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COMPOSITION OPERATORS ON UNIFORM ALGEBRAS AND THE PSEUDOHYPERBOLIC METRIC

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ABSTRACT. Let A be a uniform algebra, and let ϕ be a self-map of the spectrum M_A of A that induces a composition operator C_ϕ on A. It is shown that the image of M_A under some iterate ϕ^n of ϕ is hyperbolically bounded if and only if ϕ has a finite number of attracting cycles to which the iterates of ϕ converge. On the other hand, the image of the spectrum of A under ϕ is not hyperbolically bounded if and only if there is a subspace of A^{**} "almost" isometric to ℓ_∞ on which C_ϕ^{**} is "almost" an isometry. A corollary of these characterizations is that if C_ϕ is weakly compact, and if the spectrum of A is connected, then ϕ has a unique fixed point, to which the iterates of ϕ converge. The corresponding theorem for compact composition operators was proved in 1980 by H. Kamowitz [17].

1. Background

Let A be a uniform algebra, with spectrum M_A. We regard A as an algebra of continuous functions on M_A, so that A is a closed unital subalgebra of $C(M_A)$.

Recall that the pseudohyperbolic metric ρ on the open unit disk $\mathbb{D} = \{|z| < 1\}$ in the complex plane is defined by

$$\rho(z, w) = \frac{|z-w|}{|1-\overline{w}z|}, \qquad z, w \in \mathbb{D}.$$

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The pseudohyperbolic metric of $\mathbf{D}$ is invariant under the conformal self-maps of $\mathbf{D}$. It satisfies a sharpened form of the triangle inequality,

$$\rho(z,w) \le \frac{\rho(z,\zeta)+\rho(\zeta,w)}{1+\rho(z,\zeta)\rho(\zeta,w)}, \qquad z,\zeta,w \in \mathbf{D}.$$

(To see this, proceed as follows. Map ζ to 0 and w to $s > 0$ by a conformal self-map of $\mathbf{D}$, to reduce to the estimate $\rho(z,s) \le (|z|+s)/(1+s|z|)$. Then use the fact that the hyperbolic circle centered at s and passing through $-r$ is a Euclidean circle to argue that the maximum of $\rho(z,s)$ over the circle $|z| = r$ is attained at $z = -r$.)

We use the pseudohyperbolic metric on $\mathbf{D}$ to define the pseudohyperbolic metric ρ_A on the spectrum M_A by

$$\rho_A(x,y) = \sup\{\rho(f(x),f(y)) : f \in A, ||f|| < 1\}.$$

Evidently $\rho_A(x,y) \le 1$. Since $(s+t)/(1+st)$ is an increasing function of s and t for $0 \le s,t \le 1$, the sharpened form of the triangle inequality for $\rho(z,w)$ easily yields the same inequality for $\rho_A(x,y)$,

$$(1) \qquad \rho_A(x,y) \le \frac{\rho_A(x,u)+\rho_A(u,y)}{1+\rho_A(x,u)\rho_A(u,y)}, \qquad x,u,y \in M_A.$$

This inequality was introduced in the context of uniform algebras by H. König [20]. It shows in particular that ρ_A is a metric on M_A.

The open unit ball of A is invariant under post-composition with conformal self-maps of $\mathbf{D}$. By composing f with a conformal self-map of $\mathbf{D}$ that sends $f(y)$ to 0, we obtain

$$\rho_A(x,y) = \sup\{|f(x)| : f \in A, ||f|| < 1, f(y) = 0\}.$$

Thus $\rho_A(x,y)$ is the norm of the evaluation functional at x on the null-space of the evaluation functional at y,

$$\rho_A(x,y) = ||x|_{y^{-1}(0)}||, \qquad x,y \in M_A.$$

It follows that $\rho_A(x,y) \le ||x-y||$. Since $\rho(z,w) \ge |z-w|/2$, also $\rho_A(x,y) \ge ||x-y||/2$, and we obtain

$$\frac{||x-y||}{2} \le \rho_A(x,y) \le ||x-y||, \qquad x,y \in M_A.$$

Thus convergence in the pseudohyperbolic metric of M_A is tantamount to convergence in the norm of A^*.

From König's inequality (1) it is easy to see that any two open pseudohyperbolic balls in M_A of radius 1 either are disjoint or coincide. (See the proof of Lemma 2.1.) These open balls are called the *Gleason parts* of A. For a discussion of Gleason parts, see Chapter VI of [11].

The bidual A^{**} of A is also a uniform algebra. For a description of the bidual of A, see [12]. The evaluation functionals at points of M_A extend uniquely to be weak-star continuous multiplicative functionals on A^{**}, so we can regard M_A as a subset of the spectrum of A^{**}. The restrictions of the functions in A^{**} to M_A are the pointwise limits of bounded nets in A. These restrictions are not necessarily continuous on M_A. According to work of B. Cole (see [12]), the restriction algebra $A^{**}|_{M_A}$ includes all bounded functions on M_A that are constant on each Gleason part. It follows that each Gleason part of A is relatively weakly open and closed in M_A (the weak topology being the A^{**}-topology). Consequently each weakly precompact subset of M_A meets only finitely many Gleason parts (Theorem 1.1(c) of [21]; see also [22] or [9]).

Under the canonical embedding of A in A^{**}, the unit ball of A is weak-star dense in the unit ball of A^{**}. It follows that the canonical embedding induces an isometry with respect to hyperbolic metrics,

$$\rho_A(x,y) \;=\; \rho_{A^{**}}(x,y), \qquad x,y \in M_A.$$

Hence each Gleason part of $M_{A^{**}}$ either meets M_A in a Gleason part for A or is disjoint from M_A.

2. Hyperbolically bounded sets

A subset of the open unit disk is bounded with respect to the hyperbolic metric if and only if it is contained in a pseudohyperbolic ball of radius strictly less than 1. This occurs just as soon as it is contained in a finite union of pseudohyperbolic balls of radii strictly less than 1. Proceeding in analogy with the disk case, we define a subset E of M_A to be *hyperbolically bounded* if it is contained in a finite union of pseudohyperbolic balls whose radii are strictly less than 1. Each such ball is contained in a single Gleason part, so that a hyperbolically bounded subset of M_A meets only a finite number of Gleason parts of M_A.

LEMMA 2.1. *Let E be a hyperbolically bounded subset of M_A. If E is contained in a single Gleason part, then there is a constant $c < 1$ such that $\rho_A(x,y) \leq c$ for all $x, y \in E$.*

Proof. Suppose E is contained in the union of the pseudohyperbolic balls with centers x_j and radii r_j, where $r_j < 1$, $1 \leq j \leq n$. Let r be the maximum of the r_j's and the distances $\rho_A(x_1, x_j)$, $1 \leq j \leq n$. Thus $r < 1$. König's inequality (1) shows that $\rho_A(x_1, y) \leq 2r/(1+r^2) = b < 1$

for any y in the jth ball, hence for any $y \in E$. If $\rho_A(x_1,x) \leq b$ and $\rho_A(x_1,y) \leq b$, then $\rho_A(x,y) \leq 2b/(1+b^2) = c < 1$, again by (1). □

2.1. Interpolating sequences for A^{**}

A sequence $\{x_n\}$ in M_A is an *interpolating sequence for* A^{**} if the restriction of A^{**} to the sequence is isomorphic to ℓ_∞. Since the unit ball of A is weak-star dense in the unit ball of A^{**}, this occurs if and only if for a given $\lambda = \{\lambda_n\} \in \ell_\infty$, there are $C \geq 0$ and a sequence $\{f_m\}$ in A such that $||f_m|| \leq C$ for $m \geq 1$, and $f_m(x_n) \to \lambda_n$ as $m \to \infty$.

By duality of ℓ_1 and ℓ_∞, the sequence $\{x_n\}$ is an interpolating sequence for A^{**} if and only if $\{x_n\}$ is an ℓ_1-sequence, that is, the correspondence $e_n \mapsto x_n$, where e_n is the nth canonical basis element of ℓ_1, extends to an isomorphism of ℓ_1 onto the closed linear span of the x_n's in A^*. Let M be the norm of the operator $\ell_1 \mapsto A^*$. The duality shows that each $\lambda \in \ell_\infty$ can be interpolated by a function $F \in A^{**}$ satisfying $||F|| \leq (M+\varepsilon)||\lambda||_\infty$. By taking a weak-star limit of interpolating functions as $\varepsilon \to 0$, we can find an interpolating function F such that $||F|| \leq M||\lambda||_\infty$. The constant M is best possible; it is called the *interpolation constant* for the interpolating sequence $\{x_n\}$.

THEOREM 2.2. *Let A be a uniform algebra, let E be a subset of M_A, and let $\varepsilon > 0$. If E is not hyperbolically bounded, then E contains an interpolating sequence for A^{**} with interpolation constant $M < 1 + \varepsilon$.*

Proof. The bounded functions on M_A that are constant on each Gleason part belong to A^{**}. Thus any sequence of points from different Gleason parts is an interpolating sequence for A^{**} with interpolation constant $M = 1$.

We assume then that E is contained in a single Gleason part, and we follow the line of proof of Theorem 5.5 of [4]. By hypothesis, there is a sequence $\{x_n\}_{n=0}^\infty$ in E such that $\rho_A(x_n,x_0) \to 1$ as $n \to \infty$. According to Chapter VI of [11], there are functions $f_n \in A$ that satisfy $\mathrm{Re}\, f_n > 0$, $f_n(x_0) = 1$, and $\mathrm{Re}\, f_n(x_n) \to +\infty$. Passing to a subsequence, we can assume that $2^{-n}\mathrm{Re}\, f_n(x_n) \to +\infty$. Let

$$g_n = \sum_{k=1}^{n} 2^{-k} f_k.$$

Then $g_n \in A$, $\mathrm{Re}\, g_n > 0$, $g_n(x_0) \to 1$, and $\mathrm{Re}\, g_n(x_n) \to +\infty$. Set $G_n = (g_n-1)/(g_n+1)$. Then $G_n \in A$, $|G_n| < 1$, $G_n(x_0) \to 0$, and $G_n(x_n) \to 1$ as $n \to \infty$. Let $G \in A^{**}$ be a weak-star adherent point of the G_n's as $n \to \infty$. Then $|G| \leq 1$, and $G(x_0) = 0$. If $m \geq n$, then $\mathrm{Re}\, g_m \geq \mathrm{Re}\, g_n$. Composing with the map $w = (z-1)/(z+1)$, we see that $G_m(x_n)$ lies in the disk with diameter on the real axis having endpoints $(\mathrm{Re}\, g_n(x_n) - 1)/(\mathrm{Re}\, g_n(x_n) + 1)$ and 1. Since the length of this diameter does not exceed $2/\mathrm{Re}\, g_n(x_n)$, we obtain

$$|G_m(x_n) - 1| \leq 2/\mathrm{Re}\, g_n(x_n), \qquad m \geq n.$$

In the limit we obtain the same estimate for $|G(x_n) - 1|$. Hence $|G(x_n) - 1| \to 0$, and $G(x_n) \to 1$ as $n \to \infty$. Let P denote the Gleason part of x_0 in M_A. Since $G(x_0) = 0$, and the x_n's belong to P, we have $|G(x_n)| < 1$ for $n \geq 1$. Passing to a subsequence, we can assume that $\{G(x_n)\}$ is an interpolating sequence for the algebra $H^\infty(\mathbf{D})$ of bounded analytic functions on the unit disk $\mathbf{D}$, with interpolation constant $M < 1 + \varepsilon$. (See [15].) Since any function in $H^\infty(\mathbf{D})$ is a pointwise limit of a bounded sequence of polynomials with the same sup-norm over $\mathbf{D}$, and since any polynomial in G belongs to A^{**}, the composition of $G|_P$ with any function $g \in H^\infty(\mathbf{D})$ is the restriction to P of a function in A^{**} whose norm coincides with that of g. By composing G with interpolating functions in $H^\infty(\mathbf{D})$, we see that $\{x_n\}$ is an interpolating sequence for A^{**}, with interpolation constant $M < 1 + \varepsilon$. □

The converse of Theorem 2.2 is trivially true. Indeed, interpolation of the values 0 and 1 at two points $x, y \in E$ by a function of norm at most $1 + \varepsilon$ already implies $\rho_A(x, y) \geq 1/(1 + \varepsilon)$.

COROLLARY 2.3. *Let E be a subset of M_A. If every sequence in E has a weak Cauchy subsequence, then E is hyperbolically bounded. In particular, if E is weakly precompact, then E is hyperbolically bounded.*

Proof. Here the weak topology of E is the A^{**}-topology. For the first statement, note that the interpolating sequence of the theorem does not have a weak Cauchy subsequence. For the second statement, apply Eberlein's theorem. □

2.2. Linear interpolation operators

Davie's example [7] shows that there are algebras A with an interpolating sequence in M_A for A^{**}, for which there is no linear interpolation (extension) operator from ℓ_∞ to A^{**}. Towards finding linear interpolation operators, we begin with the following.

LEMMA 2.4. *Let A be a uniform algebra, and let $\{x_j\}_{j=1}^\infty$ be a sequence of points in M_A. Suppose there is $M \geq 1$ such that for each finite collection $\{\lambda_1, \ldots, \lambda_n\}$ of complex numbers of unit modulus, there is $f \in A$ satisfying $f(x_j) = \lambda_j$, $1 \leq j \leq n$, and $||f|| \leq M$. Then there is* *a sequence of functions* $\{F_k\}_{k=1}^{\infty}$ *in* A^{**} *such that* $F_k(x_j) = 0$ *for* $j \neq k$, $F_k(x_k) = 1$, *and* $\sum_{k=1}^{\infty} |F_k| \leq M^2$ *on* $M_{A^{**}}$.

Proof. The proof depends on a theorem of Varopoulos (see p.298 of [15]). Given $n \geq 1$ and $\varepsilon_n > 0$, that theorem provides functions $f_{n1}, \dots, f_{nn} \in A$ that satisfy $f_{nk}(x_k) = 1$ for $1 \leq k \leq n$, $f_{nk}(x_j) = 0$ for $j \neq k$, $1 \leq j, k \leq n$, and $\sum_{k=1}^{n} |f_{nk}| \leq M^2 + \varepsilon_n$ on M_A. Since bounded sets in A are weak-star precompact in A^{**}, we can find a net n_α such that $f_{n_\alpha k}$ converges weak-star to $F_k \in A^{**}$ for $1 \leq k < \infty$. Evidently F_k satisfies the interpolation conditions. Fix $m \geq 1$, and let $a_1, \dots, a_m$ be complex numbers of unit modulus. For any $n \geq m$ we have $|\sum_{k=1}^{m} a_k f_{nk}| \leq \sum_{k=1}^{m} |f_{nk}| \leq M^2 + \varepsilon_n$. Passing to the weak-star limit, we obtain $||\sum_{k=1}^{m} a_k F_k|| \leq M^2$. Since this is true for all such choices of the a_k's, we obtain $\sum_{k=1}^{m} |F_k| \leq M^2$. Since this is true for all m, we may sum to ∞, and the lemma is proved. □

If we apply this lemma in the situation in the proof of Theorem 2.2, we obtain the following result, where we have set $M^2 = 1 + \varepsilon$.

THEOREM 2.5. *Let A be a uniform algebra, with spectrum M_A and bidual A^{**}. Let E be a subset of M_A that is not hyperbolically bounded. Then for each $\varepsilon > 0$, there are a sequence of points $\{x_j\}_{j=1}^{\infty}$ in E and a sequence of functions $\{F_k\}_{k=1}^{\infty}$ in A^{**} such that $F_k(x_j) = 0$ for $j \neq k$, $F_k(x_k) = 1$, and $\sum_{k=1}^{\infty} |F_k| \leq 1 + \varepsilon$ on $M_{A^{**}}$.*

Proof. If E meets infinitely many Gleason parts, we select the x_j's from different Gleason parts, and we take $F_j \in A^{**}$ to be the idempotent corresponding to the part containing x_j. These do the trick, with $\sum |F_k| = 1$. If E has infinitely many points in the same Gleason part, we take $1 < M < \sqrt{1+\varepsilon}$, and we let G and the x_j's be as in the proof of Theorem 2.2. Any interpolation problem on a finite subset of the sequence $\{G(x_j)\}$ can be solved with interpolation constant M equal to the interpolation constant associated with the infinite sequence $\{G(x_j)\}$ in the open unit disk. Composing G with analytic interpolating functions, we are able to solve any interpolation problem for a finite subset of the x_j's with functions in A^{**} and with the same interpolation constant M. By approximating the interpolating functions in A^{**} weak-star by functions in A, we obtain interpolating functions in A for the finite interpolation problem, with possibly a small increase in the interpolation constant, say to $\sqrt{1+\varepsilon}$. Now we apply Lemma 2.4, and we are done. □

We will denote the pairing of $L \in A^*$ and $F \in A^{**}$ by $\langle L, F\rangle$. Regarded as a functional on A^{**}, L is represented by a finite measure on $M_{A^{**}}$. The condition $\sum |F_k| \le M$ in Theorem 2.5 then guarantees that $\sum |\langle L, F_k\rangle| \le M||L||$ for $L \in A^*$. This leads to the following corollary, where the notation is the same as above, and the projection of A^{**} onto the subspace spanned by the F_k's is given by $F \mapsto \sum F(x_k)F_k$.

COROLLARY 2.6. *Let the F_k's be as above. The map $V : L \mapsto \{\langle L, F_k\rangle\}$ is a continuous linear operator from A^* onto ℓ_1, with norm $||V|| \le 1 + \varepsilon$. Its adjoint V^* is an embedding $\lambda \mapsto \sum \lambda_k F_k$ of ℓ_∞ onto a complemented subspace of A^{**}. The operator V^* is a linear interpolation operator, in the sense that $F = V^*(\lambda)$ solves the interpolation problem $F(x_k) = \lambda_k$, $1 \le k < \infty$.*

2.3. Hyperbolically separated sequences

We say that a sequence $\{x_n\}$ in M_A is *hyperbolically separated* if there is $\varepsilon > 0$ such that $\rho_A(x_n, x_m) \ge \varepsilon$ for all $n \ne m$. If E is a subset of M_A that is not precompact with respect to the pseudohyperbolic metric, then E contains a sequence $\{x_n\}$ that is hyperbolically separated.

We are interested in conditions on a subset E of M_A that guarantee that E contains an interpolating sequence for A^{**}, that is, that E contains an ℓ_1-sequence. We might begin by asking whether each hyperbolically separated sequence $\{x_n\}$ in M_A has a subsequence that is interpolating for A^{**}. The answer turns out to be "yes" for some algebras A and "no" for others.

Let D be a bounded domain in the complex plane, and let $A(D)$ be the algebra of continuous functions on the closure $\overline{D}$ of D that are analytic on D. The spectrum of $A(D)$ is $\overline{D}$. From [14] it follows that any hyperbolically separated sequence in $\overline{D}$ has a subsequence that is an ℓ_1-sequence. The same result holds for the algebra $R(K)$ generated by the functions analytic in a neighborhood of some fixed compact subset K of the complex plane.

For another class of examples, let B be the open unit ball of a Banach space X, and let A be a uniform algebra on B that contains the functions in X^*. One such algebra is the algebra $H^\infty(B)$ of bounded analytic functions on B. Another such algebra is the algebra $A(B)$ of analytic functions on B that extend to be weak-star continuous on the closed unit ball of the bidual X^{**} of X. (See [2].) For any such algebra A, the points of B belong to the same Gleason part of A, and the usual Schwarz estimate for the intersections of B with one-dimensional subspaces shows that $\rho_A(0, x) = ||x||$ for $x \in B$.

For a very simple example of a hyperbolically bounded interpolating sequence, we take $X = \ell_1$, with standard basis $\{e_n\}$, and set $x_n = e_n/2$. Since $\rho_A(x_n, 0) = ||x_n|| = 1/2$, the sequence is hyperbolically bounded. We may regard any $\alpha \in \ell_\infty$ as an element of $(\ell_1)^* \subseteq A(B)$. The function $f = 2\alpha \in A(B)$ then interpolates α on $\{x_n\}$. Thus $\{x_n\}$ is an interpolating sequence for $A(B)$, and also for $H^\infty(B)$. Note that the interpolation constant for any subsequence of this sequence is $M = 2$.

To find an example of a hyperbolically separated sequence with no interpolating subsequence, we take X to be the (original) Tsirelson space. If $0 < r < 1$, the closed ball $r\overline{B}$ is a weakly compact subset of $A(B)^*$. (This is because polynomials on X are X^*-continuous on bounded sets, and the Taylor series of a bounded analytic function on B converges uniformly on $r\overline{B}$; see [1], [3], [10].) Hence $r\overline{B}$ has no interpolating sequence for $A(B)^{**}$. However, $r\overline{B}$ is not norm compact, and consequently it contains hyperbolically separated sequences. In fact, any sequence in $r\overline{B}$ with no norm-convergent subsequence is hyperbolically separated.

3. Unital homomorphisms of a uniform algebra

We focus now on unital homomorphisms of the uniform algebra A. These are in one-to-one correspondence with continuous maps $\phi : M_A \mapsto M_A$ such that $f \circ \phi \in A$ whenever $f \in A$. For such a map ϕ, the composition operator

$$C_\phi(f) = f \circ \phi, \qquad f \in A,$$

is evidently a unital homomorphism of A, that is, C_ϕ is multiplicative and $C_\phi(1) = 1$. For a given homomorphism $T : A \mapsto A$, the restriction of the adjoint T^* to M_A yields the mapping ϕ such that $T = C_\phi$.

U. Klein [19] has shown that such a map ϕ is nonexpanding with respect to the pseudohyperbolic metric of M_A. More precisely, he obtained the sharp estimate

$$\rho_A(\phi(x), \phi(y)) \leq c\rho_A(x, y), \qquad x, y \in M_A,$$

where c is the pseudohyperbolic diameter of $\phi(M_A)$,

$$c = \sup\{\rho_A(\phi(x), \phi(y)) : x, y \in M_A\}.$$

This estimate, which is valid for unital homomorphisms from one uniform algebra to another, is established as follows. Suppose $f \in A$ satisfies $||f|| < 1$ and $f(\phi(y)) = 0$. If $u \in M_A$, then $|f(\phi(u))| =$

$\rho(f(\phi(u)), f(\phi(y))) \leq \rho_A(\phi(u), \phi(y)) \leq c$. Thus $g = (f \circ \phi)/c$ satisfies $||g|| \leq 1$ and $g(y) = 0$. Hence $|g(x)| \leq \rho_A(x,y)$, and $|f(\phi(x))| \leq c\rho_A(x,y)$. Now take the supremum over such f.

According to Klein's theorem, ϕ is a (strict) contraction mapping with respect to the pseudohyperbolic metric if and only if its image $\phi(M_A)$ is a hyperbolically bounded subset of a single Gleason part. In this case, the contraction mapping theorem applies, and the iterates of ϕ converge uniformly in the pseudohyperbolic metric (or equivalently, in the norm of A^*) to a unique fixed point for ϕ.

It is easy to check that the pseudohyperbolic diameter of $\phi(M_A)$ is strictly less than 1 whenever the homomorphism C_ϕ is compact and M_A is connected. By invoking the contraction mapping theorem, Klein [19] obtained as a corollary a theorem of H. Kamowitz [17], that in this case the iterates of ϕ converge in the norm of A^* to a unique fixed point for ϕ. See [18] for more references on compact endomorphisms of Banach algebras, and see [13] for an exposition of Klein's work.

3.1. Weakly compact homomorphisms

To extend Klein's theorems on compact homomorphisms to a more general setting, it is natural to focus on weakly compact homomorphisms.

THEOREM 3.1. *Let A be a uniform algebra with connected spectrum M_A, and let C_ϕ be a unital homomorphism of A. If C_ϕ is weakly compact, then $\phi(M_A)$ is hyperbolically bounded, and ϕ is a (strict) contraction mapping with respect to the pseudohyperbolic metric. Consequently ϕ has a unique fixed point x_0, and the iterates of ϕ converge uniformly on M_A to x_0 in the pseudohyperbolic metric (or, equivalently, in the norm of A^*).*

Proof. Since C_ϕ is a weakly compact operator, so is C_ϕ^*. Since ϕ is the restriction of C_ϕ^* to M_A, $\phi(M_A)$ is a weakly compact subset of A^*, and consequently $\phi(M_A)$ meets only finitely many Gleason parts. Since M_A is connected, $\phi(M_A)$ is connected in the weak topology. As observed before, Gleason parts are relatively weakly open, hence $\phi(M_A)$ is contained in a single Gleason part. That ϕ is a contraction now follows from Corollary 2.3 and Klein's theorem cited above. □

We might also ask what can be said about the spectrum of a weakly compact homomorphism. Unlike compact homomorphisms, weakly compact homomorphisms can have nonzero eigenvalues of infinite multiplicity. For such an example, we return to the uniform algebra $A(B)$ on the open unit ball B of the Tsirelson space. Fix a complex number λ such that $0 < |\lambda| < 1$, and consider the unital homomorphism C_ϕ determined by the analytic map $\phi(x) = \lambda x$, $x \in B$. As shown in [3], the operator C_ϕ is weakly compact though not compact. The spectrum of C_ϕ consists of 0 together with the sequence of eigenvalues $\{\lambda^m\}_{m=0}^{\infty}$. The eigenspace corresponding to the eigenvalue λ^m is the restriction to B of the space of m-homogeneous analytic functions on X. For $m \geq 1$, these eigenspaces are infinite dimensional.

This example can be modified to obtain the following.

THEOREM 3.2. *Any sequence of complex numbers $\{\lambda_n\}$ satisfying $\sup|\lambda_n| < 1$ can be eigenvalues for a weakly compact composition operator on a uniform algebra.*

Proof. We take X to be the Tsirelson space, with unit ball B, and we take $A = H^\infty(B)$. By construction, X is a sequence space with a natural lattice structure. (See p.17 of [5].) For $x = \{x_n\} \in B$, we define $\phi(x) = \{\lambda_n x_n\}$. Then $\phi : B \mapsto B$ is well defined, linear, and continuous. The argument in [3] (see also [10]) shows that the operator C_ϕ is weakly compact. The nth coordinate projection π_n is an eigenfunction of the composition operator C_ϕ with eigenvalue λ_n. □

In connection with these examples, we might ask the following question: if the spectrum of a weakly compact homomorphism contains points other than 0 and 1, does the spectrum have an eigenvalue other than 1?

Another question has to do with the existence of point derivations. Suppose that M_A is connected. Klein [19] proved that if C_ϕ is compact, and if the spectrum of C_ϕ is larger than $\{0, 1\}$, then there is a nonzero continuous point derivation of A at the fixed point x_0 of ϕ. The point derivation can be regarded as a vestige of an analytic structure at x_0 in M_A. Question: is there an analog of Klein's result for weakly compact homomorphisms?

3.2. Homomorphisms with attracting cycles

In [19], Klein focuses on power-compact homomorphisms of uniform algebras. In the next two theorems, we modify Klein's development to extend certain of his results to their natural boundaries. First we clarify notation.

We denote the kth iterate of ϕ by ϕ^k, so that $\phi^1 = \phi$, and $\phi^k = \phi^{k-1} \circ \phi$ for $k \geq 2$. With this notation, the kth power of C_ϕ coincides with the operator of composition with ϕ^k, $C_\phi^k = C_{\phi^k}$.

A point $x \in M_A$ is a *periodic point* of ϕ if $\phi^k(x) = x$ for some $k \geq 1$. The least such k is called the *period* of x. The points $\{x, \phi(x), \phi^2(x), \ldots, \phi^{k-1}(x)\}$ are said to form a *cycle of length k*.

THEOREM 3.3. *Let C_ϕ be a unital homomorphism of the uniform algebra A. Then $\phi^n(M_A)$ is hyperbolically bounded for some $n \geq 1$ if and only if there is a decomposition of M_A into disjoint clopen subsets $F_1, \ldots, F_m$ such that the iterates of ϕ converge uniformly on each F_j in the pseudohyperbolic metric to a cycle C_j in F_j for ϕ.*

Proof. Suppose first that $\phi^n(M_A)$ is hyperbolically bounded. Then $\phi^n(M_A)$ is contained in finitely many Gleason parts $Q_1, \ldots, Q_p$. Since ϕ is nonexpanding, each image $\phi^i(Q_j)$ of Q_j is contained in a single Gleason part, and further $\phi^n(Q_j)$ is contained in one of the Q_i's. Thus there is a collection of at most np Gleason parts such that ϕ maps each of them to another. Consequently the images of a given Q_j under the iterates of ϕ must eventually cycle around a subset of the Gleason parts in the collection. Since the collection is finite, there is a subset $\{G_1, \ldots, G_q\}$ of the collection consisting of Gleason parts that are permuted by ϕ, such that the iterates of each $x \in M_A$ eventually land in the G_j's. Choose N so large that $\phi^N(M_A) \subset \cup G_j$. By taking N to be a multiple of the periods of the cycles of G_j's, we can also assume that $\phi^N(G_j) \subset G_j$ for $1 \leq j \leq N$. Let E_j be the set of $x \in M_A$ such that $\phi^N(x) \in G_j$. From the definition of the pseudohyperbolic metric in M_A, we see that for fixed $r < 1$ and $y \in M_A$, the pseudohyperbolic ball consisting of $x \in M_A$ satisfying $\rho_A(x, y) \leq r$ is a closed subset of M_A. Using the fact that $\phi^N(M_A)$ is hyperbolically bounded and closed, we see that each $\phi^N(M_A) \cap G_j$ is closed in M_A. Consequently each E_j is a closed subset of M_A. Thus the sets $E_1, \ldots, E_q$ form a decomposition of M_A into disjoint clopen subsets. By the Shilov idempotent theorem, there is a corresponding decomposition of the algebra A as a finite direct sum of subalgebras, $A = B_1 \oplus \cdots \oplus B_q$, such that the spectrum of B_j is E_j. Since each of the E_j's is invariant under ϕ^N, each of the algebras B_j is invariant under the operator C_ϕ^N of composition with ϕ^N. The image $\phi^N(E_j)$ is a hyperbolically bounded subset of G_j, so by Klein's estimate, ϕ^N is a pseudohyperbolic contraction of G_j. Hence there is a unique fixed point $x_j \in G_j$ for ϕ^N, such that the iterates ϕ^{kN} of ϕ^N converge uniformly on E_j to x_j as $k \to \infty$. The fixed points $\{x_1, \ldots, x_q\}$ of ϕ^N are permuted by ϕ. Thus we can partition them into a finite number of cycles $C_1, \ldots, C_m$. Let F_j be the union of the E_i's corresponding to the points in the jth cycle C_j. The sets $F_1, \ldots, F_m$ form a decomposition of

M_A into disjoint clopen subsets, and the iterates of ϕ converge uniformly on F_j to the cycle C_j.

The proof of the converse is easy. Suppose there are a finite number of cycles $C_1, \dots, C_m$ to which the iterates of points of M_A converge. Then for large n, $\phi^n(M_A)$ is contained in the union of pseudohyperbolic balls centered at points of $\cup C_j$ of small radii, and in particular $\phi^n(M_A)$ is hyperbolically bounded. □

We refer to C_j as an *attracting cycle* for ϕ, and we refer to the clopen set F_j as the *basin of attraction* of C_j.

THEOREM 3.4. *Let C_ϕ be a unital homomorphism of the uniform algebra A, and suppose $\phi^n(M_A)$ is hyperbolically bounded. Let $x_1, \dots, x_k$ be the periodic points in M_A of ϕ. Let N be a common multiple of the periods of the x_j's, and define*

$$\kappa \;=\; \max_{1\le j\le k} \limsup_{x\to x_j} \left[\frac{\rho_A(\phi^N(x), x_j)}{\rho_A(x, x_j)}\right]^{1/N} \;<\; 1.$$

Then the spectrum of C_ϕ is the union of a subset of the disk $\{|\lambda| \le \kappa\}$ and a finite set of eigenvalues of finite multiplicity lying on the unit circle. Further, the eigenvalues of C_ϕ lying on the unit circle are roots of unity. The multiplicity of the eigenvalue 1 is the number of cycles of ϕ, and the corresponding eigenspace is spanned by the characteristic functions of the basins of attraction of the attracting cycles of ϕ.

Proof. We continue with the same notation as in the preceding proof. Let S_j be the operator obtained by restricting C_ϕ^N to the functions in B_j that vanish at x_j. According to one of Klein's main results (Theorem 9 in [13]), the spectral radius $||S_j||_r$ of S_j is estimated by the local contraction constant at x_j,

$$||S_j||_r \;\le\; \limsup_{x\to x_j} \frac{\rho_{B_j}(\phi^N(x), x_j)}{\rho_{B_j}(x, x_j)}, \qquad 1 \le j \le n.$$

Let A_0 be the ideal of functions $f \in A$ such that $f(x_j) = 0$ for $1 \le j \le n$. The ideal A_0 is invariant under C_ϕ. Let $T_0 = C_\phi|_{A_0}$ be the restriction of C_ϕ to A_0. The spectral radius of the restriction of C_ϕ^N to A_0 is the maximum of the spectral radii of the S_j's. Consequently the spectral radius of T_0 is given by

$$||T_0||_r \;=\; (\max\{||S_1||_r, \dots, ||S_1||_r\})^{1/N}.$$

Since the invariant subspace A_0 has finite codimension in A, the spectrum of C_ϕ is obtained from the spectrum of T_0 by adjoining the eigenvalues of the quotient operator $T(f + A_0) = C_\phi f + A_0$ on the quotient space A/A_0. Since C_ϕ^N is the identity map on $\cup C_j$, $C_\phi^N(f) - f \in A_0$ for all $f \in A$. Hence T^N is the identity operator on A/A_0, and the spectrum of T consists only of Nth roots of unity.

It is easy to identify explicitly the spectrum of T in terms of the lengths of the cycles $C_1, \dots, C_m$. The quotient space A/A_0 is a direct sum of m subspaces corresponding to the functions supported on the jth cycle for $1 \leq j \leq m$. The subspace corresponding to the jth cycle is an invariant subspace of the operator T on the quotient space A/A_0, whose dimension is the length m_j of C_j. On this subspace, T is essentially the composition operator induced by the action of ϕ on the cycle. The eigenvalues of T on this subspace are the m_jth roots of unity, and each of these is a simple eigenvalue of the restriction of T to the subspace. In particular, the multiplicity of the eigenvalue 1 is the number of cycles m. The corresponding eigenspace includes the characteristic functions of the clopen sets $F_1, \dots, F_m$ of Theorem 3.3. Since these functions are linearly independent, they span the eigenspace. □

Recall that an operator V is *quasi-compact* if there is an integer $m \geq 1$ and a compact operator K such that $||V^m + K|| < 1$. We mention the following corollary of the preceding analysis.

COROLLARY 3.5. *Let C_ϕ be a unital homomorphism of the uniform algebra A. If $\phi^n(M_A)$ is hyperbolically bounded for some $n \geq 1$, then C_ϕ is quasi-compact.*

Proof. Let S be the finite-dimensional operator defined so that $S = 0$ on A_0, while S coincides with C_ϕ on the eigenspaces of C_ϕ corresponding to eigenvalues of unit modulus. Then $||C_\phi - S||_r < 1$, so $||(C_\phi - S)^m|| < 1$ for large m. Thus $||C_\phi^m + K|| < 1$ for some finite-dimensional operator K, and C_ϕ is quasi-compact. □

3.3. Factorization of operators

We say that an operator S factors through an operator T if there are operators U and V with appropriate domains and ranges such that $S = U \circ T \circ V$. We say that S factors *almost isometrically* through T if, moreover, for any $\varepsilon > 0$ we can choose the operators U and V so that $(1-\varepsilon)||x|| \leq ||Ux|| \leq (1+\varepsilon)||x||$ and $(1-\varepsilon)||x|| \leq ||Vx|| \leq (1+\varepsilon)||x||$.

We will be interested in factoring the identity operator of ℓ_p through an operator T. This boils down to finding a subspace "almost" isometric to ℓ_p on which T is "almost" an isometry.

THEOREM 3.6. *Let C_ϕ be a composition operator on the uniform algebra A. The following are equivalent:*

(i) *$\phi(M_A)$ is* not *hyperbolically bounded,*

(ii) *for each $n \geq 1$, the identity operator of ℓ_∞^n factors almost isometrically through C_ϕ,*

(iii) *the identity operator of ℓ_∞ factors almost isometrically through C_ϕ^{**},*

(iv) *the identity operator of ℓ_1 factors almost isometrically through C_ϕ^*.*

Proof. Suppose first that (i) holds. Let $\varepsilon > 0$, and let $\{y_j\}_{j=1}^\infty$ be a sequence of points in M_A whose images $x_j = \phi(y_j)$ have the properties of Theorem 2.5. Thus there are functions $\{F_k\}$ in A^{**} such that $F_k(x_j) = 0$ for $j \neq k$, $F_k(x_k) = 1$, and $\sum |F_k| \leq 1 + \varepsilon$. Define the operator $U : \ell_1 \mapsto A^*$ by

$$U(\xi) = \sum \xi_j y_j, \qquad \xi \in \ell_1.$$

Then $||U(\xi)|| \leq ||\xi||$, so $||U|| \leq 1$. Let $V : A^* \mapsto \ell_1$ be defined as in Corollary 2.6, so that $V(L)$ is the sequence $\{\langle L, F_k \rangle\} \in \ell_1$ for $L \in A^*$. By representing L as a measure on $M_{A^{**}}$ and estimating an integral, we obtain $||V(L)|| \leq (1+\varepsilon)||L||$, or $||V|| \leq 1+\varepsilon$. Since $C_\phi^*(y_j) = \phi(y_j) = x_j$, we have $V(C_\phi^*(U(\xi)))_k = \langle C_\phi^*(\sum \xi_j y_j), F_k \rangle = \sum \xi_j \langle x_j, F_k \rangle = \xi_k$, and $V \circ C_\phi^* \circ U$ is the identity map of ℓ_1. Using $||C_\phi^*|| = 1$, we obtain $||\xi|| = ||V(C_\phi^*(U(\xi)))|| \leq ||V|| \, ||C_\phi^*|| \, ||U(\xi)|| \leq (1+\varepsilon)||U(\xi)||$ and $||\xi|| = ||V(L)|| \leq (1+\varepsilon)||L||$ for $L = C_\phi^*(U(\xi))$. These show that U is close to isometry on ℓ_1, and V is close to isometry from the range of $C_\phi^* \circ U$ onto ℓ_1. Thus (iv) holds.

We obtain (iii) from (iv) by taking adjoints. We obtain (ii) from (iii) and the principle of local reflexivity, or by using the functions from the proof of Lemma 2.4.

Suppose finally that (ii) holds. Let $r < 1$ and $n \geq 1$, and let $\varepsilon > 0$ be small. Let $f_j \in A$ be the image of the jth basis element e_j of ℓ_∞^n under the operator that is close to being an isometry. Then $1 - \varepsilon < ||f_j \circ \phi|| < 1+\varepsilon$, so we can find $y_j = \phi(x_j)$ such that $1-\varepsilon < |f_j(y_j)| < 1+\varepsilon$. If $k \neq j$, then $||e_j \pm e_k|| = 1$, so that $||f_j \pm f_k|| < 1 + \varepsilon$, and $|f_j(y_j) \pm f_k(y_j)| < 1 + \varepsilon$. It follows that $|f_k(y_j)| < \tau(\varepsilon)$, where $\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$. (Take $\tau(\varepsilon) = 2\sqrt{\varepsilon}$.) Thus $\rho_A(y_j, y_k) \geq 1 - \delta(\varepsilon)$, where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Consequently the points $y_1, \dots, y_n$ are not contained in any collection of $n-1$ hyperbolic balls of radius r, and $\phi(M_A)$ is not hyperbolically bounded. □

COROLLARY 3.7. *Let C_ϕ be a composition operator on the uniform algebra A. If C_ϕ^* belongs to an operator ideal to which the identity operator on ℓ_1 does not belong, then $\phi(M_A)$ is hyperbolically bounded.*

This corollary applies, in particular, when C_ϕ is weakly compact.

3.4. Uniform algebras on arbitrary sets

We say that A is a *uniform algebra on a set* Y if A is a uniformly closed subalgebra of bounded functions on Y that contains the constants and separates the points of Y. The norm of A is given by

$$||f|| \ = \ \sup\{|f(y)| : y \in Y\}, \qquad f \in A. \tag{2}$$

We can identify Y with a subset of the spectrum M_A of A, and we take the topology of Y to be the weak topology determined by the functions in A, that is, the topology inherited from M_A. If Y is any subset of M_A such that (2) holds, then we may view A as a uniform algebra on Y.

If $\phi : Y \mapsto Y$ is such that $f \circ \phi \in A$ whenever $f \in A$, then the operator $f \mapsto f \circ \phi$ is a unital homomorphism of A, and consequently ϕ extends to a self-map of M_A. We denote this extension also by ϕ, so that the homomorphism coincides with C_ϕ.

We say that ϕ is *hyperbolically bounded on* Y if $\phi(Y)$ is a hyperbolically bounded subset of M_A, that is, if $\phi(Y)$ is contained in a finite union of pseudohyperbolic balls whose radii are strictly less than 1.

LEMMA 3.8. *Let A be a uniform algebra on Y, and suppose $\phi : Y \mapsto Y$ is such that $f \circ \phi \in A$ whenever $f \in A$. If ϕ is hyperbolically bounded on Y, then (the extended) ϕ is hyperbolically bounded on M_A. Further, $\phi(M_A)$ is contained in the Gleason parts that meet Y.*

Proof. Choose $r < 1$ and points $y_j \in Y$, $1 \leq j \leq m$, such that $\phi(Y)$ is contained in the pseudohyperbolic balls $\{\rho_A(\phi(y_j), x) \leq r\}$. We may assume that the $\phi(y_j)$'s belong to different Gleason parts of M_A. Suppose that $\phi(M_A)$ is not hyperbolically bounded. Then there are points $w_i \in M_A$ and functions $f_i \in A$ such that $||f_i|| < 1$, $f_i(\phi(y_j)) = 0$ for $1 \leq j \leq m$, and $f_i(\phi(w_i)) \to 1$. Then $|f_i(x)| \leq r$ for all $x \in \phi(Y)$, so $|f_i \circ \phi| \leq r$ on Y. By (2), $||f_i \circ \phi|| \leq r$. In particular, $|f_i(\phi(w_i))| \leq r$. This contradiction shows that $\phi(M_A)$ is hyperbolically bounded. The same argument shows that $\phi(M_A)$ is contained in the Gleason parts of the $\phi(y_j)$'s. □

THEOREM 3.9. *Let A be a uniform algebra on Y. Suppose $\phi : Y \mapsto Y$ is such that $f \circ \phi \in A$ whenever $f \in A$. If $\phi^n(Y)$ is hyperbolically bounded for some $n \geq 1$, then (the extended) ϕ has a finite number of periodic points in M_A, all of which belong to the norm closure of Y. Further, the iterates ϕ^k of ϕ converge uniformly on Y to the set of periodic points.*

Proof. By the preceding lemma, each point u of $\phi^n(M_A)$ is contained in the same Gleason part as a point v of $\phi^n(Y)$. Since $\rho_A(\phi^k(u), \phi^k(v)) \to 0$ as $k \to \infty$, the iterates of any point of M_A accumulate on the norm-closure of Y in M_A. Thus the periodic points belong to the norm-closure of Y. The remaining assertions of the theorem follow from Theorem 3.3. □

For certain algebras it is possible to improve on Theorem 3.6 by factoring the identity operator of ℓ_∞ through C_ϕ rather than through its double dual C_ϕ^{**}.

THEOREM 3.10. *Let A be a uniform algebra on Y such that the limit of any bounded net of functions in A that converges pointwise on Y also belongs to A. Let $\phi : Y \mapsto Y$ be such that $f \circ \phi \in A$ whenever $f \in A$. Suppose ϕ is <u>not</u> hyperbolically bounded on Y. Then the identity operator of ℓ_∞ factors almost isometrically through C_ϕ. Further, A is a dual Banach space, C_ϕ is the adjoint of an operator on the predual of A, and the identity operator of ℓ_1 factors almost isometrically through the predual of C_ϕ.*

Proof. The condition on pointwise bounded limits guarantees that A is a weak-star closed subspace of $\ell_\infty(Y)$, by the Krein-Schmulian theorem. (For a similar argument, see p.100 of [11].) It follows that the restriction of any function in A^{**} to Y coincides on Y with a function in A. Thus we can replace the functions F_k in the proof of Theorem 3.6 by functions $f_k \in A$ that satisfy the interpolation conditions and the estimate $\sum |f_k(y)| \leq 1 + \varepsilon$ for $y \in Y$. Thus $|\sum_{k=1}^m a_k f_k| \leq 1 + \varepsilon$ on Y for any choice of the unimodular constants $a_1, \ldots, a_m$, and since by (2) the norm on A is the sup-norm over Y, $||\sum_{k=1}^m a_k f_k|| \leq 1 + \varepsilon$. It follows as before that $\sum |f_k| \leq 1 + \varepsilon$ on M_A. We define $R : \ell_\infty \mapsto A$ and $S : A \mapsto \ell_\infty$ by $R\lambda = \sum \lambda_j f_j$ and $Sf = \{f(y_k)\}$, and we compute that $(S \circ C_\phi \circ R)(\lambda)_k = C_\phi((\sum_j \lambda_j f_j)(y_k)) = \sum_j (\lambda_j f_j)(x_k) = \lambda_k$, so that $S \circ C_\phi \circ R$ is the identity on ℓ_∞. The estimates on the norms of R and S are obtained as before.

Since A is a weak-star closed subspace of $\ell_\infty(Y)$, the quotient Banach space $A_* = \ell_1(Y)/(\ell_1(Y) \cap A^\perp)$ has A as its dual. Let $\delta_y \in \ell_1(Y)$ be the characteristic function of the singleton $\{y\}$. The correspondence $\delta_y \to \delta_{\phi(y)}$ induces an operator on $\ell_1(Y)$ that leaves $A^\perp$ invariant. It induces a quotient operator on A_*, which is readily seen to have C_ϕ as its dual. We define an operator $R_* : A_* \to \ell_1$ by

$$R_*(\mu + A^\perp) \;=\; \left\{ \sum_{y\in Y} \mu_y f_j(y) \right\}_{j=1}^{\infty}, \qquad \text{where } \mu = \sum \mu_y \delta_y \in \ell_1(Y),$$

and we define an operator $S_* : \ell_1 \to A_*$ by

$$S_*(\lambda) \;=\; \sum_j \lambda_j \delta_{y_j} \;+\; A^\perp, \qquad \lambda \in \ell_1.$$

A straightforward computation reveals that the adjoint operators of R_* and S_* are respectively R and S, and that R_* and S_* implement a factorization of the identity operator of ℓ_1 through the predual of C_ϕ. Estimates on the norms for R_* and S_* follow from those for R and S, so that the identity of ℓ_1 factors almost isometrically through the predual of C_ϕ. □

3.5. Uniform algebras of analytic functions

Let A be a uniform algebra on Y. A subset D of Y is an *analytic disk* if there is a one-to-one map z of D onto an open disk in the complex plane such that the functions in A are analytic functions of the coordinate map z on D. We say that Y is *analytic-diskwise connected* if given any two points $x, y \in Y$, there is a finite collection of analytic disks $D_1, \ldots, D_m$ in Y such that $x \in D_1$, $y \in D_m$, and $D_j \cap D_{j+1} \neq \emptyset$ for $1 \leq j < m$. We state for emphasis the following corollary to Theorem 3.9.

THEOREM 3.11. *Let A be a uniform algebra on Y. Suppose that Y is analytic-diskwise connected. Let $\phi : Y \mapsto Y$ be a map such that $f \circ \phi \in A$ whenever $f \in A$. If ϕ is hyperbolically bounded on Y, then the iterates of (the extended) ϕ converge uniformly to a fixed point x_0 of ϕ, which belongs to the norm closure of Y in A^*.*

Proof. The connectedness hypothesis implies that A has no nontrivial idempotents. Consequently M_A is connected. Since the iterates of ϕ converge uniformly on M_A to the set of periodic points, the set of periodic points is connected, and hence it consists of only one point, which is a fixed point. □

Theorem 3.11 applies to $H^\infty(Y)$, the algebra of bounded analytic functions on Y, where Y is any connected complex analytic variety (possibly infinite dimensional) such that $H^\infty(Y)$ separates the points of Y. In this case the metric ρ_A is essentially the Carathéodory metric of Y.

In order to apply the theorem, we would like to have on hand a criterion for a subset of Y to be hyperbolically bounded. One such obvious criterion is obtained by modifying the definition of analytic-diskwise connectedness of a subset E of Y. Suppose there are $N \geq 1$ and $r < 1$ such that for any two points $x, y \in E$, there is a collection of $m \leq N$ analytic disks $D_1, \ldots, D_m$ in Y so that x is the center of D_1, y is the center of D_m, and for $1 \leq j < m$, D_j meets D_{j+1} at a point whose pseudohyperbolic distance from the center of each disk is less than r. Then König's inequality (1) shows that the pseudohyperbolic diameter of E is strictly less than 1, so that E is hyperbolically bounded.

THEOREM 3.12. *Let D be a bounded convex domain in a Banach space X, and let A be a uniform algebra on D that includes the functions in X^*. Then a subset E of D is hyperbolically bounded if and only if E is at a positive (norm) distance from the boundary ∂D of D.*

Proof. Without loss of generality we may assume $0 \in E$. Suppose first that the distance from E to ∂D is $\delta > 0$. Let $y \in E$, and consider the intersection of the subspace spanned by y with D. By considering disks of radius δ centered on the line segment joining 0 and y, we see that the above criterion applies, and E is hyperbolically bounded. For the converse, suppose there is a sequence $x_n \in E$ whose distances to ∂D tend to 0. Choose $y_n \in X$ such that y_n does not belong to the closure of D, and $||x_n - y_n|| \to 0$. By the separation theorem for convex sets, there is $L_n \in X^*$ such that $\sup\{\mathrm{Re}\,(L_n(x)) : x \in D\} < 1 = L_n(y_n)$. Since D contains the ball centered at 0 of radius δ, the norms $||L_n||$ are uniformly bounded by $1/\delta$. Hence $|L_n(x_n - y_n)| \leq ||x_n - y_n||/\delta \to 0$, and $L_n(x_n) \to 1$. Let $f_n = e^{L_n - 1} \in A$. Then $||f_n|| < 1$, $f_n(0) = 1/e$, and $f_n(x_n) \to 1$. It follows that $\rho_A(0, x_n) \to 1$, and E is not hyperbolically bounded. □

As a special case, let B be the open unit ball of a Banach space X, and let A be a uniform algebra on B that contains the functions in X^*. Let $\phi : B \mapsto B$ be an analytic self-map of B such that $f \circ \phi \in A$ whenever $f \in A$. It was proved in [3] that if $\phi(B) \subset rB$ for some $r < 1$, and if C_ϕ is compact, then ϕ has a unique fixed point x_0 in B, to which the iterates of ϕ converge uniformly; further, the spectrum of C_ϕ is the unital semigroup generated by the spectrum of the Frechet derivative $dC_\phi(x_0)$ of C_ϕ at x_0. The first of these two conclusions is contained in Theorems 3.11 and 3.12, and further the result holds for bounded convex domains just as soon as the composition operator is weakly compact. With respect to the second conclusion, it would be of interest to say something about the spectrum of C_ϕ in the case that C_ϕ is only weakly compact.

The proof in [3] depends on the Earle-Hamilton fixed point theorem (see p.187 of [6], p.192 of [8]), which asserts that if D is a bounded domain in a Banach space, and if ϕ is an analytic self-map of D such that the distance from $\phi(D)$ to the boundary of D is strictly positive, then ϕ has a unique fixed point in D, to which the iterates of ϕ converge uniformly. The proof of the Earle-Hamilton fixed point theorem depends on the contraction properties of analytic maps with respect to a certain "hyperbolic" metric. As we have seen above, the Earle-Hamilton fixed point theorem in the (simple) case of a bounded convex domain in a Banach space is a consequence of Theorems 3.11 and 3.12. This is related to work of L. Harris (see Proposition 23 on p. 381 of [16]), who treats a class of metrics on domains in Banach spaces that includes the metric ρ_A.

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