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FUNCTIONS ATTAINING THE SUPREMUM AND ISOMORPHIC PROPERTIES OF A BANACH SPACE

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Dedicated to the memory of Prof. Klaus Floret.

ABSTRACT. We prove that a Banach space that is convex-transitive and such that for some element u in the unit sphere, and for every subspace M containing u, it happens that the subset of norm attaining functionals on M is second Baire category in M^* is, in fact, almost-transitive and superreflexive. We also obtain a characterization of finite-dimensional spaces in terms of functions that attain their supremum: a Banach space is finite-dimensional if, for every equivalent norm, every rank-one operator attains its numerical radius. Finally, we describe the subset of norm attaining functionals on a space isomorphic to ℓ_1, where the norm is the restriction of a Luxembourg norm on L_1. In fact, the subset of norm attaining functionals for this norm coincides with the subset of norm attaining functionals for the usual norm.

Introduction

The spirit of this note is to find some relationship between isomorphic or isometric assumptions of a Banach space and the size of the set of norm attaining functionals. Along this line, James' Theorem is remarkable: a Banach space is reflexive as soon as the set of norm attaining functionals becomes the topological dual space [16]. If the Banach space satisfies the Radon-Nikodym property, then the set of norm attaining

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functionals contains a G_δ-dense set (see [12], [25], [26]). On the other extreme, if the unit ball of the space X is not dentable, a result due to Bourgain and Stegall states that the set of norm attaining functionals is of the first Baire category in the case that X is separable ([13, Problem 3.5.6]). In the general case, it is not known if the same result holds. However, for any compact and Hausdorff topological space K, the set of norm attaining functionals on $C(K)$ is first Baire category in the dual (norm topology) (see [18, Theorem 4]).

We are not going to consider in the dual space any topologies other than the norm topology. However, there are interesting results for the w^*-topology or the weak topology in the dual (see for instance [22], [15, Lemma 11] and [17, Proposition 3.2]). A survey of some of the previous results can be found in [3].

In the following, we will denote by $NA(X)$ the set of norm attaining functionals on a Banach space X, that is,

$$NA(X) := \{x^* \in X^* : \exists x \in X, \|x\| = 1, x^*(x) = \|x^*\|\},$$

where X^* is the topological dual of X.

Before stating the new results, we will mention that any Banach space X can be renormed in such a way that $NA(X)$ has non empty interior [4, Corollary 2]. Under some special assumptions of smoothness, if $NA(X)$ has non empty interior, then the space X has to be reflexive. For instance, this happens for Banach spaces with a Hahn-Banach smooth norm [5] or for spaces having the Mazur intersection property [17]. In [2] the authors proved that a Banach space X is reflexive if it does not contain an isomorphic copy of ℓ_1 and for some $r > 0$, it holds that the dual unit ball B_{X^*} is the w^*-closure of the set

$$\{x^* \in X^* : \|x^*\| = 1, x^* + rB_{X^*} \subset NA(X)\}.$$

It is also known that a Banach space whose dual unit ball can be generated as above is reflexive if the norm is not rough (see [2, Proposition 5]). As a consequence of previous results, spaces having a lot of isometries (convex-transitive) and such that $NA(X)$ has non empty interior are superreflexive, given that they do not contain isomorphic copies of ℓ_1.

In Section 1 we will prove that a certain abundance of norm attaining functionals and isometries on X will imply that the space is superreflexive.

We consider some other functions instead of functionals in the second section. Acosta and Ruiz Galán proved in [6] that a Banach space such that every rank one operator attains its numerical radius has to be reflexive. On the other hand, they showed that every infinite dimensional Banach space with a Schauder basis admits an equivalent norm for which some rank one operator does not attain its numerical radius. Here we extend this result to the general case.

In the last section we describe the subset of norm attaining functionals of a space isomorphic to ℓ_1 with a norm coming from an Orlicz norm on L_1, and we pose some open problems that arise naturally.

Section 1

The result appearing in the first section is motivated by two facts. The first of them is that spaces which are convex transitive (see definition below) are either superreflexive, or the space and the dual are rough [8, Theorem 3.2]. On the other hand, the transpose of any surjective isometry preserves the subset $NA(X)$ and so, by assuming that $NA(X)$ has non empty interior, the isometries will provide a lot of norm attaining functionals if the space is convex-transitive. Under these assumptions we can expect reflexivity, and by using the first result we mentioned, we actually get superreflexivity.

The presence of points with special properties in the interior of $NA(X)$ will force reflexivity. In order to be more precise, we recall some definitions.

DEFINITION 1. A point u in S_X (the unit sphere of X) is a *big point* if

$$B_X = \overline{\text{co}}\,\Big(\bigcup_{T\in\mathcal{G}(X)} Tu\Big),$$

where $\mathcal{G}(X)$ denotes the set of all surjective linear isometries on X.

If all the points in the unit sphere of X are big, the space is *convex-transitive* (see [24, §9]). X is *almost-transitive* if $\overline{\bigcup_{T\in\mathcal{G}(X)}(Tu)} = S_X$ for any element u in the unit sphere of X.

For instance, the space $L_1[0,1]$ is almost-transitive [24, Theorem 9.6.4]. $L_\infty[0,1]$ and the Calkin algebra are convex-transitive [9, Corollary 4.6].

Up to this point, we know that a certain abundance of norm attaining functionals and some other condition imply reflexivity. For instance, a separable convex-transitive Banach space X, satisfying that $NA(X)$ is second Baire category, is in fact, superreflexive [2, Proposition 7]. We arrived at the same assertion for a convex-transitive Banach space not containing a copy of ℓ_1 and such that $NA(X)$ has non empty interior [2, Proposition 8].

We do not know any non superreflexive Banach space X that is convex-transitive and such that $NA(X)$ has non empty interior. That is why we asked in [2, Open problem 9] whether or not superreflexivity is satisfied if X is convex-transitive and the set of norm attaining functionals on X has non empty interior. We will use a localization argument to give a new partial answer to the previous problem. For this purpose, some topological property of the set of big points will be needed.

LEMMA 2. *Let X be a Banach space, then the set of big points is closed.*

Proof. Let $B \subset S_X$ be the subset of big points and $y \in \overline{B}$. If we fix $x \in B_X, \varepsilon > 0$, there is $b \in B$ satisfying $\|y - b\| < \varepsilon$. Since b is a big point, there are elements $T_1, \dots, T_n \in \mathcal{G}(X)$ so that

$$\left\| x - \sum_{i=1}^{n} c_i T_i(b) \right\| < \varepsilon,$$

where $c_i \geq 0$ for all i and $\sum_{i=1}^{n} c_i = 1$. Hence, we have

$$\begin{aligned} \left\| x - \sum_{i=1}^{n} c_i T_i(y) \right\| &\leq \left\| x - \sum_{i=1}^{n} c_i T_i(b) \right\| + \left\| \sum_{i=1}^{n} c_i T_i(b - y) \right\| \\ &\leq \varepsilon + \sum_{i=1}^{n} c_i \|T_i\| \; \|b - y\| \leq 2\varepsilon. \end{aligned}$$

Since x is any element in the unit ball and ε any positive real number, y is a big point, as we wanted to prove. □

PROPOSITION 3. *Let X be a convex-transitive Banach space. For any separable linear subspace $M \subset X$, there is a linear space $\tilde{M} \subset X$, which is closed, separable, convex-transitive and contains M.*

Proof. We use a similar argument to the one appearing in [14, Theorem 1.2]. We will construct an increasing sequence of separable subspaces of X whose union will be dense in the desired subspace $\tilde{M}$.

To begin with, we will take $Y_1 = M$. Let us consider a countable subset $D_1 \subset S_M$, which is dense in the unit sphere. Since X is convex-transitive and M is separable, there is a countable subgroup $G_1 \subset \mathcal{G}(X)$ so that

$$x \in \overline{\text{co}}\{Ty : T \in G_1\}, \quad \forall x, y \in D_1.$$

Now, let Y_2 be given by

$$Y_2 = \overline{\mathrm{Lin}}\Big(\bigcup_{T \in G_1} T(Y_1) \Big).$$

Hence, Y_2 is a closed linear subspace of X, which is separable and contains Y_1. Now, we will repeat the procedure by considering a countable subset $D_2 \subset S_{Y_2}$ that is dense in S_{Y_2} and contains D_1. Since X is convex-transitive, we can find a countable subgroup $G_2 \subset \mathcal{G}(X)$ containing G_1 so that

$$x \in \overline{\mathrm{co}}\{Ty : T \in G_2\}, \quad \forall x, y \in D_2.$$

In this manner, we will inductively construct sequences $\{Y_n\}$, $\{D_n\}$, $\{G_n\}$ such that

i) Y_n is a separable linear subspace of X, for each n.
ii) $M \subset Y_n \subset Y_{n+1}, \quad \forall n$.
iii) D_n is a countable dense subset of S_{Y_n} and $D_n \subset D_{n+1}, \forall n$.
iv) G_n is a countable subgroup of $\mathcal{G}(X)$ and $G_n \subset G_{n+1}$ for each n.
v) $T(Y_n) \subset Y_{n+1}$, $\forall\, T \in G_n$.
vi) $D_n \subset \overline{\mathrm{co}}\{Tz : T \in G_n\}, \quad \forall z \in D_n$.

We will take $\tilde{M} = \overline{\cup_n Y_n}$, which is clearly a separable closed subspace of X and contains M by i) and ii). In order to check that $\tilde{M}$ is convex-transitive, we will show that every element in the set $D = \cup_n D_n \subset \tilde{M}$ is a big point of $\tilde{M}$.

First, $\tilde{M}$ is $\tilde{G}$-invariant, since for any $T \in G_m$ by using ii) and v), we have

$$T(\tilde{M}) = T\left(\overline{\cup_{n \geq m} Y_n}\right) \subset \overline{\cup_n T(Y_n)} \subset \tilde{M}.$$

Since G_m is a subgroup of $\mathcal{G}(X)$ and $T^{-1}(\tilde{M}) \subset \tilde{M}$, $T_{|\tilde{M}}$ is a surjective isometry on $\tilde{M}$.

For any $x, y \in D$, condition iii) gives us that $x, y \in D_n$ (some n). Therefore, by using vi),

$$y \in \overline{\mathrm{co}}\{Tx : T \in G_n\} \subset \overline{\mathrm{co}}\{Tx : T \in \tilde{G}\}.$$

Since D is dense in $S_{\tilde{M}}$, then $B_{\tilde{M}} \subset \overline{\mathrm{co}}\, \tilde{G}(x)$. By Lemma 2 and the denseness of D in $S_{\tilde{M}}$ (condition iii)), it follows that $B_{\tilde{M}} = \overline{\mathrm{co}}\, \left(\bigcup_{T \in \tilde{G}} T(x)\right)$ for any $x \in S_{\tilde{M}}$, so $\tilde{M}$ is convex-transitive. □

DEFINITION 4. A Banach space X has property $\mathcal{H}NA(X)$ if there is $u \in S_X$ such that $NA(M)$ is second Baire category (in the norm topology of M^*) for any closed, separable and linear subspace $M \subset X$ containing u.

Property $\mathcal{H}NA(X)$ is not restrictive at all from an isomorphic point of view. Every Banach space can be renormed to have $\mathcal{H}NA$ [4, Proposition 1]. It is easily checked that, for instance, ℓ_1 has this property for $u = e_1$.

THEOREM 5. *Let X be a convex-transitive Banach space satisfying property $\mathcal{H}NA(X)$. Then X is almost-transitive and superreflexive.*

Proof. Let $N \subset X$ be a closed and separable linear space of X. If $u \in S_X$ is the element in X such that $NA(M)$ is second Baire category in M^* for any closed and separable subspace M containing u, we use as M the linear span of u and N, $M = \text{Lin}\{u, N\}$.

By using Proposition 3, there is a separable, closed and convex-transitive subspace $\tilde{M} \subset X$ containing M, so $NA(\tilde{M})$ is second Baire category. By using the result by Bourgain-Stegall [13, Theorem 3.5.5 and Problem 3.5.6], $B_{\tilde{M}}$ is dentable. Since $\tilde{M}$ is also convex-transitive, by using [8, Theorem 3.2], it follows that $\tilde{M}$ is superreflexive, so N is also superreflexive.

Since superreflexivity is separably determined, then X is superreflexive, and by using [8, Corollary 3.3], we get that X is indeed almost-transitive. □

COROLLARY 6. *If X is a convex-transitive Banach space which is not Asplund, then for every u in the unit sphere, there is a separable and convex-transitive closed subspace M containing u such that $NA(M)$ is first Baire category and so $NA(M)$ has empty interior in M^*.*

Section 2

In this section we pay attention to the parallel version of James' Theorem for numerical radius. In order to be more precise, let us recall that the *numerical radius* of an operator $T \in L(X)$ (bounded and linear operators on X) is the real number $v(T)$ given by

$$v(T) := \sup\{|x^*Tx| : (x, x^*) \in \Pi(X)\},$$

where $\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$ (S_X is the unit sphere of X). We say that such an operator T *attains the numerical radius* if for some $(x_0, x_0^*) \in \Pi(X)$ it holds that

$$|x_0^*(Tx_0)| = v(T).$$

A good survey of results on numerical radius can be found in the monographs [10, 11].

James' Theorem can be stated as follows: a Banach space is reflexive if, and only if, every rank-one operator on it attains the norm. In [6] it is shown that a Banach space is reflexive provided that every rank-one operator attains the numerical radius. However, the converse does not hold. In fact, we have obtained the following general renorming result, previously shown for Banach spaces with a Schauder basis [6, Example]:

THEOREM 7. *A Banach space is finite-dimensional if, and only if, for any equivalent norm, every rank-one operator attains its numerical radius.*

Proof. A simple compactness argument gives us one implication. Thus, we just have to prove that an infinite-dimensional Banach space X admits an equivalent norm for which some rank-one operator does not attain the numerical radius. In view of [6, Theorem] we can assume X to be reflexive. Otherwise, the original norm satisfies the desired condition.

Proof of the separable case. If X is separable and infinite-dimensional, we can find a positive number $K > 0$ and a biorthogonal system $\{(e_n, e_n^*)\}$ in $S_X \times K B_{X^*}$ such that the space generated by $\{e_n : n \in \mathbb{N}\}$ is (norm) dense in X and the subset $\{e_n^* : n \in \mathbb{N}\}$ separates the elements in X (see [20] or [21]). We will use a similar argument to the one appearing in [6, Example]. Let us consider the following subset of X:

$$A := \left\{ x \in X : \sum_{n=1}^{\infty} \frac{1}{\varepsilon_n^2} |e_n^*(x)|^2 \leq 1 \right\},$$

where $\{\varepsilon_n\}$ is a fixed sequence in ℓ_1 satisfying

$$\varepsilon_1 = 1, \ 0 < \varepsilon_{n+1} < \varepsilon_n, \qquad \forall n \in \mathbb{N}.$$

Because of $\{\varepsilon_n\} \in \ell_1$ and the fact that the set $\{e_n^*\}$ separates the points in X, it holds that

$$x = \sum_{n=1}^{\infty} e_n^*(x) e_n, \qquad \forall x \in A.$$

It is also clear that the set A is (norm) compact. Now let us consider the subset B given by

$$B = \mathrm{co} \left\{ \frac{1}{2K} B_X \cup \overline{\mathrm{aco}}\{e_2, e_2 + e_n : n \geq 3\} \cup A \right\},$$

which is the unit ball of an equivalent norm $\|\cdot\|$ on X (aco is the absolutely convex hull). Let Y be the space X endowed with the new norm. Then its dual norm is given by

$$\begin{aligned}
&\|y^*\| \\
&= \max\left\{\frac{1}{2K}|y^*|, |y^*(e_2)|, \max_{n\geq 3}|y^*(e_2)+y^*(e_n)|, \max_{a\in A}|y^*(a)|\right\} \\
&= \max\left\{\frac{1}{2K}|y^*|, |y^*(e_2)|, \max_{n\geq 3}|y^*(e_2)+y^*(e_n)|, \left(\sum_{n=1}^{\infty}\varepsilon_n^2|y^*(e_n)|^2\right)^{\frac{1}{2}}\right\},
\end{aligned}$$

for any $y^* \in Y^*$, where we denote by $|\cdot|$ the original norm in X. Now we take the elements

$$z_0 = e_1, \; x_0 = e_2, \; x_n = e_2 + e_n \qquad (n \geq 3).$$

We finish the proof in the separable case by checking that $z_0, x_n \in B_Y$, $x_0 \in S_Y$ and also the following four conditions:

i) $\{\|x_n + z_0\|\} \to 2$,

ii) z_0 is a point of smoothness of the norm and the unique functional $z_0^* \in S_{Y^*}$ such that $z_0^*(z_0) = 1$ is also a point of smoothness (of the dual norm),

iii) $\{x_n\} \xrightarrow{w} x_0$,

iv) $z_0 \notin \mathbb{K}x_0$.

The previous assumptions imply that the operator $x_0^* \otimes z_0$ does not attain its numerical radius, where $x_0^* \in S_{Y^*}$ is any support functional of the unit ball at x_0 (see [6, Proposition 2]).

Now we will check the previous four conditions. We know by the definition of B that $z_0, x_0, x_n \in B_Y$ $(n \geq 3)$ and, in fact, $\|x_0\| = 1$ because $e_2^*(x_0) = 1$ and $\|e_2^*\| = 1$. Conditions iii) and iv) also hold since $\{e_n^* : n \in \mathbb{N}\}$ generates a w^*-dense subspace of X^*, and, since X is reflexive, it is a dense subspace of X^*. In order to check condition i) we simply consider the functionals

$$x_n^* := (1 - \varepsilon_n)e_1^* + e_n^* \qquad (n \geq 3),$$

then $\|x_n^*\| \leq 1$ and

$$\{x_n^*(x_n + z_0)\} = \{2 - \varepsilon_n\} \to 2.$$

But $2 - \varepsilon_n \leq \|x_n + z_0\| \leq 2$, so

$$\{\|x_n + z_0\|\} \to 2.$$

Finally, we check condition ii). If $z_0^* \in S_{Y^*}$ satisfies $z_0^*(z_0) = z_0^*(e_1) = 1$, then

$$1 = \|z_0^*\|^2 \geq \sum_{n=1}^{\infty} \varepsilon_n^2 |z_0^*(e_n)|^2 = |z_0^*(e_1)|^2 + \sum_{n=2}^{\infty} \varepsilon_n^2 |z_0^*(e_n)|^2,$$

so $z_0^*(e_n) = 0$ for $n \geq 2$. The linear span of $\{e_n : n \in \mathbb{N}\}$ is dense in X and the previous condition therefore implies that $z_0^* = e_1^*$ and z_0 is a point of smoothness. On the other hand, we have

$$|e_1^*(x)| < 1, \quad \forall x \in \frac{1}{2K} B_X \cup \overline{\text{aco}}\{e_2, e_2 + e_n\}_{n \geq 3},$$

and so e_1^* attains its norm only at elements in A. If $a \in A$ is such an element, then $e_1^*(a) = 1$, and the fact that $\sum_{n=1}^{\infty} \frac{1}{\varepsilon_n^2} |e_n^*(a)|^2 \leq 1$ $(\varepsilon_1 = 1)$ gives us $e_n^*(a) = 0$ for $n \geq 2$. But the set of functionals $\{e_n^* : n \in \mathbb{N}\}$ separates the points of X, so $a = e_1 = z_0$, and z_0^* is also smooth.

Extension to the general case. Assume now that X is reflexive and infinite-dimensional. Then there exists an infinite-dimensional and separable complemented subspace X_0 of X (see [19, Proposition 1]). In view of the proof in the separable case, we can assume that there is an equivalent norm on X_0, satisfying the conditions i) to iv) previously checked. Hence, there is an element $z_0 \in S_{X_0}$ and a functional $x_0^* \in S_{X_0^*}$ such that the rank-one operator $x_0^* \otimes z_0 \in L(X_0)$ does not attain its numerical radius.

By renorming X we can assume that we have the decomposition

$$X = X_0 \oplus_2 Y$$

for some closed subspace Y of X. Since the ℓ_2-sum preserves smoothness, one can directly check that the operator $x_0^* \otimes z_0$ still satisfies the conditions i) to iv) (as an operator in X) previously checked in the separable case. Therefore, there is a space isomorphic to X such that the operator $x_0^* \otimes z_0$ does not attain its numerical radius. □

By considering some classical reflexive spaces, it is known that any compact operator on ℓ_p $(1 < p < \infty)$ attains its numerical radius (see [7] or [1]). In general, one has to renorm in order to get a rank-one operator not attaining its numerical radius. Not too much is known about the rest of the spaces, even in the case of $L_p[0,1]$. We pose the following problem: Can every reflexive space be renormed so that every rank-one operator on it attains its numerical radius?

Section 3

An explicit description of the subset of norm attaining functionals is known just in a few cases. In this section we will consider ℓ_1 embedded into L_1 with an appropriate Luxembourg norm and we will describe the set of norm attaining functionals on ℓ_1, endowed with the restriction of the Luxembourg norm on L_1.

Let us consider the function $M : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ given by

$$M(t) = \begin{cases} t^2 & \text{if} \quad 0 \le t \le 1, \\ 2t - 1 & \text{if} \quad\quad t > 1. \end{cases}$$

Then M is an Orlicz function and the Orlicz space $L_M[0,1]$ is given by the subset of measurable functions $f : [0,1] \longrightarrow \mathbb{R}$ such that for some positive real s it holds that

$$\int_0^1 M\left(\frac{|f(t)|}{s}\right)\, dt < +\infty.$$

We will consider this space endowed with the Luxembourg norm, that is,

$$\|f\|_M = \inf\left\{ s > 0 : \int_0^1 M\left(\frac{|f(t)|}{s}\right)\, dt \le 1 \right\} \qquad (f \in L_M[0,1]).$$

Since the subset $\{\frac{M(t)}{t} : t > 0\}$ is bounded above, it is well-known that the space $L_M[0,1]$ is isomorphic to $L_1[0,1]$ and, in fact, the identity is an isomorphism.

Since we can embed ℓ_1 into $L_1[0,1]$ in a natural way, from now on we will take $X = \ell_1$ endowed with the Luxembourg norm on L_1. More precisely, we will fix a sequence $\{A_n\}$ of disjoint measurable subsets of $[0,1]$ such that $\lambda(A_n) = \frac{1}{2^n}$, and consider the mapping $T : \ell_1 \longrightarrow L_1[0,1]$ given by

$$T(x) = \sum_{n=1}^{\infty} x(n) 2^n \chi_{A_n} \qquad (x \in \ell_1),$$

which is clearly an embedding from ℓ_1 into L_1. From now on, the norm considered on X is given by

$$\|x\| := \|Tx\|_M \qquad (x \in X).$$

First we will get an expression for the norm on X in terms of the space.

LEMMA 8. *For any $x \in X$ it holds that*

$$\|x\| = \min\left\{ s \geq 0 : \sum_{2^n|x(n)|\leq s} 2^n \frac{x(n)^2}{s^2} + \sum_{2^n|x(n)|>s} 2^{\frac{|x(n)|}{s}} - \sum_{2^n|x(n)|>s} \frac{1}{2^n} \leq 1 \right\}.$$

Proof. By definition of the function M, if we write $f = Tx$, then

$$\int_0^1 M\left(\frac{|f(t)|}{s}\right) dt = \sum_{|x(n)|2^n\leq s} \frac{(x(n)2^n)^2}{s^2}\frac{1}{2^n} + \sum_{|x(n)|2^n>s} \left(2^{\frac{|x(n)2^n|}{s}} - 1\right)\frac{1}{2^n},$$

since for $t \in A_n$, $|x(n)|2^n \leq s \Rightarrow M\left(\frac{|f(t)|}{s}\right) = \frac{\left(x(n)2^n\right)^2}{s^2}$; in the case that $|x(n)|2^n > s$, then $M(|f(t)|) = 2^{\frac{|x(n)|2^n}{s}} - 1$ and $\lambda(A_n) = \frac{1}{2^n}$ and now it suffices to use the definition of the Luxembourg norm of f to get

$$\|x\| = \inf\left\{ s \geq 0 : \sum_{2^n|x(n)|\leq s} 2^n \frac{x(n)^2}{s^2} + \sum_{2^n|x(n)|>s} 2^{\frac{|x(n)|}{s}} - \sum_{2^n|x(n)|>s} \frac{1}{2^n} \leq 1 \right\}.$$

Since M is continuous, by the monotone convergence Theorem, it holds that for any $f \in L_M[0,1]$ the infimum defining $\|f\|_M$ is actually a maximum and so, as a consequence, we obtain the announced statement. □

LEMMA 9. *The Young conjugate of M is the function $\varphi : \mathbb{R}_0^+ \longrightarrow [0, +\infty]$ given by*

$$\varphi(t) = \begin{cases} \frac{t^2}{4} & \text{if } 0 \leq t \leq 2, \\ +\infty & \text{if } \quad t > 2. \end{cases}$$

Proof. By definition of the Young conjugate, it is sufficient to compute

$$\varphi(t) := \sup\{xt - M(x) : x \geq 0\}.$$

□

In the next result we compute the dual norm of the space X.

PROPOSITION 10. *Let X be $(\ell_1, \|\cdot\|)$. Then, for any $z \in \ell_\infty = X^*$, $z \neq 0$, the dual norm is given by*

$$\|z\| = \inf_{k>0}\left\{\frac{1}{k}\left(1+\frac{k^2}{4}\sum_{n=1}^{\infty}\frac{z(n)^2}{2^n}\right)\right\}$$
$$= \frac{1}{2}\left(\|z\|_\infty + \frac{1}{\|z\|_\infty}\sum_{n=1}^{\infty}\frac{z(n)^2}{2^n}\right).$$

Proof. If the two functions M and its Young conjugate φ were N-complementaries the result would be a consequence of a general result, appearing, for instance, in [23, Theorem 13].

In the following, we will use the identification $X^* \equiv \ell_\infty$. Let x be an element in X satisfying $\|x\| \leq 1$ and $z \in X^*\backslash\{0\}$. We will call $f := \sum_{n=1}^{\infty} x(n)2^n\chi_{A_n}$ and $g := \sum_{n=1}^{\infty} z(n)\chi_{A_n}$. Since f is the image of x under T, $\|x\| \leq 1$ and M is continuous, we get $\int_0^1 M(|f|)\, d\lambda \leq 1$. Then we have, for any $k \in \mathbb{R}^+$, the following inequality:

$$\sum_{n=1}^{\infty} z(n)x(n) = \int_0^1 g(t)f(t)\, dt$$
$$= \frac{1}{k}\int_0^1 kg(t)f(t)\, dt \leq \frac{1}{k}\int_0^1 k|g(t)f(t)|\, dt$$
$$\leq \frac{1}{k}\int_0^1 \Big(\varphi(k|g(t)|) + M(|f(t)|)\Big)\, dt$$
$$\leq \frac{1}{k}\left(1 + \int_0^1 \varphi(k|g(t)|)\, dt\right).$$

In the case that the previous expression is finite, that is, if $k\|z\|_\infty \leq 2$, then the above expression coincides with

$$\frac{1}{k}\left(1+\sum_{n=1}^{\infty}\frac{k^2 z(n)^2}{2^{n+2}}\right) = \frac{1}{k} + k\sum_{n=1}^{\infty}\frac{z(n)^2}{2^{n+2}}.$$

Hence, we have proven that

$$\|z\| \leq \inf_{0<k\leq \frac{2}{\|z\|_\infty}}\left\{\frac{1}{k} + k\sum_{n=1}^{\infty}\frac{z(n)^2}{2^{n+2}}\right\}.$$

Now, simple computations show that the above infimum is attained at the value $k = \frac{2}{\|z\|_\infty}$ and so, we obtain that

$$\|z\| \leq \frac{1}{2}\Big(\|z\|_\infty + \frac{1}{\|z\|_\infty}\sum_{n=1}^{\infty}\frac{z(n)^2}{2^n}\Big).$$

Now we will check that the previous inequality becomes an equality for the set of norm attaining functionals. For the function M used, the Luxembourg norm in L_M is smooth (see [27, Lemma 3.3]), and so, we will show that for any element $x \in \ell_1$ satisfying that $\|x\|_M = 1$, there is an element $z \in \ell_\infty$ such that

$$z(x) = \frac{1}{2}\Big(\|z\|_\infty + \frac{1}{\|z\|_\infty}\sum_{n=1}^{\infty}\frac{z(n)^2}{2^n}\Big).$$

Since the expression on the right hand side is continuous in ℓ_∞, it will give the dual norm if both norms coincide in the set of norm attaining functionals, which is a (norm) dense subset of X^*, in view of the Bishop-Phelps Theorem.

For $x \in X$ satisfying $\|x\| = 1$, we define the following element $z \in \ell_\infty$

$$z(n) = \begin{cases} 2^n x(n), & \text{if } |x(n)|2^n \leq 1 \\ \operatorname{sign} x(n), & \text{if } |x(n)|2^n > 1. \end{cases}$$

Because of the choice of z we get

$$z(x) = \sum_{n=1}^{\infty} z(n)x(n) = \sum_{|x(n)|2^n \leq 1} 2^n x(n)^2 + \sum_{|x(n)|2^n > 1} |x(n)|. \tag{1}$$

On the other hand, since $\|x\| = 1$, then by using Lemma 8, it is satisfied that $|x(n)|2^n \geq 1$ for some n and so, $\|z\|_\infty = 1$. As a consequence, we have

$$\begin{aligned} &\frac{1}{2}\Big(\|z\|_\infty + \frac{1}{\|z\|_\infty}\sum_{n=1}^{\infty}\frac{z(n)^2}{2^n}\Big) \\ &= \frac{1}{2}\Big(1 + \sum_{|x(n)|2^n \leq 1} 2^n x(n)^2 + \sum_{|x(n)|2^n > 1} \frac{1}{2^n}\Big). \end{aligned} \tag{2}$$

By using that $\|x\| = 1$, from Lemma 8 we know that

$$1 = \sum_{|x(n)|2^n \leq 1} 2^n x(n)^2 + 2\sum_{|x(n)|2^n > 1} |x(n)| - \sum_{|x(n)|2^n > 1} \frac{1}{2^n},$$

and it follows that

$$2\Big(\sum_{|x(n)|2^n\leq 1} 2^n x(n)^2 + \sum_{|x(n)|2^n>1} |x(n)| \Big) = 1 + \sum_{|x(n)|2^n\leq 1} 2^n x(n)^2 + \sum_{|x(n)|2^n>1} \frac{1}{2^n}. \tag{3}$$

By using (1), the above identity and (2), we get

$$z(x) = \frac{1}{2}\Big(\|z\|_\infty + \frac{1}{\|z\|_\infty} \sum_{n=1}^{\infty} \frac{z(n)^2}{2^n} \Big).$$

□

PROPOSITION 11. *The space $(\ell_1, \|\cdot\|)$ is smooth, and for any $x \in S_X$, its normalized support functional is the element z given by*

$$z(n) = \begin{cases} 2^n x(n) M, & \text{if } |x(n)|2^n \leq 1 \\ M\text{sign } x(n), & \text{if } |x(n)|2^n > 1, \end{cases}$$

where

$$M = \frac{2}{1 + \sum_{|x(n)|2^n\leq 1} 2^n x(n)^2 + \sum_{|x(n)|2^n>1} \frac{1}{2^n}}.$$

Proof. By using [27, Lemma 3.3], L_M is smooth, and so $(\ell_1, \|\cdot\|)$ is also smooth. By Proposition 10, the functional z has norm one, since the subset $\{n \in \mathbb{N} : |x(n)| \geq 1\}$ is not empty (see Lemma 8) and hence

$$\begin{aligned} \|z\| &= \frac{1}{2}\Big(\|z\|_\infty + \frac{1}{\|z\|_\infty} \sum_{n=1}^{\infty} \frac{z(n)^2}{2^n} \Big) \\ &= \frac{M}{2}\Big(1 + \sum_{|x(n)|2^n\leq 1} 2^n x(n)^2 + \sum_{|x(n)|2^n>1} \frac{1}{2^n} \Big) \\ &= 1, \end{aligned}$$

where we used that $\|x\| = 1$.

By using identity (3) of the proof of Proposition 10, we get

$$\begin{aligned} z(x) &= \sum_{n=1}^{\infty} z(n)x(n) \\ &= \sum_{|x(n)|2^n\leq 1} 2^n x(n)^2 M + \sum_{|x(n)|2^n>1} M|x(n)| \\ &= 1. \end{aligned}$$

□

THEOREM 12. *The set of norm attaining functionals on X has non empty interior. In fact,*

$$NA(X) = NA(\ell_1) = \{z \in \ell_\infty : |z(n)| = \|z\|_\infty \text{ (some } n)\}.$$

Proof. Since the set of norm attaining functionals is a cone, assume that an element $z \in \ell_\infty$ satisfies that $\|z\|_\infty = 1$, $|z(n)| < 1$, for every n, and z attains the norm at some element $x \in S_X$. By the description of the duality mapping given in the previous result, then

$$2^n |x(n)| < 1, \qquad \forall n \in \mathbb{N},$$

and so

$$1 = \|x\| = \sum_{n=1}^{\infty} 2^n x(n)^2 < \sum_{n=1}^{\infty} |x(n)| < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Therefore, if $z \in NA(X)$, then for some coordinate n it holds that $|z(n)| = \|z\|_\infty$.

Now assume that

$$\{n \in \mathbb{N} : |z(n)| = \|z\|_\infty = 1\} \neq \emptyset.$$

In the case that

$$|z(n)| = 1, \qquad \forall n \in \mathbb{N},$$

we will check that z attains its norm. Let us take the element $x \in X$ given by

$$x(n) = \text{sign } z(n) \frac{1}{2^n}, \qquad \forall n \in \mathbb{N}.$$

From the expression of the norm in X (see Lemma 8), it holds that $\|x\| = 1$ and

$$z(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Since in this case

$$\|z\| = \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} \frac{z(n)^2}{2^n}\right) = 1,$$

then $z(x) = 1 = \|x\| = \|z\|$ and z attains its norm at x.

Assume now that

$$A := \left\{m \in \mathbb{N} : |z(m)| < 1\right\} \neq \emptyset.$$

We put

$$C := \mathbb{N} \backslash A := \left\{n \in \mathbb{N} : |z(n)| = 1\right\},$$

and by assumption, C is not empty. We will define an element $x \in X$ as follows:

$$x(n) = \begin{cases} \frac{z(n)}{2^n} & \text{if } n \in A \\ \text{sign } z(n)\frac{1+r}{2^n} & \text{if } n \in C, \end{cases}$$

where $r = \dfrac{1 - \sum_{n=1}^{\infty} \frac{z(n)^2}{2^n}}{2\sum_{n\in C} \frac{1}{2^n}} > 0.$

We check that $x \in B_X$, since

$$\begin{aligned} &\sum_{|x(n)|2^n \leq 1} 2^n x(n)^2 + 2 \sum_{|x(n)|2^n > 1} |x(n)| - \sum_{|x(n)|2^n > 1} \frac{1}{2^n} \\ &= \sum_{n\in A} \frac{z(n)^2}{2^n} + 2(1+r)\sum_{n\in C} \frac{1}{2^n} - \sum_{n\in C} \frac{1}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{z(n)^2}{2^n} + 2r\sum_{n\in C} \frac{1}{2^n} \quad \text{(by the choice of } r\text{)} \\ &= \sum_{n=1}^{\infty} \frac{z(n)^2}{2^n} + 1 - \sum_{n=1}^{\infty} \frac{z(n)^2}{2^n} \\ &= 1, \end{aligned}$$

and so $\|x\| \leq 1$. Also

$$\begin{aligned} z(x) &= \sum_{n\in A} \frac{z(n)^2}{2^n} + \sum_{n\in C} \frac{1+r}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{z(n)^2}{2^n} + r\sum_{n\in C} \frac{1}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{z(n)^2}{2^n} + \frac{1}{2}\left(1 - \sum_{n=1}^{\infty} \frac{z(n)^2}{2^n}\right) \\ &= \frac{1}{2}\left(1 + \sum_{n=1}^{\infty} \frac{z(n)^2}{2^n}\right) \\ &= \|z\|, \end{aligned}$$

(having used in the last inequality $\|z\|_\infty = 1$). We checked that z attains its norm.

As proven earlier

$$NA(X) = \{z \in \ell_\infty : \exists n, |z(n) = \|z\|_\infty\} = NA(\ell_1).$$

Therefore, the set of norm attaining functionals has non empty interior in the dual space (norm topology). □

It is not known whether a Banach space X satisfying that for every equivalent norm, the subset $NA(X)$ has non empty interior, has to be reflexive. This problem was posed by I. Namioka during a workshop held in Murcia (1999).

As proven in [4, Corollary 7], any separable Banach space X that is not weakly sequentially complete admits an equivalent norm for which $NA(X)$ has empty interior. Even for the space ℓ_1, we do not know if it admits a norm satisfying that the set of norm attaining functionals has empty interior. In order to arrive at such a norm, it seems to be reasonable to work with smooth norms. However, we have proven here that the Luxembourg norm does not work.

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References

[1] M. D. Acosta, *Algunos resultados sobre operadores que alcanzan su radio numérico.* In: Analysis, Actas XV Jornadas Luso-Espanholas de Matemática, Evora (Portugal) 199, Vol. **II**, Universidade de Evora, Departamento de Matemática, Evora, 1991, pp. 319–324.

[2] M. D. Acosta, J. Becerra Guerrero and M. Ruiz Galán, *Dual spaces generated by the interior of the set of norm attaining functionals,* Studia Math. **149** (2002), 175–183.

[3] ———, *Characterizations of the reflexive spaces in the spirit of James' Theorem*, Contemp. Math., Vol. 321, (Kaminska, Ed.), Amer. Math. Soc., Providence, Phode Island, 2003, pp. 1–14.

[4] M. D. Acosta and M. Ruiz Galán, *New characterizations of the reflexivity in terms of the set of norm attaining functionals*, Canad. Math. Bull. **41** (1998), 279–289.

[5] ———, *Norm attaining operators and reflexivity,* Rend. Circ. Mat. Palermo **56** (1998), 171–177.

[6] ———, *A version of James' Theorem for numerical radius*, Bull. London Math. Soc. **31** (1999), 67–74.

[7] G. de Barra, J. R. Giles and B. Sims, *On the numerical range of compact operators on Hilbert spaces*, J. London Math. Soc. **5** (1972), 704–706.

[8] J. Becerra Guerrero and A. Rodríguez Palacios, *The geometry of convex-transitive Banach spaces*, Bull. London Math. Soc. **31** (1999), 323–331.

[9] ———, *Transitivity of the norm on Banach spaces having a Jordan structure,* Manuscripta Math. **102** (2000), 111–127.

[10] F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras,* London Math. Soc. Lecture Note Ser. **2**, Cambridge University Press, 1971.

[11] ———, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. **10**, Cambridge University Press, 1973.
[12] J. Bourgain, *On dentability and the Bishop-Phelps property*, Israel J. Math. **28** (1977), 265–271.
[13] R. D. Bourgin, *Geometric aspects of convex sets with the Radon-Nikodym property*, Lecture Notes in Math. **993**, Springer-Verlag, Berlin, 1983.
[14] F. Cabello Sánchez, *Transitivity of M spaces and Wood's conjecture*, Math. Proc. Cambridge Philos Soc. **124** (1998), 513–520.
[15] G. Debs, G. Godefroy and J. Saint Raymond, *Topological properties of the set of norm-attaining linear functionals*, Canad. J. Math. **47** (1995), 318–329.
[16] R. C. James, *Weak compactness and reflexivity*, Israel J. Math. **2** (1964), 101–119.
[17] M. Jiménez Sevilla and J. P. Moreno, *A note on norm attaining functionals*, Proc. Amer. Math. Soc. **126** (1998), 1989–1997.
[18] P. S. Kenderov, W. B. Moors and S. Sciffer, *Norm attaining functionals on* $C(T)$, Proc. Amer. Math. Soc. **126** (1998), 153–157.
[19] J. Lindenstrauss, *On nonseparable reflexive Banach spaces*, Bull. Amer. Math. Soc. **72** (1966), 967–970.
[20] R. I. Ovsepian and A. Pełczyński, *The existence in every separable Banach space of a fundamental total and bounded biorthogonal sequence and related constructions of uniformly bounded orthogonal systems in* L^2, Studia Math. **54** (1975), 149–159.
[21] A. Pełczyński, *All separable Banach spaces admit for every* $\varepsilon > 0$ *fundamental and total biorthogonal sequences bounded* $1+\varepsilon$, Studia Math. **55** (1976), 295–304.
[22] J. I. Petunin and A. N. Plichko, *Some properties of the set of functionals that attain a supremum on the unit sphere*, Ukrain. Math. Zh. **26** (1974), 102–106.
[23] M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces*, Monogr. Textbooks Pure Appl. Math. **146**, Marcel Dekker, New York, 1991.
[24] S. Rolewicz, *Metric linear spaces*, Mathematics and Its Applications, Reidel, Dordrecht, 1985.
[25] C. Stegall, *Optimization of functions on certain subsets of Banach spaces*, Math. Ann. **236** (1978), 171–176.
[26] ———, *Optimization and differentiation in Banach spaces*, Linear Algebra Appl. **84** (1986), 191–211.
[27] S. L. Troyanski, *Gateaux differentiable norms in* L_p, Math. Ann. **287** (1990), 221–227.

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