J. Korean Math. Soc. **41** (2004), No. 1, pp. 39–50

HOLOMORPHIC FUNCTIONS AND THE BB-PROPERTY ON PRODUCT SPACES

CHRISTOPHER BOYD

ABSTRACT. In [25] Taskinen shows that if $\{E_n\}_n$ and $\{F_n\}_n$ are two sequences of Fréchet spaces such that (E_m, F_n) has the BB-property for all m and n then $\left(\prod_m E_m, \prod_n F_n\right)$ also has the BB-property. Here we investigate when this result extends to
(i) arbitrary products of Fréchet spaces,
(ii) countable products of $\mathcal{D}FN$ spaces,
(iii) countable direct sums of Fréchet nuclear spaces.
We also look at topologies properties of $(\mathcal{H}(U), \tau)$ for U balanced open in a product of Fréchet spaces and $\tau = \tau_o, \tau_\omega$ or τ_δ.

1. Introduction

Our main motivation in this paper is in studying holomorphic functions on products of $\mathcal{D}FN$ and Fréchet spaces. For Fréchet spaces our main interest is in the case of uncountable products while for $\mathcal{D}FN$ spaces countable products are sufficient to uncover new interesting results. If B_1 (resp. B_2) is a bounded subset of the locally convex space E (resp. F) then $\bar{\Gamma}(B_1 \bigotimes_\pi B_2)$ is a bounded subset of $E\widehat{\bigotimes}_\pi F$. We shall say that the pair of spaces (E, F) has the BB-property if every bounded subset of $E\widehat{\bigotimes}_\pi F$ is contained in a set of this form. The BB-property is fundamental in the study of equivalence of certain topologies on certain locally topologies (see [16]) and in the establishment of exponential laws for certain spaces of holomorphic functions (see [8]). In our investigation of holomorphic functions on products of $\mathcal{D}FN$ and Fréchet spaces we discover that the problems under consideration are closely related to extending the following result of Taskinen [25]: if $\{E_m\}_m$ and $\{F_n\}_n$ are two sequences of Fréchet spaces such that (E_m, F_n) has the

Received November 9, 2002.
2000 Mathematics Subject Classification: Primary 46G20, 46G25; Secondary 46A12.
Key words and phrases: BB-property, products, direct sums.

BB-property for all m and n then $(\prod_m E_m, \prod_n F_n)$ also has the BB-property. In fact we characterise arbitrary products of Fréchet spaces and products of $\mathcal{D}FN$ spaces for which various extensions of Taskinen's result hold. The crucial examples which arise in our study are the spaces $\mathbf{C}^I$ (I uncountable), $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ or $\phi\omega$ and $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ or $\omega\phi$.

In section 1 we collect the main known results for the spaces $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ and $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ and prove some new results. In section 2 we discuss the BB-property and a weaker concept — the BB-property for points. This section contains our main results and include the following:

1. (1) if E is a countable product of $\mathcal{D}FN$ spaces then (E, E) has the BB-property if and only if E is isomorphic to $\mathbf{C}^{\mathbf{N}}$,
2. (2) uncountable products do not have the BB-property,
3. (3) if E is a direct sum of Fréchet Nuclear spaces then (E, E) has the BB-property if and only if it admits a continuous norm.

In the final section we prove various results for holomorphic functions on products. We refer the reader to [20] for further information on locally convex spaces and to [14] for further information on infinite dimensional holomorphy.

2. Holomorphic functions on the spaces $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ and $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$

The aim of this section is to investigate holomorphic functions on the spaces $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ and $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$. These space have proved useful in the linear theory and in constructing counterexamples to many conjectures. They are considered by Köthe in [22, § 13.5] where he uses ω to denote $\mathbf{C}^{\mathbf{N}}$, ϕ to denote $\mathbf{C}^{(\mathbf{N})}$, $\phi\omega$ to denote $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ and $\omega\phi$ to denote $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$. The spaces $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ and $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ are both reflexive A-nuclear spaces with a basis and hence are fully nuclear spaces. It is clear that $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ is the strong dual of $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$. Since $\mathbf{C}$ is a closed subspace of $\mathbf{C}^{(\mathbf{N})}$ it will follow that $\mathbf{C}^{(\mathbf{N})} \times \mathbf{C}^{\mathbf{N}} \simeq \mathbf{C}^{(\mathbf{N})} \times \mathbf{C} \times \cdots \times \mathbf{C} \cdots$ is a closed complemented subspace of $\mathbf{C}^{(\mathbf{N})} \times \mathbf{C}^{(\mathbf{N})} \times \cdots \times \mathbf{C}^{(\mathbf{N})} \cdots \simeq (\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$. As $\mathbf{C}^{\mathbf{N}} \times \mathbf{C}^{(\mathbf{N})}$ is isomorphic to its dual we have that $\mathbf{C}^{\mathbf{N}} \times \mathbf{C}^{(\mathbf{N})}$ is a closed complemented subspace of $\mathbf{C}^{\mathbf{N}} \oplus \mathbf{C}^{\mathbf{N}} \oplus \cdots \oplus \mathbf{C}^{\mathbf{N}} \cdots \simeq (\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$.

Let E a locally convex space and n a positive integer. In [16], four topologies are considered on $P(^nE) = (\widehat{\bigotimes_{s,n,\pi}} E)'$. We use τ_o (resp. τ_b) to denote the compact-open (resp. strong) topology — the topology of uniform convergence on compact (resp. bounded) subsets of E and τ_ω to denote the Nachbin ported topology. We define the topology β

by $(P(^nE),\beta) := \left(\widehat{\bigotimes_{s,n,\pi}} E\right)'_b$. We always have that $\tau_o \leq \tau_b \leq \beta \leq \tau_\omega$ on $P(^nE)$. We know that $\tau_o = \tau_b$ on $P(^nE)$ if and only if E is semi-Montel and that $\beta = \tau_b$ on $P(^nE)$ if and only if each bounded subset of $\widehat{\bigotimes_{s,n,\pi}} E$ is contained in a set of the form $\overline{\Gamma}\left(\bigotimes_{s,n,\pi} B\right)$ for B a bounded subset of E. When this happens we shall say that E has $(BB)_n$. In particular, as noted in [16], if (E,E) has BB then $\beta = \tau_b$ on $P(^2E)$. Since $(\mathcal{P}(^nE),\tau_\omega) = \left(\widehat{\bigotimes_{s,n,\pi}} E\right)'_i$, we see that $\beta = \tau_\omega$ on $P(^nE)$ if and only if $\left(\widehat{\bigotimes_{s,n,\pi}} E\right)'_b = \left(\widehat{\bigotimes_{s,n,\pi}} E\right)'_i$.

We now list properties of spaces of holomorphic functions on $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ and $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$.

THEOREM 1.

1. *For* $E = (\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ *or* $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ *and* $n \geq 2$, *we have* $\tau_o \neq \tau_\omega$ *on* $P(^nE)$. *For* U *open in* $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ *or* $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ *we have* $\tau_\omega \neq \tau_\delta$ *on* $\mathcal{H}(U)$.
2. *For any open subset of* $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ *or* $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ *we have* $\mathcal{H}_{HY}(U) = \mathcal{H}_M(U)$. *However for* $E = (\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ *or* $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ *and* $n \geq 2$ *we have* $P(^nE) \neq P_{HY}(^nE)$. *In particular neither* $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ *or* $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ *are k-spaces.*
3. *For* U *an open polydisc in* $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ *or* $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$, $(\mathcal{H}(U),\tau_o)$ *is not complete and the* τ_o *bounded sets of* $\mathcal{H}(U)$ *are not locally bounded.*
4. *We have the identity*
$$(\mathcal{H}((\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}),\tau_\delta) = \mathrm{ind}_n(\mathcal{H}((\mathbf{C}^{(\mathbf{N})})^n),\tau_o)$$
and $(\mathcal{H}((\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}),\tau_\delta)$ *and* $(\mathcal{H}((\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}),\tau_\omega)$ *are both complete.*

Proof.

1. Since $\tau_o \neq \tau_\omega$ on $P(^2\mathbf{C}^{\mathbf{N}} \times \mathbf{C}^{(\mathbf{N})})$ it will follow, by a modification of [15, Theorems 5 and 9], that $\tau_o \neq \tau_\omega$ on $P(^2E)$ for $E = (\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ or $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$.

   Since $\mathbf{C}^{\mathbf{N}}$ contain a nontrivial very strongly convergent sequence it follows that $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ and $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ both contain a nontrivial very strongly convergent sequences. Therefore from [13, 2.52] we have that $\tau_\omega \neq \tau_\delta$ on $\mathcal{H}(U)$ for U open in either $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ or $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$.

2. For every compact subset K of $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ (resp. $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$) there is a bounded subset B such that K is contained and compact in E_B. It follows from [13, Example 2.19] that for any open subset U of either of these spaces $\mathcal{H}_{HY}(U) = \mathcal{H}_M(U)$. ($\mathcal{H}_M(U)$ is the space G-holomorphic functions on U such that $\frac{\hat{d}^n f(x)}{n!}$ sends bounded sets to bounded sets for each x in U and each integer n.) From [7, Corollary] and [24, Proposition 2] we conclude that for E equal to $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$ or $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ we have that $P(^nE) \neq P_{HY}(^nE)$.
3. Apply (2) and [6, Proposition 9].
4. Since $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$ is an open and compact surjective limit of $\mathcal{D}FN$ spaces we may apply [13, Proposition 6.28] to conclude that

$$(\mathcal{H}((\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}), \tau_\delta) = \operatorname{ind}_n(\mathcal{H}((\mathbf{C}^{(\mathbf{N})})^n), \tau_o)$$

and that $(\mathcal{H}((\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}), \tau_\delta)$ is complete. By [3, Theorem 4.1] the τ_ω-bounded sets of $\mathcal{H}(U)$ are locally bounded for every open subset U of $(\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}$. Applying [13, Proposition 5.37] we see that $(\mathcal{H}((\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}), \tau_\omega)$ is complete. This means in particular that the τ_o-bounded sets of $\mathcal{H}((\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}})$ are not τ_ω-bounded. □

In [9] the authors show that $\mathcal{H}(U)$ is complete and the τ_ω-bounded subsets of $\mathcal{H}(U)$ are locally bounded for U an open subset of $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$. In the class of direct sums of Fréchet nuclear space, spaces with this properties are the exception. In fact $(\mathbf{C}^{\mathbf{N}})^{(\mathbf{N})}$, $\mathbf{C}^{\mathbf{N}}$and $\mathbf{C}^{\mathbf{N}} \times \mathbf{C}^{(\mathbf{N})}$ are the only three spaces in this class with the property that the τ_ω-bounded sets are locally bounded (see [9]).

3. The BB-property for points

Let us recall that the pair of locally convex space E and F has the BB-property if each bounded set in $E\widehat{\bigotimes}_\pi F$ is contained in a set of the form $\bar{\Gamma}\,(B_1 \bigotimes_\pi B_2)$ for some bounded sets B_1 in E and B_2 in F.

We shall say that a locally convex space E has the *BB-property for points* if each point in $E\widehat{\bigotimes}_\pi E$ is contained in a set of the form $\bar{\Gamma}\,(B \bigotimes B)$ for some bounded set B in E.

Given locally convex spaces E and F we define $\Sigma(E'_b\widehat{\bigotimes}_\pi F'_b)$

$$\Sigma(E'_b\widehat{\bigotimes}_\pi F'_b)$$

$$:= \left\{ \sum_{n=1}^{\infty} \lambda_n x'_n \otimes y'_n \in E'_b \widehat{\bigotimes}_\pi F'_b : \lambda \in \ell_1, \right.$$
$$\left. \{x'_n\}_n, \{y'_n\}_n \text{ are equicontinuous of } E'_b \text{ and } F'_b \text{ respectively} \right\}.$$

Defant and Floret [12] call $\Sigma(E'_b\widehat{\otimes}_\pi F'_b)$ the set of all points of $E'_b\widehat{\otimes}_\pi F'_b$ which have series representations. Mangino, [23], characterises when each point in the projective tensor product of two complete countable inductive limits of Banach spaces with the approximation property has a series representation. We note that if every point of $E'_b\widehat{\otimes}_\pi F'_b$ has a series representation then (E'_b, F'_b) has the BB-property for points.

In [25] Corollary 3.6, Taskinen shows that if $\{E_n\}_{n\in\mathbf{N}}$ and $\{F_n\}_{n\in\mathbf{N}}$ are two sequences of Fréchet spaces such that (E_m, F_n) has BB for each pair of integers (m,n) then $(\prod_{m\in\mathbf{N}} E_m, \prod_{n\in\mathbf{N}} F_n)$ has BB. We will now show that this result does not extend to countable products of $\mathcal{D}FN$-spaces, or to uncountable products of Fréchet spaces. We prove that a countable product of $\mathcal{D}FN$ spaces has the BB-property for points if and only if the space is isomorphic to $\mathbf{C}^{\mathbf{N}}$.

PROPOSITION 2. *Let $E = \prod_{n=0}^{\infty} E_n$ be a product of $\mathcal{D}FN$ spaces, then $\tau_o = \tau_\omega$ on $P(^2E)$ if and only if $E = \mathbf{C}^{\mathbf{N}}$.*

Proof. If $E = \mathbf{C}^{\mathbf{N}}$ then E is a Fréchet nuclear space and hence $\tau_o = \tau_\omega$ on $P(^2E)$. If $E \neq \mathbf{C}^{\mathbf{N}}$ then we may assume without loss of generality that $(E_0)'_b$ is not a normed space. By [7, Corollary 2] it follows that $P(^2E'_b) \neq P_{HY}(^2E'_b)$ and therefore since E is fully nuclear, $P(^2E'_b) \neq P_M(^2E'_b)$. Applying [13, Corollary 1.50] we get $\tau_o \neq \tau_\omega$ on $P(^2E)$. □

We now show that we cannot replace the Fréchet spaces in [25, Corollary 3.6] by $\mathcal{D}FN$ spaces.

The τ_o topology on $P(^nE)$ may be defined in terms of the duality between $P(^nE)$ and $\widehat{\bigotimes}_{s,n,\pi} E$ as the topology on $\left(\widehat{\bigotimes}_{s,n,\pi} E\right)'$ of uniform convergence on sets of the form $\bar{\Gamma}\left(\bigotimes_{n,s} K\right)$ where K is a compact subset of E. It is therefore weaker than the topology on $P(^nE)$ of uniform convergence on all compact subsets of $\widehat{\bigotimes}_{s,n,\pi} E$ and hence is weaker than the Mackey topology $\tau(P(^nE), \widehat{\bigotimes}_{s,n,\pi} E)$. If every $x \in \widehat{\bigotimes}_{s,n,\pi} E$ is contained in a set of the form $\bar{\Gamma}\left(\bigotimes_{n,s}K\right)$ for K compact in E, then the τ_o-topology on $P(^nE)$ is finer than the $\sigma(P(^nE), \widehat{\bigotimes_{s,n,\pi}}E)$-topology on $P(^nE)$. This implies that τ_o is compatible with the duality between $P(^nE)$ and $\widehat{\bigotimes_{s,n,\pi}}E$ and hence we have that $(\mathcal{P}(^nE), \tau_o)' = \widehat{\bigotimes_{s,n,\pi}}E$.

Let us now suppose that E is a fully nuclear space in which the τ_ω-bounded sets are locally bounded. An adaptation of the proof of [17, Theorem 8] gives us that $\widehat{\bigotimes_{s,n,\pi}}E = (\mathcal{P}(^nE), \tau_\omega)'$. Therefore, if every point in $\widehat{\bigotimes_{s,n,\pi}}E$ is contained in a set of the form $\bar{\Gamma}\left(\bigotimes_{n,s}K\right)$ for K compact in E we have that $(\mathcal{P}(^nE), \tau_o)' = (\mathcal{P}(^nE), \tau_\omega)'$. Applying [13, Propositions 1.56 and 1.67] this implies that $P(^nE'_b) = P_{HY}(^nE'_b)$.

Let $E = \prod_n E_n$ be a product of $\mathcal{D}FN$ spaces at least one of which is infinite dimensional. It follows from [3, Theorem 4.1] that the τ_ω-bounded subsets of $P(^nE)$ are locally bounded. Therefore, if E has the the BB-property for points the above paragraph implies that $P(^nE'_b) = P_{HY}(^nE'_b)$. Since E'_b is a direct sum of Fréchet nuclear spaces at least one of which is infinite dimensional this contradicts [7, Corollary 2] and we obtain the following result.

THEOREM 3. *Let E be a countable product of $\mathcal{D}FN$ spaces, then (E, E) has the BB property if and only if $E = \mathbf{C}^{\mathbf{N}}$.*

REMARKS.

1. This means that $((\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}}, (\mathbf{C}^{(\mathbf{N})})^{\mathbf{N}})$ does not have the BB-property for points, to see this set $E_n = \mathbf{C}^{(\mathbf{N})}$.
2. We denote by $\mathcal{D}'$ the space of distributions on the real line. Since $\mathcal{D}' = \prod_n s'$ (s the space of rapidly decreasing sequences) we see that $(\mathcal{D}', \mathcal{D}')$ does not have the BB-property for points.
3. If we set $E_0 = \mathbf{C}^{(\mathbf{N})}$ and $E_n = \mathbf{C}$ for $n \geq 1$ we get that $(\mathbf{C}^{(\mathbf{N})} \times \mathbf{C}^{\mathbf{N}}, \mathbf{C}^{(\mathbf{N})} \times \mathbf{C}^{\mathbf{N}})$ does not have the BB-property for points.

If E and F are a pair of Fréchet or DF spaces, one of which is nuclear, then it follows from [19] that $(E\widehat{\bigotimes}_\epsilon F)'_b = E'_b\widehat{\bigotimes}_\pi F'_b$. In [11, Theorem 3] Defant shows that if E and F are two locally convex spaces, such that the dual of one of these spaces has the local RNP and if E'_b has the approximation property then

$$(E\widehat{\bigotimes}_\epsilon F)' = \Sigma(E'_b\widehat{\bigotimes}_\pi F'_b).$$

(For the definition of what it means that a space to have dual with the local RNP see [10]).

If $E = \coprod_{n=1}^{\infty} E_n$ is a direct sum of Fréchet nuclear spaces, E'_b is a locally convex space whose dual has the local RNP and the approximation property. If at least one of these spaces is infinite dimensional then it follows from the above that

$$\Sigma(E'_b\widehat{\bigotimes}_\pi E'_b) \neq E'_b\widehat{\bigotimes}_\pi E'_b.$$

Hence we have

$$(E\widehat{\bigotimes}_\epsilon E)' = E'_b\widehat{\bigotimes}_\pi E'_b$$

if and only if $E = \mathbf{C}^{(\mathbf{N})}$.

THEOREM 4. *$(\mathbf{C}^I, \mathbf{C}^I)$ has the BB-property for points if and only if I is countable.*

Proof. If $I = \mathbf{N}$ then $\mathbf{C}^{\mathbf{N}}$ is a Fréchet nuclear space and therefore has the (BB) property. As we shall see in Proposition 10, $(P(^n\mathbf{C}^I), \tau_\omega)$ is a Montel space. It follows by the argument used in [17, Theorem 8] that $\widehat{\bigotimes}_{s,2,\pi}\mathbf{C}^I = (P(^2\mathbf{C}^I), \tau_\omega)'_b$. Therefore, if every point in $C^I\widehat{\bigotimes}_\pi C^I$ is contained in a set of the form $\left(\bar{\Gamma}\bigotimes_{s,2}K\right)$ for K compact in C^I then $(P(^2C^I), \tau_o)' = (P(^2C^I), \tau_\omega)'$. If I is uncountable this contradicts [2, Lemma 13] and so (C^I, C^I) does not have the BB-property for points when I is uncountable. ([2, Lemma 13] says that the two topologies are not compatible for I having cardinality at least equal to the continuum. By [21] the same result holds for I uncountable.) □

Since the space $\mathbf{C}^I$ is complemented in $\prod_{i\in I} E_i$ for any choice of E_i, we also have:

PROPOSITION 5. *Let $E = \prod_{i\in I} E_i$ be a product of Fréchet space. Then (E, E) has the BB-property if and only I is countable and (E_i, E_j) has the BB-property for every pair (i, j) in I^2.*

To finish this section we consider the BB-property on countable direct sums of Fréchet nuclear spaces.

THEOREM 6. *Let $E = \coprod_{n\in\mathbf{N}} E_n$ be a direct sum of Fréchet nuclear spaces, the following are equivalent:*

1. *$\tau_o = \tau_\omega$ on $\mathcal{H}(E)$,*
2. *$\tau_o = \tau_\omega$ on $P(^nE)$ for each integer n,*
3. *E has $(BB)_n$ for each integer n,*
4. *each E_n admits a continuous norm.*

Proof. Clearly (1) implies (2) and (2) implies (3). If each E_n admits a continuous norm then E admits a continuous norm. Applying [24, Corollary 2 to Proposition 6] we see that $\tau_o = \tau_\omega$ on $\mathcal{H}(E)$ and hence (4) implies (1). If (4) is not true, then we may suppose without loss of generality that E_1 does not admit a continuous norm. By [5, Theorem 1] E_1 contains $\mathbf{C}^{\mathbf{N}}$ as a complemented subspace and therefore E contains $\mathbf{C}^{\mathbf{N}} \times \mathbf{C}^{(\mathbf{N})}$ as a complemented subspace. Since $\mathbf{C}^{\mathbf{N}} \times \mathbf{C}^{(\mathbf{N})}$ does not have $(BB)_2$, it will follow that E does not have $(BB)_2$ and therefore we see that (3) implies (4). □

4. Topological properties of spaces of holomorphic functions on Products of Fréchet spaces

In [1] it is shown that the spaces $(\mathcal{H}(U), \tau_o)$, $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(U), \tau_\delta)$ are complete for U a balanced open subset of $\mathbf{C}^I$. In this section we extend this result to arbitrary products of Fréchet spaces and look at further topological properties of these spaces. We begin by looking at completeness for the τ_o-topology. The proof of this result is an adaptation of [1, Proposition 3].

PROPOSITION 7. *Let $E = \prod_{i\in I} E_i$ be a product of Fréchet spaces and U be a balanced open subset of E then $(\mathcal{H}(U), \tau_o)$ is complete.*

For $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(U), \tau_\delta)$ we have.

PROPOSITION 8. *Let $E = \prod_{i\in I} E_i$ be a product of Fréchet spaces E_i and U be a balanced open subset of E, then $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(U), \tau_\delta)$ are complete.*

Proof. By Proposition 7 $\mathcal{H}(U)$ is both τ_ω and τ_δ-T.S. complete. It suffices to complete the proof to show that $(\mathcal{P}(^nE), \tau_\omega)$ is complete for each integer n.

By [18, Lemma 2] we have

$$\left(\widehat{\bigotimes_{n,\pi}} E\right)'_i = \coprod_{i_1,\ldots,i_n} \left(E_{i_1} \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_{i_n}\right)'_i.$$

Since each $E_{i_1}\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi E_{i_n}$ is a Fréchet space, using [22, p.400], we see that $\left(\widehat{\bigotimes_{n,\pi}} E\right)'_i$ is complete and is equal to

$$\left(\left(\widehat{\bigotimes_{n,\pi}} E\right)', \beta\left(\left(\widehat{\bigotimes_{n,\pi}} E\right)', \left(\widehat{\bigotimes_{n,\pi}} E\right)''\right)\right).$$

We denote by $C(^nE)$ the complement of $\widehat{\bigotimes_{s,n,\pi}} E'$ in $\widehat{\bigotimes_{n,\pi}} E$. We have

$$\begin{aligned}&\left(\left(\widehat{\bigotimes_{n,\pi}} E\right)', \beta\left(\left(\widehat{\bigotimes_{n,\pi}} E\right)', \left(\widehat{\bigotimes_{n,\pi}} E\right)''\right)\right)\\ &= \left(\left(\widehat{\bigotimes_{s,n,\pi}} E\right)', \beta\left(\left(\widehat{\bigotimes_{s,n,\pi}} E\right)', \left(\widehat{\bigotimes_{s,n,\pi}} E\right)''\right)\right)\\ &\quad\bigoplus\left((C(^nE))', \beta\left((C(^nE))', (C(^nE))''\right)\right).\end{aligned}$$

It follows from [18, Lemma 1] that

$$(\mathcal{P}(^nE), \tau_\omega) = \left(\widehat{\bigotimes_{s,n,\pi}} E\right)'_i = \left(\left(\widehat{\bigotimes_{s,n,\pi}} E\right)', \beta\left(\left(\widehat{\bigotimes_{s,n,\pi}} E\right)', \left(\widehat{\bigotimes_{s,n,\pi}} E\right)''\right)\right)$$

is complete. □

To consider Montelness and the approximation property on these spaces we need the following Theorem.

Proposition 9. *Let $E = \prod_{i\in I} E_i$ where each E_i is a Fréchet space and let n be a positive integer such that $E_{i_1}\hat{\otimes}_\pi E_{i_2}\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi E_{i_n}$ is distinguished for each $(i_1,\ldots,i_n)\in I^n$. Then $\beta = \tau_\omega$ on $P(^nE)$.*

Proof. By [20, Theorem 15.4.1] we have that

$$\widehat{\bigotimes_{n,\pi}} E = \prod_{i_1,\ldots,i_n} E_{i_1}\hat{\otimes}_\pi E_{i_2}\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi E_{i_n}.$$

Taking strong duals we get that

$$\left(\widehat{\bigotimes_{n,\pi}} E\right)'_b = \coprod_{i_1,\ldots,i_n}\left(E_{i_1}\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi E_{i_n}\right)'_b.$$

As in Proposition 8 we also have

$$\left(\widehat{\bigotimes_{n,\pi}} E\right)_i' = \coprod_{i_1,\ldots i_n} (E_{i_1}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_{i_n})_i'.$$

Hence

$$\left(\widehat{\bigotimes_{n,\pi}} E\right)_b' = \left(\widehat{\bigotimes_{n,\pi}} E\right)_i'.$$

In particular $\left(\widehat{\bigotimes_{n,\pi}} E\right)_b'$ is bornological. Since $\widehat{\bigotimes_{s,n,\pi}} E$ is complemented in $\widehat{\bigotimes_{n,\pi}} E$ we have that $\left(\widehat{\bigotimes_{s,n,\pi}} E\right)_b'$ is complemented in $\left(\widehat{\bigotimes_{n,\pi}} E\right)_b'$ and therefore it also is bornological. Since $\widehat{\bigotimes_{s,n,\pi}} E$ is barrelled we can apply Proposition 3.3 of [4] to conclude that

$$(P(^nE), \beta) = \left(\widehat{\bigotimes_{s,n,\pi}} E\right)_b' = \left(\widehat{\bigotimes_{s,n,\pi}} E\right)_i' = (\mathcal{P}(^nE), \tau_\omega).$$

□

Remark. Any product of Fréchet Schwartz spaces will satisfy the above condition as will an arbitrary product of Banach spaces.

Using Proposition 9 we can easily show the following results.

Proposition 10. *Let* $E = \prod_{i\in I} E_i$ *be a product of Fréchet spaces such that* $E_{i_1}\hat{\otimes}_\pi E_{i_2}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_{i_n}$ *is reflexive (resp. Montel) for each* $(i_1, \ldots, i_n) \in I^n$ *and let* U *be a balanced open subset of* E *then* $(\mathcal{H}(U), \tau_\omega)$ *is semi-reflexive (resp. semi-Montel) and* $(\mathcal{H}(U), \tau_\delta)$ *is reflexive (resp. Montel).*

Acknowledgements. The author would like to thank Prof. A. Defant for drawing his attention to [10] and [11].

References

[1] J. A. Barroso and S. Dineen, *Holomorphic functions on* $\mathbf{C}^I$*, I uncountable*, Note Mat. Vol. X (1990), no 1, 65–71.

[2] J. A. Barroso and L. Nachbin, *Some topological properties of spaces of holomorphic mappings in infinitely many variables*, Advances in Holomorphy, (ed J.A. Barroso), North-Holland Math. Studies **34** (1979), 67–91.

[3] P. Berner, *Topologies on spaces of holomorphic functions of certain surjective limits*, Infinite Dimensional Holomorphy and Applications, (Ed. M. Matos) North–Holland Math. Studies **12** (1977), 75–92.

[4] I. A. Berezanskiĭ, *Inductively reflexive locally convex spaces*, Soviet Math. Dokl. **9** (1968), 1080–1082.

[5] C. Bessaga and A. Pełczyński, *On a class of B_0-Spaces*, Bull. Acad. Pol. Sci. **5** (1957), 375–377.

[6] P. Boland and S. Dineen, *Duality theory for spaces of germs and holomorphic functions on nuclear spaces*, Advances in Holomorphy, (Ed. J.A. Barroso) North–Holland Math. Studies **34** (1979), 179–207.

[7] ———, *Holomorphy on spaces of distributions*, Pacific J. Math. **92** (1981), no. 1, 27–34.

[8] C. Boyd, *Exponential laws for the Nachbin ported topology*, Canad. Math. Bull. **43** (2000), no. 2, 138–144.

[9] C. Boyd and S. Dineen, *Locally bounded subsets of holomorphic functions*, Matematica Aplicada e computacional **13** (1994), no. 3, 189–194.

[10] A. Defant, *The local Radon-Nikodym property for duals of locally convex spaces*, Bull. Soc. Roy. Sci. Liége **53** (1984), 233–246.

[11] ———, *A duality Theorem for locally convex Tensor Products*, Math. Z. **190** (1985), 45–53.

[12] A. Defant and K. Floret, *Topological tensor products and the approximation property of locally convex spaces*, Bull. Soc. Roy. Sci. Liége **58** (1989), no. 1, 29–51.

[13] S. Dineen, *Complex analysis on locally convex spaces*, North–Holland Math. Studies **57** (1981).

[14] ———, *Complex analysis on infinite dimensional spaces*, Monogr. Math., Springer-Verlag, 1999.

[15] ———, *Topological properties inherited by subspaces of holomorphic functions*, Mathematical analysis and applications, Part A, pp. 317–326, Adv. in Math. Suppl. Stud., **7a**, Academic Press, New York-London, 1981.

[16] ———, *Holomorphic Functions and the (BB)-property*, Math. Scand. **74** (1994), 215–236.

[17] S. Dineen, P. Galindo, D. García and M. Maestre, *Linearisation of holomorphic functions on fully nuclear spaces with a basis*, Glasgow Math. J. **36** (1994), 201–208.

[18] K. Floret, *Über den Dualraum eines lokalkonvexen Unterraumes*, Arch. Math. **25** (1974), no. 6, 646–648.

[19] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).

[20] H. Jarchow, *Locally convex spaces*, B.G. Teubner, Stuttgart, 1981.

[21] T. J. Jech, *On a problem of L. Nachbin*, Proc. Amer. Math. Soc. **79** (1980), 341–342.

[22] G. Köthe, *Topological Vector Spaces. Vol. 1*, Springer-Verlag, Berlin-Heidleberg-New York, 1969.

[23] E. Mangino, *Complete projective tensor products of (LB)-spaces*, Arch. Math. (Basel) **64** (1995), no. 1, 33–41.
[24] L. A. Moraes, *Holomorphic functions on strict inductive limits*, Res. Math. **4** (1981), 201–212.
[25] J. Taskinen, *Counterexamples to "probléme des topologies" of Grothendieck*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes **63** (1986), 25 pp.

Department of Mathematics
University College Dublin
Belfield, Dublin 4, Ireland
E-mail: Christopher.Boyd@ucd.ie