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PROPERTIES OF Q-REFLEXIVE LOCALLY CONVEX SPACES

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ABSTRACT. Q-reflexive locally convex spaces are spaces where $\widehat{\bigotimes}_{n,s,\pi} E''_e$ and $\overline{(\mathcal{P}({}^nE), \tau_b)'_i}$ are isomorphic in a canonical way for every n. We investigate properties and find examples of such spaces.

A Banach space E is Q-reflexive if for every n the space $\mathcal{P}({}^nE)''$ is isomorphic to $\mathcal{P}({}^nE'')$ in a canonical way. Q-reflexive Banach spaces were defined by R. Aron and S. Dineen in [3], and in [11] the definition was changed by González to its present form. Q-reflexive locally convex space were defined in [7], also there were given examples of such spaces. In this paper we investigate properties of Q-reflexive locally convex spaces and give further examples. Many of the results of this paper appear in [16].

We refer to [9] and [10] for background information on polynomials over locally convex spaces, and to [13] and [14] for the general theory of locally convex spaces.

1. In this section we give some known results about Q-reflexive locally convex spaces and introduce notation that will be used throughout the article. If E is a locally convex space over the complex numbers $\mathbb{C}$, we let $\overline{E}$ denote the completion of E, and let E' denote the space of all continuous linear functionals on E. If E' is endowed with the strong topology (i.e. the topology of uniform convergence over the bounded subsets of E), we denote it by E'_β. We say that E is *infrabarrelled* if the canonical inclusion of E into $E''_\beta := (E'_\beta)'_\beta$ is continuous. Let $\mathcal{V}$ be a fundamental 0-neighbourhood basis of E, the collection $(V^{\circ\circ})_{V\in\mathcal{V}}$ is a fundamental 0-neighbourhood basis for the *natural topology* on E''. The bidual of E endowed with the natural topology is denoted by E''_e. It is well known that E is infrabarrelled if and only if the bounded subsets of E'_β are equicontinuous. A locally convex space E is *barrelled*

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if and only if the $\sigma(E', E)$-bounded subsets in E' are equicontinuous (thus every barrelled space is infrabarrelled), and is $\aleph_0$-*barrelled* if and only if every $\sigma(E', E)$-bounded subset of E' which is a countable union of equicontinuous sets is itself equicontinuous.

For E a locally convex space we let $\mathcal{P}(^nE)$ denote the space of all continuous n-homogeneous polynomials on E. The topology on $\mathcal{P}(^nE)$ of uniform convergence over the bounded subsets of E is denoted by τ_b. Clearly when $n = 1$ we have $E'_\beta := (\mathcal{P}(^1E), \tau_b)$. If $\mathcal{U}$ is a fundamental system of absolutely convex 0-neighbourhoods of E, the *inductive dual* of E, E'_i, is defined as the inductive limit

$$E'_i = \varinjlim_{U \in \mathcal{U}} E'_{U^\circ}.$$

If $\widehat{\bigotimes}_{n,s,\pi} E$ denotes the completed symmetric n-fold tensor product of E endowed with the projective tensor topology, then $(\widehat{\bigotimes}_{n,s,\pi} E)'_\beta$ and $(\mathcal{P}(^nE), \beta)$ are isomorphic, where β is the topology of uniform convergence over the bounded subsets of $\widehat{\bigotimes}_{n,s,\pi} E$. The space E has $(BB)_n$ if the closed convex hull of $\otimes_{n,s} B$ forms a fundamental system of bounded subsets of $\widehat{\bigotimes}_{n,s,\pi} E$ as B ranges over the bounded subsets of E. Clearly E has $(BB)_n$ if and only if $(\widehat{\bigotimes}_{n,s,\pi} E)'_\beta$ and $(\mathcal{P}(^nE), \tau_b)$ are isomorphic. The locally convex space E has $(BB)_\infty$ if and only if it has $(BB)_n$ for every n. A locally convex space E has the *strict approximation property* if it admits a fundamental system $\mathcal{A}$ of semi-norms such that $E_\alpha = (E, \alpha)/\alpha^{-1}(0)$ has the approximation property for each $\alpha \in \mathcal{A}$.

Let $P \in \mathcal{P}(^nE)$ and let $AB_n(P)$ denote the Aron-Berner extension of P to $E'' := (E'_\beta)'$ (see [2]). The mapping

$$J_n : \bigotimes_{n,s} E'' \longrightarrow (\mathcal{P}(^nE), \tau_b)',$$

given by $\left[J_n(\underset{n}{\otimes} x'')\right](P) = \left[AB_n(P)\right](x'')$ for all $P \in \mathcal{P}(^nE)$ and all $x'' \in E''$, and extended by linearity, is well defined. Let

$$J_n^{bw} : \bigotimes_{n,s,\pi} E''_e \longrightarrow (\mathcal{P}(^nE), \tau_b)'_i. \tag{1}$$

The following definition is given in [7].

DEFINITION 1.1. The locally convex space E is *Q-reflexive* if for every positive integer n,

1. the mapping J_n^{bw} is continuous,
2. the extension of J_n^{bw} to the completion is an isomorphism between $\widehat{\bigotimes}_{n,s,\pi} E_e''$ and $\overline{(\mathcal{P}(^nE),\tau_b)_i'}$.

By [7] Q-reflexive spaces are infrabarrelled.

Next we consider certain subspaces of $\mathcal{P}(^nE)$. An n-homogeneous polynomial P on E is called *nuclear* if there exist an equicontinuous sequence $(\psi_i)_i$ in E' and $(\lambda_i)_i$ in l_1 such that

$$P(x) = \sum_{i=1}^{\infty} \lambda_i {\psi_i}^n(x)$$

for all $x \in E$. Let $\mathcal{P}_N(^nE)$ denote the space of all nuclear polynomials on E. If A is a subset of E let

$$\pi_{N,A}(P) = \|P\|_{N,A} := \inf\Big[\sum_{i=1}^{\infty} |\lambda_i| \|\psi_i\|_A^n : P = \sum_{i=1}^{\infty} \lambda_i \psi_i^n\Big].$$

As A ranges over the bounded sets of E we obtain the π_b topology. If E'_β has the strict approximation property then by ([6], p.186)

$$\overline{(\mathcal{P}_N(^nE),\pi_b)} = \widehat{\bigotimes}_{n,s,\pi} E'_\beta. \tag{2}$$

We let

$$(\mathcal{P}_N(^nE),\pi_w) = \varinjlim_{\alpha\in cs(E)} (\mathcal{P}_N(^nE_\alpha),\pi_b).$$

An n-homogeneous polynomial P on a locally convex space E is *integral* if there is an absolutely convex closed neighbourhood of 0, U, and a finite regular Borel measure μ on U° endowed with the w^*-topology, so that

$$P(x) = \int_{U^\circ} \psi^n(x)d\mu(\psi)$$

for all $x \in E$. The space of all n-homogeneous integral polynomials on E is denoted by $\mathcal{P}_I(^nE)$, and the topology τ_I on $\mathcal{P}_I(^nE)$ is defined as the locally convex inductive limit

$$(\mathcal{P}_I(^nE),\tau_I) = \varinjlim_{U\in\mathcal{U}} (\mathcal{P}(^nE_U),\|\cdot\|_{U,I}),$$

where

$$\|P\|_{U,I} = \inf\{\|\mu\|_{U^\circ} : P(x) = \int_{U^\circ} \psi^n(x)d\mu(\psi)\}.$$

A polynomial $P \in \mathcal{P}(^nE)$ has *finite rank* if there exists a finite subset $\{\varphi_i\}_{i=1}^l$ in E' such that

$$P(x) = \sum_{i=1}^{l} \varphi_i^n(x)$$

for all $x \in E$. We let $\mathcal{P}_f(^nE)$ denote the space of all n-homogeneous polynomials of finite rank on E. Polynomials in $\mathcal{P}_A(^nE)$, the closure of $\mathcal{P}_f(^nE)$ in $(\mathcal{P}(^nE), \tau_b)$, are called *continuous approximable polynomials.* By ([6], Propositions 1 and 2)

$$(\mathcal{P}_A(^nE), \tau_b) = \widehat{\bigotimes_{n,s,\varepsilon}} E'_\beta$$

and

$$(\mathcal{P}_I(^nE'_\beta), \tau_I) = (\mathcal{P}_A(^nE), \tau_b)'_i.$$

We have the following characterization of Q-reflexivity ([7], Proposition 4.2):

PROPOSITION 1.2. *If E is an infrabarrelled locally convex space whose strong bidual has the strict approximation property, then the following conditions are equivalent*:

1. *E is Q-reflexive,*
2. $\overline{(\mathcal{P}_N(^nE'_\beta), \pi_b)} = (\mathcal{P}_I(^nE'_\beta), \tau_I)$ *and* $\mathcal{P}(^nE) = \mathcal{P}_A(^nE)$.

REMARK 1.3. In the proof of (2)⇒(1) in Proposition 1.2 we do not need the assumption that E''_β has the strict approximation property.

For Fréchet spaces Proposition 1.2 can be reformulated in the following way:

PROPOSITION 1.4. *If E is a Fréchet space whose strong bidual has the strict approximation property, then the following conditions are equivalent*:

1. *E is Q-reflexive,*
2. $(\mathcal{P}_N(^nE'_\beta), \pi_b) = (\mathcal{P}_I(^nE'_\beta), \tau_I)$ *and* $\mathcal{P}(^nE) = \mathcal{P}_A(^nE)$.

Proof. By Proposition 1.2 and (2) suffices to show that every element P in $\widehat{\bigotimes_{n,s,\pi}} E''_\beta$ belongs to $\mathcal{P}_N(^nE'_\beta)$. By ([14], 15.6.4) P admits a representation $\sum_{i=1}^{\infty} \lambda_i (\otimes_n x''_i)$ where $(\lambda_i)_i \in l_1$ and $(x''_i)_i$ is a null sequence in E''_β. By ([14], Theorem 12.4.3) $(x''_i)_i$ is equicontinuous, so by identifying P with the polynomial $\sum_{i=1}^{\infty} \lambda_i (x''_i)^n$ we see that P is nuclear. □

2. We are ready now to extend some of the properties of Q-reflexive Banach spaces (see [3], [11] and [15]) to more general classes of spaces. We omit the proof of Proposition 2.1 because of its similarity to the corresponding proof in the Banach space case ([10], Corollary 2.46).

PROPOSITION 2.1. *Let E be a Q-reflexive locally convex space whose strong bidual has the strict approximation property. Then l_1 is not isomorphic to a subspace of E'_β.*

PROPOSITION 2.2. *Let E be a Q-reflexive locally convex space whose strong bidual has the strict approximation property. Then l_1 is not isomorphic to a subspace of E.*

Proof. Suppose $l_1 \hookrightarrow E$. By a result of Grothendieck the canonical inclusion $i : l_1 \to l_2$ can be factored through $L^\infty[0,1]$ and $L^2[0,1]$ in the following fashion. The classical Rademacher functions on $[0,1]$, $(r_n(t))_{n=1}^\infty$, form an orthonormal sequence in $L^2[0,1]$ and a bounded sequence in $L^\infty[0,1]$. Let $(e_n)_{n=1}^\infty$ denote the standard vector basis for l_1 and let $s : l_1 \to L^\infty[0,1]$ be defined by $s(e_n) = r_n$ and extended by linearity. Since $\|r_n\| = 1$ the mapping s is continuous. Let $T : L^\infty[0,1] \to L^2[0,1]$ denote the canonical inclusion mapping. We define $R : L^2[0,1] \to l_2$ by mapping $(r_n(t))_{n=1}^\infty$ onto the standard orthonormal basis in l_2 and extending it by linearity. The diagram

$$\begin{array}{ccc} l_1 & \xrightarrow{\;i\;} & l_2 \\ {\scriptstyle s}\downarrow & & \uparrow{\scriptstyle R} \\ L^\infty[0,1] & \xrightarrow{\;T\;} & L^2[0,1] \end{array}$$

is commutative and all mappings are continuous ([10], p.116). By ([14], Corollary 15.7.3) since l_1 is a closed subspace of E, $L^1[0,1]\widehat{\bigotimes}_\pi l_1$ is a closed subspace of $L^1[0,1]\widehat{\bigotimes}_\pi E$. Since $\mathcal{L}(l_1, L^\infty[0,1]) = (L^1[0,1]\widehat{\bigotimes}_\pi l_1)'$, we have that $s \in (L^1[0,1]\widehat{\bigotimes}_\pi l_1)'$. By the Hahn-Banach Theorem the mapping s can be extended to a continuous mapping $U : E \to L^\infty[0,1]$ so that the following diagram commutes:

$$\begin{array}{ccc} l_1 & \xrightarrow{\;i\;} & l_2 \\ \downarrow{\scriptstyle k} & & \uparrow{\scriptstyle j:=R\circ T} \\ E & \xrightarrow{\;U\;} & L^\infty[0,1] \end{array}$$

where k is the inclusion of l_1 in E.

Let $P((x_n)_n) = \sum_{n=1}^{\infty} x_n^2$ for $(x_n)_{n=1}^{\infty} \in l_2$. Then P is a 2-homogeneous continuous polynomial on l_2, and $Q := P \circ j \circ U$ is a 2-homogeneous continuous polynomial on E. We will show that Q is not an element of $\mathcal{P}_A(^2E)$.

Suppose $Q \in \mathcal{P}_A(^2E)$, then there exists a net (Q_α) in $\mathcal{P}_f(^2E)$ such that $Q = \lim_{\alpha \to \infty} Q_\alpha$ uniformly on the bounded sets of E. In particular $Q|_{l_1} = \lim_{\alpha \to \infty} Q_\alpha|_{l_1}$ on the unit ball of l_1. Since $(l_1)' = l_\infty$ has the approximation property, $\mathcal{P}_A(^2l_1) = \mathcal{P}_w(^2l_1)$ and consequently $Q|_{l_1} \in \mathcal{P}_w(^2l_1)$. By ([10], Proposition 2.6) the mapping $\hat{d}Q|_{l_1} : l_1 \to l_\infty$ is compact. If $(e_n)_n$ is the standard vector basis in l_1 then

$$Q(k(e_n)) = Q \circ k(e_n) = P \circ j \circ U \circ k(e_n) = P \circ i(e_n) = P(e_n).$$

Thus $\hat{d}Q|_{l_1}(k(e_n))(y) = 2y_n$, where $y = (y_i)_{i=1}^{\infty} \in l_1$. For $m \neq n$ we have:

$$\|\hat{d}Q|_{l_1}(k(e_n) - k(e_m))(y)\| = 2 \sup_{y \in B_{l_1}} |y_n - y_m| \geq 2.$$

Hence $\hat{d}Q|_{l_1}$ is not compact and consequently Q is not in $\mathcal{P}_A(^2E)$, which contradicts Q-reflexivity. □

LEMMA 2.3. *Let E be a complete Q-reflexive locally convex space whose strong bidual has the strict approximation property. Then the space E'_β does not contain an isomorphic copy of c_0 (or l_∞).*

Proof. Suppose $c_0 \hookrightarrow E'_\beta$. By ([5], Theorem 8) E contains a complemented copy of l_1, which contradicts Proposition 2.2. □

LEMMA 2.4. *Let E be a Q-reflexive locally convex space whose strong dual is an $\aleph_0$-barrelled complete space and whose strong bidual has the strict approximation property. Then E does not contain an isomorphic copy of c_0.*

Proof. Suppose $c_0 \hookrightarrow E$. Then $l_\infty = c_0''$ is a subspace of E''_β, hence $c_0 \hookrightarrow E''_\beta$. By ([5], Theorem 8) E'_β contains a complemented copy of l_1, which contradicts Proposition 2.1. □

REMARK 2.5. Both Fréchet and complete DF Q-reflexive spaces whose strong bidual has the strict approximation property satisfy the conditions of Lemma 2.3 and Lemma 2.4.

For the next proposition we need to impose stronger conditions on E.

DEFINITION 2.6. A topological space X is called *angelic* if every relatively countably compact subset $A \subset X$ is relatively compact in X and every $x \in \overline{A}$ is the limit of a sequence in A. We say that a locally convex space E is *weakly angelic* if $(E, \sigma(E, E'))$ is angelic.

Following the convention in [5] we say that a pair of locally convex spaces (E, F) is *admissible* if

1. both E and F are complete and weakly angelic;
2. E is $\aleph_0$-barrelled;
3. the space $\mathcal{L}_b(E, F)$ admits a strict web in the sense of De Wilde.

Pairs (E, F) where E is Fréchet space and F is complete DF space (or vice versa) are admissible ([5], p.6). We also need the following lemma:

LEMMA 2.7. *If E is a Q-reflexive locally convex space with $(BB)_\infty$ and whose strong bidual has the strict approximation property, then* $\widehat{\bigotimes}_{n,s,\varepsilon} E'_\beta = (\widehat{\bigotimes}_{n,s,\pi} E)'_\beta$ *for every n.*

Proof. Using tensor representations we have

$$\widehat{\bigotimes_{n,s,\varepsilon}} E'_\beta = (\mathcal{P}_A(^nE), \tau_b) = (\mathcal{P}(^nE), \tau_b),$$

and since E has $(BB)_\infty$-property the topologies β and τ_b are equal. Thus

$$(\mathcal{P}(^nE), \tau_b) = (\mathcal{P}(^nE), \beta) = (\widehat{\bigotimes_{n,s,\pi}} E)'_\beta.$$

□

DEFINITION 2.8. Let E be a locally convex space. The ε-product, $E\varepsilon F$, is the operator space $L_\varepsilon(E'_c, F)$ of all weak*-weakly continuous linear maps from E' into F which transform equicontinuous subsets of E' into relatively compact subsets of F, endowed with the topology of uniform convergence on the equicontinuous sets in E'.

PROPOSITION 2.9. *Let E be either a*
(a) *complete DF space, or*
(b) *Fréchet space with $(BB)_\infty$-property.*
If E is Q-reflexive space and its strong bidual has the strict approximation property, then the space l_∞ is not isomorphic to a subspace of $(\mathcal{P}(^nE), \tau_b)$ for any n.

Proof. (a) Suppose there exists an integer $n > 1$ such that $l_\infty \hookrightarrow (\mathcal{P}(^nE), \tau_b)$. By the Q-reflexivity $(\mathcal{P}(^nE), \tau_b) = (\mathcal{P}_A(^nE), \tau_b) = \widehat{\bigotimes}_{n,s,\varepsilon} E'_\beta$, hence $l_\infty \hookrightarrow \underbrace{E'_\beta\varepsilon\cdots\varepsilon E'_\beta}_{n}$. By Definition 2.8 this is equivalent to $l_\infty \hookrightarrow L_\varepsilon\big((\underbrace{E'_\beta\varepsilon\cdots\varepsilon E'_\beta}_{n-1})'_c, E'_\beta\big)$. By ([14], Proposition 16.1.2 and Theorem 16.1.5) the space $\underbrace{E'_\beta\varepsilon\cdots\varepsilon E'_\beta}_{n-1}$ is Fréchet, hence $(\underbrace{E'_\beta\varepsilon\cdots\varepsilon E'_\beta}_{n-1})'_\beta$ is a complete DF space and the pair $((\underbrace{E'_\beta\varepsilon\cdots\varepsilon E'_\beta}_{n-1})'_\beta, E'_\beta)$ is admissible. By ([5], Theorem 9) l_∞ is a subspace either of $(\underbrace{E'_\beta\varepsilon\cdots\varepsilon E'_\beta}_{n-1})$ or of E'_β. By the Q-reflexivity and Lemma 2.3 we have $l_\infty \not\hookrightarrow E'_\beta$, consequently $l_\infty \hookrightarrow \underbrace{E'_\beta\varepsilon\cdots\varepsilon E'_\beta}_{n-1}$. Repeating the above argument $n-1$ times we arrive to the conclusion that $l_\infty \hookrightarrow E'_\beta$, which contradicts Lemma 2.3.

(b) As in (a) suppose there exists an integer $n > 1$ such that $l_\infty \hookrightarrow (\mathcal{P}(^nE), \tau_b) = \widehat{\bigotimes}_{n,s,\varepsilon} E'_\beta$. By ([14], p.344)

$$\widehat{\bigotimes}_{n,s,\varepsilon} E'_\beta \hookrightarrow E'_\beta \widehat{\bigotimes}_{\varepsilon} \big(\widehat{\bigotimes}_{n-1,s,\varepsilon} E'_\beta\big) \hookrightarrow E'_\beta\varepsilon\big(\widehat{\bigotimes}_{n-1,s,\varepsilon} E'_\beta\big),$$

hence $l_\infty \hookrightarrow E'_\beta\varepsilon\big(\widehat{\bigotimes}_{n-1,s,\varepsilon} E'_\beta\big)$. By Lemma 2.7 $(\widehat{\bigotimes}_{n-1,s,\pi} E)'_\beta = \widehat{\bigotimes}_{n-1,s,\varepsilon} E'_\beta$, hence

$$l_\infty \hookrightarrow L_\varepsilon(((\widehat{\bigotimes}_{n-1,s,\pi} E)'_\beta)'_c, E'_\beta).$$

Since the space $(\widehat{\bigotimes}_{n-1,s,\pi} E)''_\beta$ is Fréchet, the pair $\big((\widehat{\bigotimes}_{n-1,s,\pi} E)''_\beta, E'_\beta\big)$ is admissible and by ([5], Theorem 9) l_∞ is a subspace either of $(\widehat{\bigotimes}_{n-1,s,\pi} E)'_\beta$ or of E'_β. By Q-reflexivity and Lemma 2.3, $l_\infty \not\hookrightarrow E'_\beta$, consequently

$$l_\infty \hookrightarrow (\widehat{\bigotimes}_{n-1,s,\pi} E)'_\beta = (\mathcal{P}(^{n-1}E), \tau_b).$$

Repeating the above argument $n-1$ times we get $l_\infty \hookrightarrow E'_\beta$, which contradicts Lemma 2.3. □

COROLLARY 2.10. *Let E be a complete DF space or a Fréchet space with $(BB)_\infty$. If E is Q-reflexive and its strong bidual has the strict* *approximation property, then the space l_1 is not complemented in $\widehat{\bigotimes}_{n,s,\pi} E$ for any n.*

Proof. Follows from Proposition 2.9 and ([5], Corollary 7). □

3. In this section we investigate the connection between (semi)reflexivity and Q-reflexivity. First we need to give some definitions. Defant, in [8], introduces locally convex spaces whose duals have the *local Radon Nikodým Property* (local RNP), as follows: E is said to have a dual with the local RNP if for every probability space (Ω, Σ, μ) all operators $T : L^1(\mu) \to E'$ which map some neighbourhood of 0 into an equicontinuous set are locally representable. In [6], Boyd renames a space which has a dual with the local RNP *locally Asplund.* Nuclear spaces, semireflexive quasinormable spaces and gDF spaces with separable duals are all locally Asplund. In [6] is shown that if E is locally Asplund then $(\mathcal{P}_I({}^nE), \tau_I) = (\mathcal{P}_N({}^nE), \pi_w)$.

PROPOSITION 3.1. *Let E be a reflexive DF space with the strict approximation property and such that E'_β is quasinormable. The following conditions are equivalent:*

1. *E is Q-reflexive,*
2. *$\mathcal{P}_A({}^nE) = \mathcal{P}({}^nE)$ for every n,*
3. *the space $(\mathcal{P}({}^nE), \tau_b)$ is reflexive for every n.*

Proof. By the hypothesis E'_β is a quasinormable reflexive space, hence by [8] it is locally Asplund. Thus $(\mathcal{P}_I({}^nE'_\beta), \tau_I) = (\mathcal{P}_N({}^nE'_\beta), \pi_w)$. Since reflexive spaces are barrelled E'_β is a distinguished Fréchet space, and by ([9], Corollary 1.53) $\pi_w = \pi_b$ on $\mathcal{P}_N({}^nE'_\beta)$. By Proposition 1.2 (1)⇔(2). (2)⇔(3) follows from ([6], Corollary 13). □

REMARK 3.2. If E is DF space which is separable or has the strict Mackey condition, then E'_β is quasinormable.

A similar result holds in the Fréchet space case.

PROPOSITION 3.3. *Let E be a reflexive Fréchet space with the strict approximation property. The following conditions are equivalent:*

1. *E is Q-reflexive,*
2. *$\mathcal{P}_A({}^nE) = \mathcal{P}({}^nE)$ for every n,*
3. *the space $(\mathcal{P}_A({}^nE), \tau_b)$ is semireflexive for every n.*

Proof. The space E'_β is a reflexive DF space and, in particular, is infrabarrelled and locally Asplund. By ([6], Theorem 3) and ([9], Corollary 1.53) $(\mathcal{P}_I({}^nE'_\beta), \tau_I) = (\mathcal{P}_N({}^nE'_\beta), \pi_b)$. Applying Proposition 1.4 gives us (1)$\Leftrightarrow$(2).
(2)$\Leftrightarrow$(3) follows from ([6], Corollary 10). □

4. The first known examples of Q-reflexive Banach spaces were the Tsirelson space T^* and the Tsirelson-James space T^*_J. By [7] $\bigoplus_{j=1}^{\infty} T^*_J$, $\prod_{k=1}^{\infty} T^*_J$ and Fréchet-Montel spaces with $(BB)_\infty$ are Q-reflexive locally convex spaces. In this section we give further examples of such spaces.

Proposition 4.1. *Let E be a DFM space. Then E is Q-reflexive.*

Proof. Since E is DFM it is reflexive and infrabarrelled, hence

$$\widehat{\bigotimes_{n,s,\pi}} E''_e = \widehat{\bigotimes_{n,s,\pi}} E''_\beta = \widehat{\bigotimes_{n,s,\pi}} E.$$

By ([14], Theorem 15.6.2) $\widehat{\bigotimes}_{n,s,\pi} E$ is a DFM space and in particular is reflexive and barrelled. Thus $(\widehat{\bigotimes}_{n,s,\pi} E)'_\beta$ is distinguished Fréchet space and, by ([4], Corollary 3.4), $(\widehat{\bigotimes}_{n,s,\pi} E)''_{\beta i} = (\widehat{\bigotimes}_{n,s,\pi} E)''_\beta$. As a DF space E has $(BB)_\infty$,

$$(\mathcal{P}({}^nE), \tau_b)'_i = (\widehat{\bigotimes_{n,s,\pi}} E)''_{\beta i} = (\widehat{\bigotimes_{n,s,\pi}} E)''_\beta = \widehat{\bigotimes_{n,s,\pi}} E.$$

By the definition of J_n it is an isomorphism. □

Lemma 4.2. *Let G be a Fréchet space with $(BB)_\infty$ and F be a Fréchet nuclear space. Then the space $E := G \times F$ has $(BB)_\infty$.*

Proof. Let B be a bounded subset of $\widehat{\bigotimes}_{n,s,\pi} E$. By ([1], Theorem 2.2)

$$\widehat{\bigotimes_{n,s,\pi}} E = \bigoplus_{k=0}^{n} [(\widehat{\bigotimes_{k,s,\pi}} G) \widehat{\bigotimes_{\pi}} (\widehat{\bigotimes_{n-k,s,\pi}} F)],$$

hence there exist $B_0, B_1, \dots, B_n$ such that $B \subset B_0 \times B_1 \times \cdots \times B_n$ and B_k is a bounded subset of $(\widehat{\bigotimes}_{k,s,\pi} G) \widehat{\bigotimes}_{\pi} (\widehat{\bigotimes}_{n-k,s,\pi} F)$. Since $\widehat{\bigotimes}_{n-k,\pi} F$ is nuclear, the pairs $\{\widehat{\bigotimes}_{k,\pi} G, \widehat{\bigotimes}_{n-k,\pi} F\}$ have the (BB) property ([14], Theorem 21.5.8), hence each B_k is contained in $\overline{\Gamma}(B'_k \otimes B''_k)$ for some B'_k bounded in $\widehat{\bigotimes}_{k,s,\pi} G$ and B''_k bounded in $\widehat{\bigotimes}_{n-k,s,\pi} F$. Since G and F have $(BB)_\infty$ there exist $\widetilde{B}'_k$ bounded in G and $\widetilde{B}''_k$ bounded in F such that $B'_k \subset \overline{\Gamma}(\otimes_{k,s} \widetilde{B}'_k)$ and $B''_k \subset \overline{\Gamma}(\otimes_{n-k,s} \widetilde{B}''_k)$. Hence $\widetilde{B}' = \cup_{k=0}^n \widetilde{B}'_k$ and $\widetilde{B}'' = \cup_{k=0}^n \widetilde{B}''_k$ are bounded subsets in G and F respectively, and $\widetilde{B} = \widetilde{B}' \times \widetilde{B}''$ is bounded in E. The set B is contained in $\overline{\Gamma}(\otimes_{n,s} \widetilde{B})$, hence E has the $(BB)_\infty$. □

We will also need the following result of Grothendieck ([12]):

Proposition 4.3. *If Z and W are Fréchet (respectively DF) spaces and one of them is nuclear then* $(Z \widehat{\otimes}_\varepsilon W)'_\beta = Z'_\beta \widehat{\otimes}_\pi W'_\beta$.

Proposition 4.4. *Let G be a Q-reflexive Fréchet space with $(BB)_\infty$ and such that G''_β has the strict approximation property, and F be a Fréchet nuclear space. Then $E = G \times F$ is Q-reflexive.*

Proof. By ([1], Theorem 2.2)

$$(\widehat{\bigotimes}_{n,s,\pi} E)'_\beta = (\widehat{\bigotimes}_{n,s,\pi} (G \times F))'_\beta = \bigoplus_{k=0}^n [(\widehat{\bigotimes}_{k,s,\pi} G) \widehat{\otimes}_\pi (\widehat{\bigotimes}_{n-k,s,\pi} F)]'_\beta.$$

If $k = n$, by Lemma 2.7 $(\widehat{\bigotimes}_{n,s,\pi} G)'_\beta = \widehat{\bigotimes}_{n,s,\varepsilon} G'_\beta$. Let $1 \le k < n$. Since $(\widehat{\bigotimes}_{k,s,\pi} G)'_\beta$ is DF and $(\widehat{\bigotimes}_{n-k,s,\pi} F)'_\beta$ a DFN, we have

$$[(\widehat{\bigotimes}_{k,s,\pi} G) \widehat{\otimes}_\pi (\widehat{\bigotimes}_{n-k,s,\pi} F)]'_\beta = [(\widehat{\bigotimes}_{k,s,\pi} G) \widehat{\otimes}_\varepsilon (\widehat{\bigotimes}_{n-k,s,\pi} F)]'_\beta.$$

By Proposition 4.3

$$[(\widehat{\bigotimes}_{k,s,\pi} G) \widehat{\otimes}_\varepsilon (\widehat{\bigotimes}_{n-k,s,\pi} F)]'_\beta = (\widehat{\bigotimes}_{k,s,\pi} G)'_\beta \widehat{\otimes}_\pi (\widehat{\bigotimes}_{n-k,s,\pi} F)'_\beta.$$

Thus,

$$\begin{aligned}(\mathcal{P}(^nE), \beta) = (\widehat{\bigotimes}_{n,s,\pi} E)'_\beta &= \bigoplus_{k=0}^n (\widehat{\bigotimes}_{k,s,\pi} G)'_\beta \widehat{\otimes}_\pi (\widehat{\bigotimes}_{n-k,s,\pi} F)'_\beta \\ &= \bigoplus_{k=0}^n (\widehat{\bigotimes}_{k,s,\pi} G)'_\beta \widehat{\otimes}_\varepsilon (\widehat{\bigotimes}_{n-k,s,\pi} F)'_\beta.\end{aligned}$$

By Lemma 2.7 applied to G and by the nuclearity of F,

$$\bigoplus_{k=0}^{n}(\widehat{\bigotimes_{k,s,\pi}} G)'_\beta \widehat{\bigotimes_{\varepsilon}}(\widehat{\bigotimes_{n-k,s,\pi}} F)'_\beta = \bigoplus_{k=0}^{n}\left[(\widehat{\bigotimes_{k,s,\varepsilon}} G'_\beta)\widehat{\bigotimes_{\varepsilon}}(\widehat{\bigotimes_{n-k,s,\varepsilon}} F'_\beta)\right],$$

hence

$$\begin{aligned}(\mathcal{P}(^nE),\beta) &= \bigoplus_{k=0}^{n}\left[(\widehat{\bigotimes_{k,s,\varepsilon}} G'_\beta)\widehat{\bigotimes_{\varepsilon}}(\widehat{\bigotimes_{n-k,s,\varepsilon}} F'_\beta)\right] \\ &= \widehat{\bigotimes_{n,s,\varepsilon}}(G\times F)'_\beta = (\mathcal{P}_A(^nE),\tau_b).\end{aligned}$$

In particular $\mathcal{P}(^nE) = \mathcal{P}_A(^nE)$. By ([6], Propositions 1 and 2),

$$(\mathcal{P}_I(^nE'_\beta),\tau_I) = (\mathcal{P}_A(^nE),\tau_b)'_i = (\widehat{\bigotimes_{n,s,\varepsilon}} E'_\beta)'_i.$$

Since $\widehat{\bigotimes_{n,s,\varepsilon}} E'_\beta = (\widehat{\bigotimes_{n,s,\pi}} E)'_\beta$ is DF we have $(\widehat{\bigotimes_{n,s,\varepsilon}} E'_\beta)'_i = (\widehat{\bigotimes_{n,s,\varepsilon}} E'_\beta)'_\beta$. Moreover

$$(\widehat{\bigotimes_{n,s,\varepsilon}} E'_\beta)'_\beta = \big(\widehat{\bigotimes_{n,s,\varepsilon}}(G\times F)'_\beta\big)'_\beta = \bigoplus_{k=0}^{n}\left[(\widehat{\bigotimes_{k,s,\varepsilon}} G'_\beta)\widehat{\bigotimes_{\varepsilon}}(\widehat{\bigotimes_{n-k,s,\varepsilon}} F'_\beta)\right]'_\beta.$$

If $k=n$, by the Q-reflexivity of G we have

$$(\widehat{\bigotimes_{n,s,\varepsilon}} G'_\beta)'_\beta = (\widehat{\bigotimes_{n,s,\varepsilon}} G'_\beta)'_i = \widehat{\bigotimes_{n,s,\pi}} G''_{\beta\beta}. \tag{3}$$

Let $1 \le k < n$. Since $\widehat{\bigotimes_{n-k,s,\varepsilon}} F'_\beta$ is DFN, by Proposition 4.3

$$\bigoplus_{k=0}^{n-1}\left[(\widehat{\bigotimes_{k,s,\varepsilon}} G'_\beta)\widehat{\bigotimes_{\varepsilon}}(\widehat{\bigotimes_{n-k,s,\varepsilon}} F'_\beta)\right]'_\beta = \bigoplus_{k=0}^{n-1}\left[(\widehat{\bigotimes_{k,s,\varepsilon}} G'_\beta)'_\beta\widehat{\bigotimes_{\pi}}(\widehat{\bigotimes_{n-k,s,\varepsilon}} F'_\beta)'_\beta\right].$$

Using (3) and the nuclearity of F,

$$\begin{aligned}\bigoplus_{k=0}^{n}\left[(\widehat{\bigotimes_{k,s,\varepsilon}} G'_\beta)'_\beta\widehat{\bigotimes_{\pi}}(\widehat{\bigotimes_{n-k,s,\varepsilon}} F'_\beta)'_\beta\right] &= \bigoplus_{k=0}^{n}\left[(\widehat{\bigotimes_{k,s,\pi}} G''_\beta)\widehat{\bigotimes_{\pi}}(\widehat{\bigotimes_{n-k,s,\pi}} F''_\beta)\right] \\ &= \widehat{\bigotimes_{n,s,\pi}}(G\times F)''_\beta.\end{aligned}$$

Hence by representation (2),

$$(\mathcal{P}_I(^nE'_\beta),\tau_I) = (\widehat{\bigotimes_{n,s,\varepsilon}} E'_\beta)'_i = \widehat{\bigotimes_{n,s,\pi}}(G\times F)''_\beta = \widehat{\bigotimes_{n,s,\pi}} E''_\beta = \overline{(\mathcal{P}_N(^nE'_\beta),\pi_b)}.$$

Applying Proposition 1.4 completes the proof. □

This proposition gives us a range of non-Banach Q-reflexive spaces. If the space G is nonreflexive, for example $T_J{}^*$ or $T^* \widehat{\bigotimes_\pi} T_J{}^*$, then E is nonreflexive and Q-reflexive.

The following result can be proved similarly to Proposition 4.4.

PROPOSITION 4.5. *Let G be a Q-reflexive DF space such that $G''_{\beta\beta}$ has the strict approximation property, and let F be a DFN space. Then $E := G \times F$ is Q-reflexive.*

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