we remarked in 2.1, the following result of interest only for non-separable spaces.

THEOREM 2.2. ([1]) *Let $X$ be a real Banach space which does not admit a positive definite 2-homogeneous polynomial. Then, for every $P \in \mathcal{P}(^2X)$, there is an infinite dimensional subspace of $X$ on which it is identically zero.*

*Proof.* Suppose $X$ does not admit a positive definite 2-homogeneous polynomial and that $P \in \mathcal{P}(^2X)$. Let $\mathcal{S} = \{S : S$ is a subspace of $X$and $P \mid_S \equiv 0\}$. Order $\mathcal{S}$ by inclusion and use Zorn's Lemma to deduce the existence of a maximal element $S$ of $\mathcal{S}$. Suppose that $S$ is finite dimensional. Let $v_1, \ldots, v_n$ be a basis for $S$ and let $T = \bigcap_{x \in S} \ker A_x = \bigcap_{i=1}^n \ker A_{v_i}$ where $A_x : X \longrightarrow \mathbf{R}$ is the linear map which sends $y$ in $X$ to $\check{P}(x, y)$. We note that $S \subset T$. To see this suppose that $y \in S$. Then for every $s \in S$, $s + y$ is also in $S$. Since

$$0 = P(s + y) = P(s) + 2A_s(y) + P(y) = 2A_s(y)$$

for every $s \in S$ we see that $y \in T$.

Since $S$ is finite dimensional we can write $T$ as $T = S \bigoplus Y$ for some subspace $Y$ of $T$. It is easy to see that all the zeros of $P\mid_T$ are contained in $S$. Therefore, either $P\mid_T$ or $-P\mid_T$ is positive definite on $Y$. Let us suppose, without loss of generality, that $P\mid_T$ is positive definite on $Y$. As $S$ is $n$-dimensional we can find $\phi_1, \dots, \phi_n$ so that $P + \sum_{i=1}^n \phi_i^2$ is positive definite on $T$. Note that $T$ has finite codimension in $X$ and hence is complemented. Let $\pi_T$ be the (continuous) projection of $X$ onto $T$. Then $\left(P + \sum_{i=1}^n \phi_i^2\right) \circ \pi_T + \sum_{i=1}^n A_{v_i}^2$ is a positive definite polynomial on $X$, contradicting the fact that $X$ does not admit such a polynomial. □

REMARK 2.2. In ([14]) it is shown that every $\mathbb{C}$-valued polynomial $P$ on an infinite dimensional complex Banach space $X$ such that $P(0) = 0$ is identically 0 on an infinite dimensional subspace.

THEOREM 2.3. ([1]) *Let $X$ be a real Banach space which does not admit a positive definite* 4*-homogeneous polynomial, and let $(\psi_k)_{k=1}^\infty$ be a sequence in $X^*$. Then for every countable family $(P_j)_{j=1}^\infty \subset \mathcal{P}(^2X)$, there is a non-separable subspace of $\bigcap_{k=1}^{\infty} \ker \psi_k$ on which each $P_j$ is identically zero.*

Note that if $X$ does not admit a positive definite 4-homogeneous polynomial, then it cannot admit a positive definite 2-homogeneous one either. An example of an $X$ satisfying the hypotheses of Theorem 2.3 is $X = l_p(I)$, where $I$ is an uncountable index set and $p > 4$.

*Proof of Theorem 2.3.* The argument begings in a similar way to our earlier proofs. As before, let $S$ be a maximal element of $\mathcal{S} = \{S :$ $S$ is subspace of $\bigcap_{k=1}^\infty \ker \psi_k$ and $P_j\mid_S \equiv 0$, all $j\}$. Suppose that $S$ is separable, with countable dense set $(v_i)_{i=1}^\infty$. Let $T = \bigcap_{k=1}^\infty \ker \psi_k \cap \bigcap_{i=1}^\infty \bigcap_{j=1}^\infty \ker (A_j)\, v_i$. As before, $S \subset T$. We can write $T$ as $T = S \bigoplus_a Y$ for some subspace $Y$ of $T$. Since all the common zeros of $P_j\mid_T, j \in \mathbf{N}$, are contained in $S, \sum_{j=1}^\infty \frac{P_j^2}{j^2\|P_j\|^2}$ is positive definite on $Y$. As $S$ is separable we can find $(\phi_i)_{i=1}^\infty$ so that $\sum_{j=1}^\infty \frac{P_j^2}{j^2\|P_j\|^2} + \sum_{i=1}^\infty \phi_i^4$ is positive definite on $T$. Then

$$\sum_{j=1}^{\infty} \frac{P_j^2}{j^2 \parallel P_j \parallel^2} + \sum_{i=1}^{\infty} \phi_i^4 + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(A_j)_{v_i}^4}{i^2 j^2 \parallel (A_j)_{v_i} \parallel^4} + \sum_{k=1}^{\infty} \frac{\psi_k^4}{k^2 \parallel \psi_k \parallel^4}$$

is a positive definite polynomial on $X$, contradicting the fact that $X$ does not admit such a polynomial. □

COROLLARY 2.1. ([1]). *Let $X$ be a real Banach space which does not admit a positive definite 4-homogeneous polynomial. Then every $P \in \mathcal{P}(^3X)$ is identically zero on a non-separable subspace of $X$.*

*Proof.* Consider $P \in \mathcal{P}(^3X)$ and let $S$ be a maximal element of $\mathcal{S} = \{S : S \text{ is a subspace of } X \text{ and } P\,|_S \equiv 0\}$. Suppose that $(v_i)_{i=1}^{\infty}$ is a countable dense subset of $S$. Let $A_{v_i,v_j} : X \to \mathbf{R}$ be the linear map which sendes $x$ in $X$ to $\check{P}\,(v_i, v_j, x)$ and $Q_{v_i} : X \to \mathbf{R}$ be the continuous 2-homogeneous polynomial which sends $x$ in $X$ to $\check{P}\left(v_i, x^2\right)$. By Theorem 2.3,

$$\bigcap_{i,j=1}^{\infty} \ker A_{v_i,v_j} \cap \bigcap_{i=1}^{\infty} \ker Q_{v_i}$$

contains a non-separable subspace which we denote by $T$. Suppose that $y \in T$ is such that $P(y) = 0$. Then for every $x = \sum_i \alpha_i v_i \in \operatorname{span} S$ and $\lambda \in \mathbf{R}$ we have

$$\begin{aligned} P(x + \lambda y) &= P(x) + 3\lambda A_{x,x}(y) + 3\lambda^2 \check{P}(x, y, y) + \lambda^3 P(y) \\ &= P(x) + 3\lambda \sum_{i,j} \alpha_i \alpha_j A_{v_i,v_j}(y) + 3\lambda^2 \sum_i \alpha_i Q_{v_i}(y) + \lambda^3 P(y) \\ &= 0. \end{aligned}$$

Hence, by continuity of $P$, $P(x+\lambda y) = 0$ for every $x \in S$. By maximality of $S$ it follows that all the zeros of $P\,|_T$ are contained in $S$. Since $S$ is separable, we can write $T$ as $T = (S \cap T) \bigoplus_a Y$ for some non-separable subspace $Y$ of $T$. Since all the zeros of $P\,|_T$ are contained in $S$, $P\,|_Y$ is a 3-homogeneous polynomial on an infinite dimensional space which has its only zero at the origin, an impossibility. □

The final theorem of this section is a natural extension of the two preceding results.

THEOREM 2.4. ([1]) *Let $X$ be a real Banach space which does not admit a positive definite homogeneous polynomial. Then, for every polynomial $P$ on $X$ such that $P(0) = 0$, there is a non-separable subspace of $X$ on which $P$ is identically zero.*

In [1, section 4] the special cases of $\mathcal{C}(K)$ and absolutely $(1,2)$-summing and nuclear polynomials is considered.

## Section 3

This section is concerned with the approximation by zeros of orthogonally additive polynomials on real $l_p$ and $L_p$ spaces. We consider $l_p$ and $L_p[0,1]$ spaces, $1 \le p < \infty$, with the usual Banach lattice. (See [13]).

DEFINITION 3.1. Let $X$ be a Banach lattice. A function $f : X \to \mathbb{R}$ is said to be orthogonally additive if $f(x+y) = f(x)+f(y)$ for all $x, y \in X$ such that $x \perp y$.

$\mathcal{P}_o(^nX)$ will denote the space of $n$-homogeneous orthogonally additive polynomials on $X$.

EXAMPLE 3.1. ([17]). Let $X = \ell_p$, $1 \le p < \infty$. $\mathcal{P}_o(^nX)$ is isometrically isomorphic to $\ell_{\frac{p}{p-n}}$ if $n < p$ and to $\ell_\infty$ if $n \ge p$. The isomorphism is given by the association $P \leftrightarrow a = (a_j = P(e_j))_{j\ge1}$.

EXAMPLE 3.2. ([17]). Let $X = L_p[0,1]$, $1 \le p < \infty$, and let $\mu$ be Lebesgue measure.

For $1 \le n < p$, $P \in \mathcal{P}_o(^nX) \Leftrightarrow$ There exists a unique function $\xi \in L_{\frac{p}{p-n}}$ such that $P(x) = \int_0^1 \xi x^n d\mu$.

For $n = p$, $P \in \mathcal{P}_o(^nX) \Leftrightarrow$ There exists a unique function $\xi \in L_\infty$ such that $P(x) = \int_0^1 \xi x^n d\mu$.

For $n > p$, $P \in \mathcal{P}_o(^nX) \Leftrightarrow P \equiv 0$.

For a Banach space $X$ and an $n$-homogeneous polynomial $P$, $P \neq 0$, we consider $Z = P^{-1}(0)$ and the sets $D_jZ = \{\sum_{i=1}^j z_i : z_i \in Z, \forall i = 1, \ldots, j\}$ $(j \ge 2)$. Finally we consider $H = \text{span}[Z]$, the subspace of $X$ generated by finite linear combinations of elements of $Z$. Note that $H = \bigcup_{j\ge1} D_jZ$.

THEOREM 3.1. ([12]). *Let $1 \le p < \infty$, $n > 1$ an odd integer, and $P$ an $n$-homogeneous orthogonally additive polynomial on $\ell_p$ associated with the sequence $a = (a_j)_{j\ge1}$.*

(i) *If there exist at least three $j$'s so that $a_j \neq 0$, then $e_j \in D_3Z$ for all $j \in \mathbb{N}$. In particular, $H$ is dense in $\ell_p$.*
(ii) *If there are at most two $j$'s with $a_j \neq 0$, then $Z$ is a closed hyperplane of $\ell_p$.*

THEOREM 3.2. ([12]). *Let $1 \leq p < \infty$, $n$ an even integer, and $P$ an $n$-homogeneous orthogonally additive polynomial on $\ell_p$ associated with the sequence $a = (a_j)_{j\geq 1}$.*

(i) *If the function sign$(a_j)$, $a_j \neq 0$, is constant, then $Z = H = \{x \in \ell_p : x_j = 0$ whenever $a_j \neq 0\}$.*

(ii) *If there exists a pair $i \neq k$ such that $a_i$, $a_k \neq 0$ and sign$(a_i) \neq$ sign$(a_k)$, then $e_j \in D_2Z$ for all $j \in \mathbb{N}$. In particular $H$ is dense in $\ell_p$.*

From what follows we consider the usual decomposition of a function a difference of two positive maps $\xi = \xi^+ - \xi^-$ and we note by $A^+, A^-$ the respective support of $\xi^+, \xi^-$ and $A_0$ the complement set of $A^+ \cup A^-$ in $[0, 1]$

THEOREM 3.3. ([12]). *Let $1 \leq p < \infty$, $n$ an odd integer, $1 < n \leq p$, and $P$ an $n$-homogeneous orthogonally additive polynomial on $L_p[0, 1]$ associated with the function $\xi \in L_{\frac{p}{p-n}}[0, 1]$.*

(i) *If $\xi \geq 0$ (or $\xi \leq 0$) a.e. then, $\chi_E \in D_3Z$, for all measurable set $E \subset [0, 1]$.*

(ii) *If $\xi$ is arbitrary then, $\chi_E \in D_4Z$, for all measurable set $E \subset [0, 1]$.*

*As a consequence of either* (i) *or* (ii), *$H$ is dense in $L_p[0, 1]$.*

THEOREM 3.4. ([12]). *Let $1 \leq p < \infty$, $n$ an even integer, $1 < n \leq p$, and $P$ an $n$-homogeneous orthogonally additive polynomial on $L_p[0, 1]$ associated with the function $\xi \in L_{\frac{p}{p-n}}[0, 1]$.*

(i) *If $\xi \geq 0$ (or $\xi \leq 0$) a.e. (i.e $\mu(A^+).\mu(A^-) = 0$), then $H = Z = \{x \in L_p[0, 1] : \mu(support(x) \cap support(\xi)) = 0\}$*

(ii) *If $\mu(A^+).\mu(A^-) > 0$ and $E \subset [0, 1]$ ia a measurable set, then*

1. *$\mu(A^+ \cap E).\mu(A^- \cap E) = 0$ implies $\chi_E \in Z$ or $\chi_E \in D_2Z$.*
2. *$\mu(A^+ \cap E).\mu(A^- \cap E) > 0$ implies $\chi_E \in D_3Z$.*

*As a consequence of either* (a) *or* (b) *in* (ii), *$H$ is dense in $L_p[0, 1]$.*

We proved that given an $n-$homogeneous orthogonally additive polynomial on $L_p$ such that $H$ is dense, it is enough to consider $D_jZ$ with, at most $j = 4$, in order to obtain the decomposition by zeros of a characteristic function. Now, the problem of when it is possible to consider only $D_2Z$, is treated.

THEOREM 3.5. ([12]). *Let $p = n$ be an odd integer, $n > 1$ and $P$ an $n$-homogeneous orthogonally additive polynomial on $L_p[0, 1]$ associated with the function $\xi \in L_\infty[0, 1]$.*

(i) *If there exists $T \subset [0, 1]$ a dense set such that $\xi(t^+) \neq 0$ for all $t \in T$, then $\chi_{[a,b]} \in \overline{D_2Z}^{\|\cdot\|_p}$ for all subinterval $[a, b] \subseteq [0, 1]$.*

(ii) *If* $\xi = \chi_F$ *for* $F \subset [0,1]$ *a measurable set, then* $\chi_E \in \overline{D_2 Z^{\|\cdot\|_p}}$ *for all measurable set* $E \subset F$.

*As a consequence of either* (i) *or* (ii), $H$ *is dense in* $L_p[0,1]$.

## References


[1] R. M. Aron, C. Boyd, R. A. Ryan and I. Zalduendo, *Zeros of polynomials on Banach spaces: The real story*, preprint.

[2] R. M. Aron, R. Gonzalo and A. Zagorodnyuk, *Zeros of real polynomials*, Linear and multilinear algebra **48** (2000), 107–115.

[3] R. M. Aron and M. P. Rueda, *A problem concerning zero - subspaces of homogeneous polynomials*, Linear topological spaces and complex analysis **3** (1997), 20–23.

[4] M. Baronti and P. Papini, *Convergence of sequences of sets*, Methods of functional analysis in approximation theory, ISNM 76, Birkhäuser, Basel, (1986), 133–155.

[5] G. Beer, *Convergence of continuous linear functionals and their level sets*, Arch. Math. **52** (1989), 482–491.

[6] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, S.M.M. Springer, (1999).

[7] J. Ferrera, *Convergence of Polynomial Level Sets*, Trans. Amer. Math. Soc. **350**, $n^0$ 12, (1998).

[8] ———, *Mosco convergence of sequences of homogeneous polynomials*, Rev. Mat. Complut. **11**, no. 1, (1998).

[9] M. González, R. Gonzalo and J. A. Jaramillo, *Symmetric polynomials on rearrangement invariant function spaces*, J. London Math. Soc. **59** (1999), no. 2, 681–697.

[10] R. C. Gunning and H. Rosi, *Analytic functions of several complex variables*, Prentice-Hall Series in modern analysis (1965).

[11] J. Harris, *Algebraic Geometry, A first course*, Grad. Texts in Math. 133, Springer Verlag, 1995.

[12] S. Lassalle and J. G. Llavona, *Zeros of orthogonally additive polynomials on* $\ell_p$ *and* $L_p$ *spaces*, preprint.

[13] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II*, Springer, 1977.

[14] J. Mújica, *Complex Analysis in Banach Spaces*, Math. Studies 120, North-Holland, Amsterdam, 1986.

[15] V. Pestov, *Two 1935 questions of Mazur about polynomials in Banach spaces: A counter example*, Questiones Mathematicae, to appear.

[16] A. Plichko and A. Zaragodnyuk, *On automatic continuity and three problems of the Scottish Book concerning the boundedness of polynomial functionals*, J. Math. Anal. Appl. **220** (1998), $n^0$ 2, 477–494.

[17] K. Sundaresan, *Geometry of Spaces of Homogeneous Polynomials on Banach Lattices*, Applied geometry and discrete mathematics, 571-586, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 4, Amer. Math. Soc., Prov., RI, 1991.

[18] B. L. van der Waerden, *Modern Algebra*, Ungar, 1964.

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