Flapwise Bending Vibration Analysis of Rotating Composite Cantilever Beams

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A modeling method for the modal analysis of a rotating composite cantilever beam is presented in this paper. Linear differential equations of motion are derived using the assumed mode method. For the modeling, hybrid deformation variables are employed and approximated to derive the equations of motion. Symmetrical laminated composite beams are considered to obtain the numerical results. The effects of the dimensionless angular velocity, the hub radius and the fiber orientation angle on the variations of modal characteristics are investigated.

Key Words : Flapwise Bending Vibration, Composite, Fiber Angle, Rotating Contilever Beam, Natural Frequency Variation, Mode Shape Variation

1. Introduction

Composite structures, especially rotating composite cantilever beams are often used in engineering applications such as helicopter blades. In order to design this type of structures, the modal characteristics of them need to be understood. The modal characteristics of rotating structures differ from those of non-rotating structures. The centrifugal inertia force due to the rotational motion causes the increment of the bending stiffness of the structure, which naturally results in the variations of natural frequencies and mode shapes. Since composite materials possess high strength and light weight characteristics, they are generally used for many structure designs. The structural properties of the composite materials can be easily controlled by adjusting fiber orientation angles and number of plies. Therefore, the modal characteristics of rotating composite cantilever beams, which are significantly affected by the stiffness change induced by rotational motion and the composite material composition, should be estimated accurately.

To estimate the modal characteristics of rotating cantilever beams, an analytical model using energy method was first introduced by Southwell and Gough (1921). Due to the simplicity of the model, it has been widely used by engineers who want to design rotating blades. Schlhansl (1958) derived the equations of motion of rotating cantilever beams and obtained more accurate analytical model to estimate the natural frequencies. These analytical models were introduced in the early stage of the rotating beam research. As the computational technology progressed since early 1970's, a large number of numerical results (see, for instance, Putter and Manor (1978)) were published. Lately, more efficient modeling methods possessing efficiency and flexibility (see reference Yoo and Shin (1998) Kuo and Lin (1998)) were introduced. All these results are, however, involved with rotating beams that are made of isotropic materials.

Modal characteristics of non-rotating composite structures were investigated in many previous works (Kapania and Raciti, 1989 ; Rand, 1991 ; Nabi and Ganesan, 1994 ; Kosmatka and Friedmann, 1989 ; Chandrashekhara et al., 1990).

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However, only a few works were involved with rotating structures that are made of composite materials. The objective of this paper, therefore, is to present an accurate modeling method that can estimate the variations of natural frequencies and mode shapes. The effects of the fiber orientation angle, the angular velocity and the hub radius of the rotating beam on the variations are investigated.

2. Equations of Motion

In this paper, the thickness of the beam is assumed to be uniform and small compared to the beam length, so that transverse shear and rotary inertia effects are ignored. Also, the flapwise bending thickness is assumed to be much smaller than the chordwise bending thickness (like helicopter blades). Thus, the bending displacement in the chordwise direction is also ignored in the following formulation. So, the elastic strain energy of a composite beam can be expressed as follows :

$$U=\frac{1}{2}\int_0^L[A_{11}\varepsilon_{x_0}^2+2B_{11}\varepsilon_{x_0}k_y+D_{11}k_y^2]\,dx \quad (1)$$

where A_{11}, B_{11}, and D_{11} can be obtained by integrating the material properties of each layer of the composite beam (as shown in Fig. 2) as follows :

$$A_{11}=b\int_{-h/2}^{h/2}Q_{11}^{(k)}dz=b\sum_{k=1}^{N}Q_{11}^{(k)}(z_k-z_{k-1})$$
$$B_{11}=b\int_{-h/2}^{h/2}Q_{11}^{(k)}zdz=\frac{b}{2}\sum_{k=1}^{N}Q_{11}^{(k)}(z_k^2-z_{k-1}^2) \quad (2)$$
$$D_{11}=b\int_{-h/2}^{h/2}Q_{11}^{(k)}z^2dz=\frac{b}{3}\sum_{k=1}^{N}Q_{11}^{(k)}(z_k^3-z_{k-1}^3)$$

where Q_{ij}^k are the off-axis stiffnesses of k^{th} layer, z_k and z_{k-1} are the distances from the mid-plane to the top and bottom surface of k^{th} layer, and N is the total number of layers. Stretching and bending strains expressed in the deformation variables s and u_3 (shown in Fig. 1) are expressed as follows :

$$\varepsilon_{x_0}=\frac{\partial s}{\partial x},\ k_y=-\frac{\partial^2 u_3}{\partial x^2} \quad (3)$$

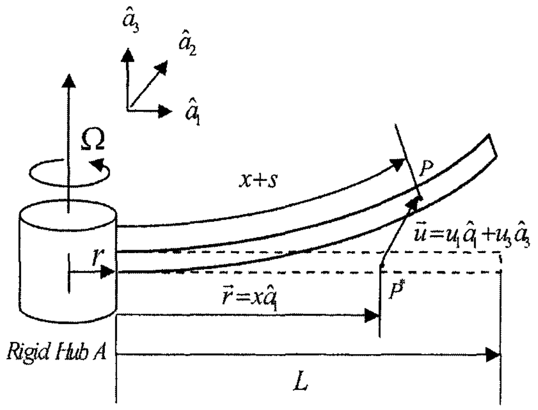


Fig. 1 Configuration of a rotating cantilever beam

Substituting Eq. (3) into Eq. (1), one obtains

$$U=\frac{1}{2}\int_0^L\left[A_{11}\left(\frac{\partial s}{\partial x}\right)^2+2B_{11}\left(\frac{\partial s}{\partial x}\right)\left(\frac{\partial^2 u_3}{\partial x^2}\right)+D_{11}\left(\frac{\partial^2 u_3}{\partial x^2}\right)^2\right]dx \quad (4)$$

To obtain ordinary differential equations of motion, deformation variables are approximated by the Rayleigh-Ritz assumed mode method as follows :

$$s(x,\ t)=\sum_{j=1}^{\mu_1}\phi_{1j}(x)\,q_{1j}(t)$$
$$u_3(x,\ t)=\sum_{j=1}^{\mu_3}\phi_{3j}(x)\,q_{3j}(t) \quad (5)$$

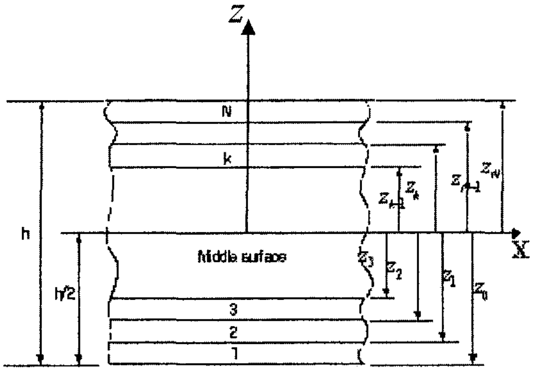
Where ϕ_{1j} and ϕ_{3j} are spatial mode functions. Any compact set of admissible functions that satisfy the geometric boundary conditions of the beam can be used as the mode functions. Meanwhile q_{1j} and q_{3j} are generalized coordinates and μ_1 and μ_3 are the number of the generalized coordinates for q_{1j} and q_{3j}, respectively.

Since s (instead of u_1) is approximated, expressions involved with u_1 need to be expressed in terms of s and u_3. For the purpose, the following approximate geometric relation is employed :

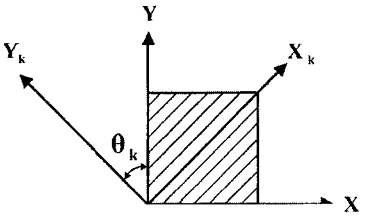
$$s=u_1+\int_0^x\left(\frac{\partial u_3}{\partial \eta}\right)^2 d\eta \quad (6)$$

By using the Kane's method, equations of motion can be obtained from the following equation given by

$$\int_0^L\rho\left(\frac{\partial \vec{v}^p}{\partial \dot{q}_i}\right)\cdot\frac{d\vec{v}^p}{dt}\,dx+\frac{\partial U}{\partial q_i}=0\quad (i=1,\ \cdots,\ \mu) \quad (7)$$



(a) Laminated composite beam geometry and layer numbering



(b) Fiber angle of the k-th layer of the laminated composite beam

Fig. 2 Constitution of a laminated composite beam

where $\mu=\mu_1+\mu_3$, $\vec{v}^P$ the velocity of the generic point P, U the strain energy of the beam, ρ the mass per unit length of the beam, L the undeformed length of the beam. When the rigid hub rotates with a constant angular speed Ω, $\vec{\omega}^A$ and $\vec{v}^P$ can be expressed as follows :

$$\vec{\omega}^A=\Omega\hat{a}_3 \tag{8}$$

$$\vec{v}^P=\dot{u}_1\hat{a}_1+\Omega(r+x+u_1)\hat{a}_2+\dot{u}_3\hat{a}_3 \tag{9}$$

where $\vec{\omega}^A$ is the angular velocity of the rigid hub attached to a beam.

By using Eqs. (5), (6) and (9)

$$\frac{\partial\vec{v}^P}{\partial\dot{q}_i}=\left[\phi_{1i}-\sum_{j=1}^{\mu}\left(\int_0^x\phi_{3i,\eta}\phi_{3j,\eta}d\eta\right)q_j\right]\hat{a}_1+\phi_{3i}\hat{a}_3 \tag{10}$$

Using Eqs. (4), (7), (9), (10) and considering composite beam that has mid-plane symmetry of cross-section, the flapwise bending vibration of the rotating composite beam is governed by the following equation which is not coupled with bending and stretching direction :

$$\begin{aligned}&\sum_{j=1}^{\mu_3}\left(\int\rho\phi_{3i}\phi_{3j}dx\right)\ddot{q}_{3j}+\sum_{j=1}^{\mu_3}\left(\int D_{11}\phi_{3i,xx}\phi_{3j,xx}dx\right)q_{3j}\\&+\sum_{j=1}^{\mu_3}\Omega^2\left[r\left(\int\rho(l-x)\,\phi_{3i,x}\phi_{3j,x}dx\right)q_{3j}\right.\\&\left.\quad+\left(\int\frac{1}{2}\rho(l^2-x^2)\,\phi_{3i,x}\phi_{3j,x}\right)q_{3j}\right]=0\ (i=1,\cdots,\mu_3)\end{aligned} \tag{11}$$

To convert the equation of motion into a dimensionless form, we introduce the dimensionless variables are defined by

$$\tau\equiv\frac{t}{T},\ \xi\equiv\frac{x}{L},\ \theta_i\equiv\frac{q_i}{L},\ \psi_i(\xi)\equiv\phi_i(x),\ \gamma\equiv T\Omega \tag{12}$$

where $T=\sqrt{\frac{\rho L^4}{D}}$ and D is D_{11} when all layers have 0 degree fiber orientation angle. Substituting these dimensionless variables into Eq. (11), one obtains

$$\begin{aligned}&\sum_{j=1}^{\mu_3}\left(\int\psi_{3i}\psi_{3j}d\xi\right)\ddot{\theta}_{3j}+\sum_{j=1}^{\mu_3}\left(\int\delta\psi_{3i,\xi\xi}\psi_{3j,\xi\xi}d\xi\right)\theta_{3j}\\&+\sum_{j=1}^{\mu_3}\gamma^2\left[\sigma\left(\int(1-\xi)\,\psi_{3i,\xi}\psi_{3j,\xi}d\xi\right)\theta_{3j}\right.\\&\left.\quad+\left(\int\frac{1}{2}(1-\xi^2)\,\psi_{3i,\xi}\psi_{3j,\xi}\right)\theta_{3j}\right]=0\ (i=1,\cdots,\mu_3)\end{aligned} \tag{13}$$

where $\delta=\frac{D_{11}}{D}$ and $\sigma=\frac{r}{L}$. In matrix form, Eq. (13) can be expressed as follows

$$\sum_{j=1}^{\mu_3}\left[M_{ij}^{33}\ddot{\theta}_{3j}+\{K_{ij}^{B3}+\gamma^3K_{ij}^{G3}\}\theta_{3j}\right]=0 \tag{14}$$

where

$$M_{ij}^{33}=\int_0^1\psi_{3i}\psi_{3j}d\xi$$

$$K_{ij}^{B3}=\delta\int_0^1\psi_{3i,\xi\xi}\psi_{3j,\xi\xi}d\xi$$

$$K_{ij}^{G3}=\int_0^1\sigma(1-\xi)\,\psi_{3i,\xi}\psi_{3j,\xi}d\xi+\frac{1}{2}\int_0^1(1-\xi^2)\,\psi_{3i,\xi}\psi_{3j,\xi}d\xi$$

If θ_{3j} is a harmonic function of τ, θ represents a column matrix which has θ_{3j} as its element :

$$\theta=e^{j\omega\tau}\Theta \tag{15}$$

where j represents a imaginary number, ω is the ratio of the flapwise bending natural frequency to the reference frequency (inverse of T) and Θ is a constant column matrix characterizing the deflection shape for synchronous motion. Using

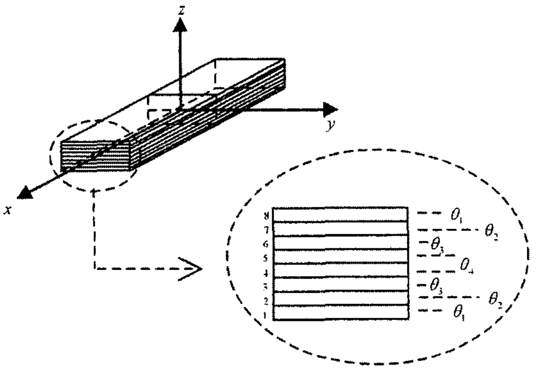


Fig. 3 Fiber direction of the eight-layer laminate

Eq. (15), Eq. (14) can be written as follows

$$\omega^2 M\Theta = K\Theta \quad (16)$$

where M and K are square matrices. Their elements and M_{ij} are K_{ij} defined as follows:

$$M_{ij} = M_{ij}^{33}$$
$$K_{ij} = K_{ij}^{B3} + \gamma^2 K_{ij}^{C3} \quad (17)$$

3. Numerical Results

In this paper, two composite materials are used for the composite beams that are composed of 8 layers. The first material is used for 1st, 4th, 5th, and 8th layers, and the second material is used for 2nd, 3rd, 6th, and 7th layers. Fiber orientation angles lie symmetrically on the mid-plane of the composite beams. Properties of the material 1 and 2 are shown in Table 1.

Figure 4 shows the variation of δ as θ_1 increases from 0° to 90° with fixed conditions of $\theta_3=45°$, $\theta_4=90°$ (for $\theta_2=30°$, 60°, and 90°). As shown in the figure, variable δ varies in the range of $0<\delta\leq 1$. Also, δ varies significantly when θ_1 and θ_2 are lower than 45°.

Figure 5 shows the variations of the lowest three natural frequencies versus the angular velocity. $\delta=0.5$ is used to obtain the results. As shown in the figure, natural frequencies increase as the angular speed increases. Also, their increasing rates become larger as the hub radius ratio increases.

Figure 6 shows the variation of the first natural frequency versus δ, where $\sigma=0.1$ is used to obtain

Table 1 Material data of the composite beam

	Material 1	Material 2
Properties	T300/5208	Kevlar Fiber -Epoxide
E_1(GPa)	181	84
E_2(GPa)	10.3	5.6
G_{12}(GPa)	7.17	2.1
v_{12}	0.28	0.34

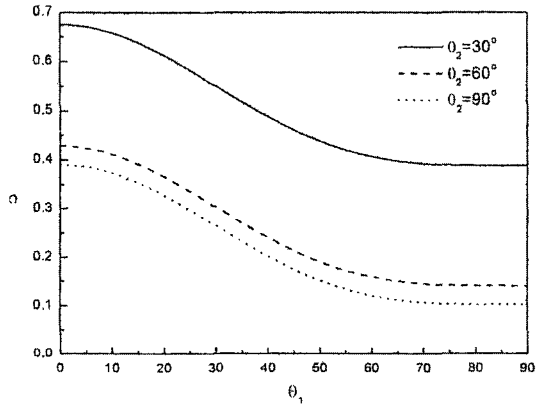


Fig. 4 Variation of δ versus fiber angles

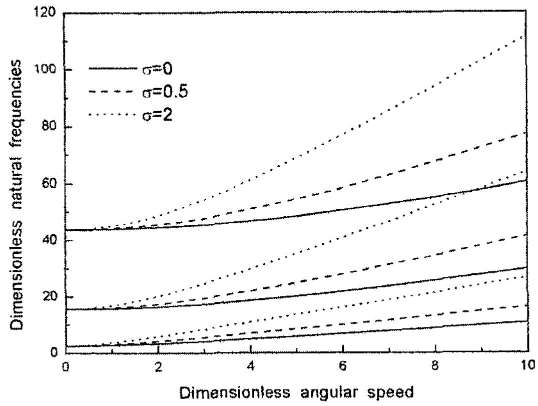
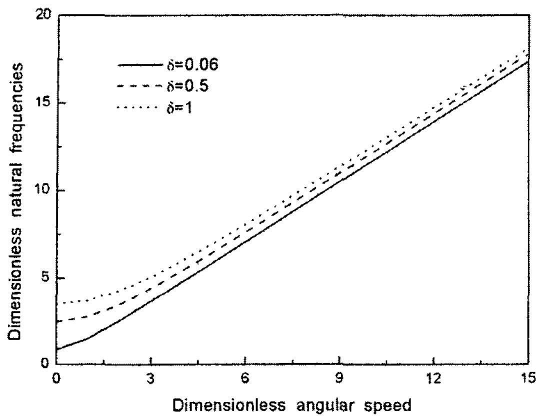


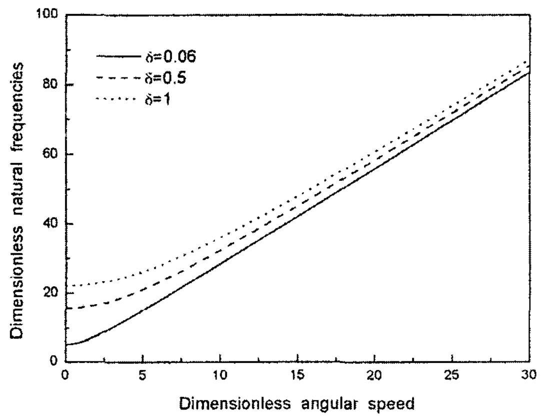
Fig. 5 Natural frequency variations versus angular speed

the results. As shown in the figure, the relative discrepancy among the three results is significant only in the lower angular speed range. As the angular speed increases the relative discrepancy becomes negligible.

Figure 7 shows the effect of δ on the lowest three natural frequencies versus the angular speed.



(a) First natural frequencies



(b) Second natural frequencies

Fig. 6 Natural frequency variations versus the angular speed for several fiber angle parameters

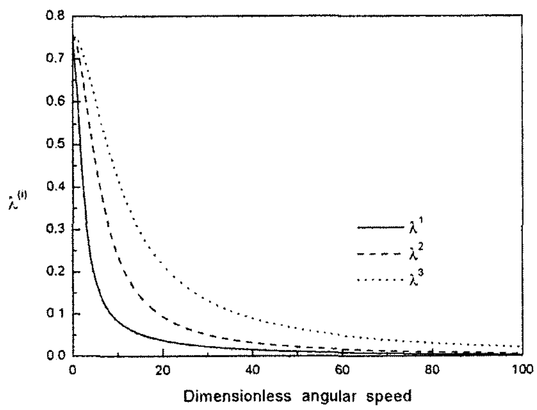
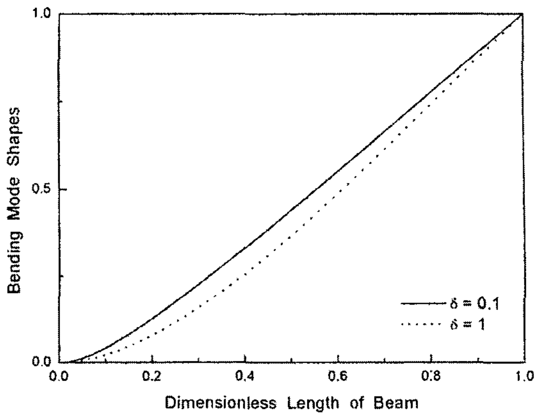
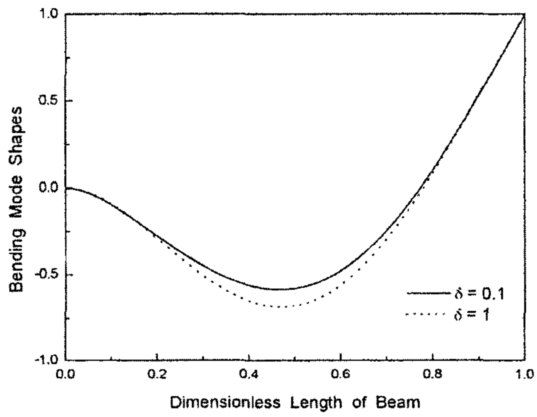


Fig. 7 Effect of fiber angle parameter δ on natural frequency variation versus the angular speed

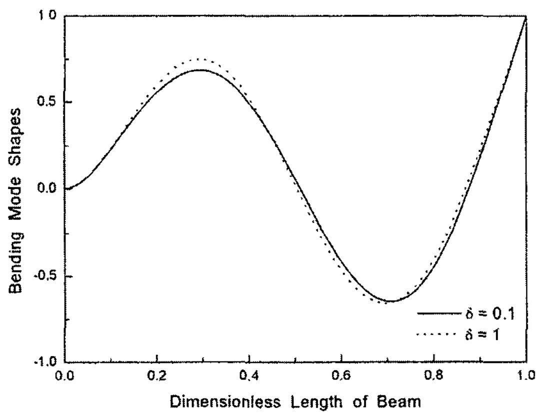
The variable $\lambda^{(i)}$ shown in the graph is defined as follows:



(a) First mode shape



(b) Second mode shape



(c) Third mode shape

Fig. 8 Comparison of mode shapes with different fiber angle parameters

$$\lambda^{(i)} \equiv (\omega_{\max}^{(i)} - \omega_{\min}^{(i)}) / \omega_{\max}^{(i)} \tag{18}$$

where $\omega_{\max}^{(i)}$ and $\omega_{\min}^{(i)}$ are i^{th} natural frequencies obtained with $\delta_{\max}$ and $\delta_{\min}$. As shown in the figure, $\lambda^{(i)}$ decrease fast as the angular speed increases. Thus, these results are consistent with the previous results of Fig. 7.

Figure 8 shows the variations of lowest two mode shapes with the conditions of $\gamma=5$ and $\sigma=0.1$. The solid lines represent the mode shapes of $\delta=0.1$ and the dotted lines represent the mode shapes of $\delta=1$. These results show that δ affects the mode shapes noticeably if not significantly.

4. Conclusion

In this paper, equations of motion for rotating composite cantilever beams are derived. Since the beam thickness is much smaller than the width and the layers have symmetric compositions, the coupling effects between stretching and bending motions are ignored and only flapwise bending motion is considered. Three dimensionless parameters are identified from the equations of motion and their effects on the modal characteristics are investigated. From numerical results, the following conclusions could be obtained. The natural frequencies increase as the angular speed increases. Furthermore, the increasing rates of the natural frequencies increase as the hub radius increases. It is found that the fiber orientation angle parameter δ noticeably affects the modal characteristics of the rotating composite cantilever beams especially in the lower angular speed range. However, the effect of δ on the modal characteristics decreases rapidly as the angular speed increases.

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