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Quotient semiring of a k-semiring by semiprimary k-fuzzy ideals

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Abstract

In this paper, we define and study the semiprimary $k-$fuzzy ideals in a commutative $k-$semiring and characterize the quotient semiring R/A of a $k-$semiring R by a semiprimary $k-$fuzzy ideal A. In particular, we show that every zero divisor of R/A is nilpotent.

Key Words : k-semiring, Semiprimary k-fuzzy ideal, Factor semiring, Extension fuzzy ideal.

1. Introduction

Chun, Kim and Kim [2] constructed an extension of a $k-$semiring and studied a $k-$ideal of a $k-$semiring. The first author et al.[3] constructed the quotient semiring of a $k-$semiring by a $k-$ideal. Liu [9] introduced and studied the notion of fuzzy ideal of a ring. Following Liu, Mukherjee and Sen [12] defined and examined fuzzy prime ideals of a ring. Kumbhojkar and Bapat [5,6] defined and studied the ring R/J of the cosets of the fuzzy ideal J.

Yue [14] introduced the concept of a primary $L-$fuzzy ideal and a prime $L-$ fuzzy ideal, and proved some fundamental propositions. Primary fuzzy ideals were further investigated by Malik and Mordeson[11].

Kumar [7] extended the concept of fuzzy ideal to fuzzy semiprimary ideals in a ring. Also Malik and Mordeson [10] gave the necessary and sufficient conditions for a fuzzy subring or a fuzzy ideal A of a commutative ring R to be extended to one A^e of a commutative ring S containing R as a subring.

Kim and Park [4] defined and studied the notion of the $k-$fuzzy ideal in a semiring, and they also introduced and studied the quotient semiring R/A of a $k-$semiring R by a $k-$fuzzy ideal A.

The purpose of this paper is to define and study the semiprimary $k-$fuzzy ideals in a commutative $k-$semiring and characterize the quotient semiring R/A of a $k-$semiring R by a semiprimary $k-$fuzzy ideal A. In particular, we show that every zero divisor of R/A is nilpotent.

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2. Preliminaries

In this section, we review some definitions and some results which will be used in the later section.

Definition 2-1[2]. A set R together with associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called a semiring provided:

(1) addition is a commutative operation,

(2) there exists $0\in R$ such that $x+0=x$ and $x0=0x=0$ for each $x\in R$,

and

(3) multiplication distributes over addition both from the left and the right.

Definition 2-2[2]. A semiring R will be called a $k-$semiring if for any $a, b\in R$ there exists a unique element c in R such that either $b=a+c$ or $a=b+c$ but not both.

Definition 2-3[3]. A non-empty subset I of a semiring R is called a subsemiring if I is itself a semiring with respect to the binary operations defined in R. A subsemiring I is called an ideal of R if $r\in R$, $a\in I$, imply ar and $ra\in I$.

Definition 2-4[3]. An ideal I of a semiring R is called a $k-$ideal if $r+a\in I$ implies $r\in I$ for each $r\in R$ and each $a\in I$.

Let R be a $k-$semiring. Let R' be a set of the same cardinality with $R-\{0\}$ such that $R\cap R'=\emptyset$ and let denote the image of $a\in R-\{0\}$ under a given bijection by a'. Let $\oplus$ and $\odot$ denote addition and multiplication respectively on a set $\overline{R}=R\cup R'$ as follows:

$$a\oplus b=\begin{cases} a+b & \text{if } a, b\in R \\ (x+y)' & \text{if } a=x', b=y'\in R' \\ c & \text{if } a\in R, b=y'\in R', a=y+c \\ c' & \text{if } a\in R, b=y'\in R', a+c=y, \end{cases}$$

where c is the unique element in R such that either $a=y+c$ or $a+c=y$ but not both, and

$$a\odot b=\begin{cases} ab & \text{if } a,b\in R \\ xy & \text{if } a=x',\ b=y'\in R' \\ (ay)' & \text{if } a\in R,\ b=y'\in R' \\ (xb)' & \text{if } a=x'\in R',\ b\in R, \end{cases}$$

It can be shown that these operations are well defined.

Theorem 2-5[2]. If R is a $k-$semiring, then $(\overline{R},\oplus,\odot)$ is a ring, called the extension ring of R.

Remark. Let $\ominus a$ denote the additive inverse of any element $a\in R$ and write $a\oplus(\ominus b)$ simply as $a\ominus b$. Then it is clear that $a'=\ominus a$ and $a=\ominus a'$ for all $a\in R$.

Note that if R is a $k-$semiring with identity, then $\overline{R}$ is a ring with identity.

Theorem 2-6[2]. Let R be a $k-$semiring, I an ideal, and $I'=\{a'\in R'|a\in I\}$. Then I is a $k-$ideal of R if and only if $\overline{I}=I\cup I'$ is an ideal of the extension ring $\overline{R}$, called the extension ideal of I.

Note that if R is a $k-$semiring and $\overline{R}$ is the extension ring of R, then each ideal of $\overline{R}$ is the extension ideal of a $k-$ideal of R and each $k-$ideal of R is the intersection of its extension ideal and R (see [2]).

Let R be a $k-$semiring and $\overline{R}$ its extension ring. Let I be a $k-$ideal of R and $\overline{I}$ its extension ideal of $\overline{R}$. Define a relation $a\equiv b$ by $a\oplus b'\in\overline{I}$, where $a,b\in R$. Then this relation is an equivalence relation on R. Let $a\oplus I$ be the equivalence class containing $a\in R$ determined by $\equiv$. Let $R/I=\{a\oplus I|a\in R\}$ be the set of all equivalence classes determined by $\equiv$.

Theorem 2-7[3]. Let I be a $k-$ideal of a $k-$semiring R and $\overline{I}$ its extension ideal of extension ring $\overline{R}$. Then $a\oplus I=(a+\overline{I})\cap R$, where $a\in R$.

Theorem 2-8[3]. Let I be a $k-$ideal of a $k-$semiring R. Then $R/I=\{a\oplus I|a\in R\}$ is a $k-$semiring under the two operations

$(a\oplus I)\oplus(b\oplus I)=(a+b)\oplus I$ and $(a\oplus I)\odot(b\oplus I)=(ab)\oplus I$.

Definition 2-9[4]. A fuzzy ideal of a semiring R is a function $A: R\rightarrow[0,1]$ satisfying the following conditions:

(1) $A(x+y)\geq\min\{A(x),\ A(y)\}$

(2) $A(xy)\geq\max\{A(x),A(y)\}$ for all $x,y\in R$

Lemma 2-10[4]. Let A be a fuzzy ideal of a semiring R. Then $A(x)\leq A(0)$ for all $x\in R$.

Definition 2-11[4]. Let A be a fuzzy ideal of a semiring R. Then A is called a $k-$fuzzy ideal of R if $A(x+y)=A(0)$ and $A(y)=A(0)$ imply $A(x)=A(0)$.

Definition 2-12[4]. Let A be a fuzzy subset of a semiring R. Then the set

$A_t=\{x\in R|A(x)\geq t\}$ $(t\in[0,1])$ is called the level subset of R with respect to A.

Theorem 2-13[4]. Let A be a fuzzy ideal of a semiring R. Then the level set $A_t\,(t\leq A(0))$ is the ideal of R.

Theorem 2-14[4]. Let A be a fuzzy ideal of a semiring R. If A_t is $k-$ideal of R for each $t(\leq A(0))$, then A is $k-$fuzzy ideal of R.

However, the converse of Theorem 2-14 does not hold by the following example.

Let $R=Z^*$, the set of nonnegative integers. Define a fuzzy subset A of R by

$$A(x)=\begin{cases} 1 & \text{if } x\in(2) \\ \frac{1}{2} & \text{if } x\in(2,3)-(2) \\ 0 & \text{if } x\in Z^*-(2,3). \end{cases}$$

Then A is a $k-$fuzzy ideal but $A_{\frac{1}{2}}=\{x\in Z^*|A(x)\geq\frac{1}{2}\}=(2,3)$ is not $k-$ideal of R.

However, if A is a $k-$fuzzy ideal of R, then $A_R=\{x\in R|A(x)=A(0)\}$ is also $k-$ideal of R.

In general, It is not true that if A is a fuzzy ideal of a semiring R, then $A_t\,(t\leq A(0))$ is $k-$ideal of R, for we have the following example.

Example 2-15[4]. Let $R=Z^*$, the set of nonnegative integers and let $I=(2,3)$ be an ideal of R generated by 2 and 3. Define a fuzzy subset A of R by

$$A(x)=\begin{cases} 1 & \text{if } x\in I \\ 0 & \text{if } x\notin I \end{cases}$$

Then A is a fuzzy ideal but $A_R=I$ is not $k-$ideal of R.

Definition 2-16[9]. A fuzzy ideal of a ring R is a function $A: R\rightarrow[0,1]$ satisfying the following axioms

(1) $A(x+y)\geq\min\{A(x),\ A(y)\}$

(2) $A(xy) \geq \max\{A(x), A(y)\}$
(3) $A(-x) = A(x)$

Let R be a commutative k-semiring, $\overline{R}$ its extension ring. If A is a fuzzy ideal of R such that all its level subsets are k-ideals of R, then $R = \bigcup_{t \in ImA} A_t$, $\overline{R} = \bigcup_{t \in ImA} \overline{A_t}$ and $s > t$ if and only if $A_s \subset A_t$ if and only if $\overline{A_s} \subset \overline{A_t}$. Thus we have the following theorem.

Theorem 2-17[4]. Let R be a commutative k-semiring, $\overline{R}$ its extension ring. Let A be a fuzzy ideal of R such that all its level subsets are k-ideals of R. Define the fuzzy subset $\overline{A}$ of $\overline{R}$ by for all $x \in \overline{R}$, $\overline{A}(x) = \sup\{t \mid x \in \overline{A}_t, t \in ImA\}$. Then $\overline{A}$ is a fuzzy ideal of $\overline{R}$.

Theorem 2-18[4]. Let A be as in Theorem 2-17. Then $\overline{A}$ is an extension of A.

Definition 2-19[4]. Let A be as in Theorem 2-17 and let $\overline{A}$ be an extension ideal of A The fuzzy subset $x + A : R \rightarrow [0,1]$ defined by $(x+A)(z) = \overline{A}(z \oplus x')$ is called a coset of the fuzzy ideal A.

Theorem 2-20[4]. Let $R, \overline{R}, A$ and $\overline{A}$ be as in Theorem 2-17. Then $x + A = y + A$ $(x, y \in R)$ if and only if $\overline{A}(x \oplus y') = A(0)$.

Theorem 2-21[4]. Let A be as in Theorem 2-17 and $\overline{A}$ an extension of A. If $x + A = u + A$ and $y + A = v + A$ $(x, y, u, v \in R)$, then
(1) $x + y + A = u + v + A$
(2) $xy + A = uv + A$

Theorem 2-21 allows us to define two binary operation "+" and " · " on the set R/A of cosets of the fuzzy ideal A as follows:

$$(x+A) + (y+A) = x + y + A$$

and

$$(x+A) \cdot (y+A) = xy + A$$

It is easy to show that R/A is a semiring under these well-defined binary operations with additive identity A and multiplicative identity $1 + A$. In this case, the semiring R/A is called the factor semiring or the quotient semiring of R by A.

Theorem 2-22[4]. Let A be as in Theorem 2-17 and $\overline{A}$ an extension of A. Then $R/A \simeq R/A_R$.

3. Main results

In this section, we have some properties of the quotient ring R/A of all fuzzy cosets of a semiprimary k-fuzzy ideal A of a commutative k-semiring R. Throughout this paper unless otherwise R is a commutative k-semiring with identity.

Definition 3-1. A fuzzy ideal A of a semiring R is called fuzzy semiprimary if for any fuzzy ideals B and C of R, the conditions $BC \subseteq A$ and $B(x) > A(x^m)$ for some $x \in R$ and for all $m \in N$ together imply that given $y \in R$, there exists $n \in N$ such that $C(y) \leq A(y^n)$ where N is the set of natural numbers.

This definition which is given in a semiring is similar to the Definition 3-1 of [7].

Theorem 3-2. An ideal I of a commutative semiring with identity R is semiprimary if and only if χ_I, the characteristic function of I is a fuzzy semiprimary ideal of R.

Proof. It is similar to the proof of Theorem 3-1 and Theorem 3-2 of [7].

Theorem 3-3. If R is a commutative semiring with identity and A is any semiprimary fuzzy ideal of R, then A_t, where $t \in Im(A)$ is a semiprimary ideal of R.

Proof. It is similar to the proof of Theorem 3-4 of [7].

However, the converse of this theorem is not true by the following example.

Example 3-4. Let Z^* denote the semiring of nonnegative integers. Define a fuzzy subset $A : Z^* \rightarrow [0,1]$ by

$$A(x) = \begin{cases} t_0 & \text{if } x \in \langle 16 \rangle, \\ t_1 & \text{if } x \in \langle 8 \rangle - \langle 16 \rangle, \\ t_2 & \text{if } x \in \langle 4 \rangle - \langle 8 \rangle, \\ t_3 & \text{if } x \in Z^* - \langle 4 \rangle, \end{cases}$$

where $1 > t_0 > t_1 > t_2 > t_3 > 0$.

Then A is a fuzzy ideal of Z^* and the chain of level ideals of A is given by $\langle 16 \rangle < \langle 8 \rangle < \langle 4 \rangle < Z^*$

Define fuzzy subsets B and $C : Z^* \rightarrow [0,1]$ by

$$B(x) = \begin{cases} \beta & \text{if } x \in \langle 16 \rangle, \ where\ t_0 < \beta < 1 \\ 0 & \text{if } x \in Z^* - \langle 16 \rangle \end{cases}$$

and $C(x)=t_0$ for all $x\in Z^*$.

Clearly B and C are fuzzy ideals of Z^* and

$$(B\cap C)(x)=\begin{cases} t_0 & \text{if } x\in\langle 16\rangle \\ 0 & \text{otherwise}\end{cases}$$

Further $BC\leq B\cap C\leq A$.

But $B(16)=\beta> t_0=A(16^m)$ for all $m\in N$ and $C(5)=t_0> t_3=A(5^n)$ for all $n\in N$. It follows that A is not fuzzy semiprimary. However, each level ideal of A is semiprimary.

Lemma 3-5. Let A be as in Theorem 2-17. If $A(x)<A(y)$ for some $x,y\in R$, then $\overline{A}(x\oplus y')=A(x)=\overline{A}(y\oplus x')$.

Proof. It is clear that $\overline{A}(x\oplus y')=\overline{A}(y\oplus x')$. We must show that $\overline{A}(x\oplus y')=A(x)$.

$$\begin{aligned} A(x)=\overline{A}(x)&=\overline{A}(x\oplus y'\oplus y)\\ &\geq \min\{\overline{A}(x\oplus y'),\ \overline{A}(y)\}\\ &=\overline{A}(x\oplus y'),\end{aligned}$$

since $\overline{A}(x)<\overline{A}(y)$ and

$$\begin{aligned}\overline{A}(x\oplus y')&\geq \min\{\overline{A}(x),\ \overline{A}(y)\}\\ &=\overline{A}(x)=A(x).\end{aligned}$$

Thus $A(x)=\overline{A}(x\oplus y')$.

Theorem 3-6. Let R, $\overline{R}$, A and $\overline{A}$ be as in Theorem 2-17. If A is a semiprimary fuzzy ideal of R, then every zero divisor of R/A is nipotent.

Proof. Let $x+A$ be any zero divisor of R/A. Then there exists nonzero fuzzy coset $y+A$ in R/A such that $(x+A)(y+A)=A$. So $A(xy)=A(0)$ by Theorem 2-20. Since A is fuzzy semiprimary, A_R is semiprimary by Theorem 3-3. Thus $x^m\in A_R$ or $y^n\in A_R$ for some $m,n\in N$. Let $x^m\in A_R$, then $A(x^m)=A(0)$. So $A(r)\leq A(0)=A(x^m)$ for all $r\in R$. If $A(r)<A(x^m)$ for some $r\in R$, then by Lemma 3-5, $\overline{A}(r\oplus(x^m)')=A(r)$. If $A(r)=A(0)=A(x^m)$, then $x^m, r\in \overline{A}_{\overline{R}}$ and so $\overline{A}(r\oplus(x^m)')=\overline{A}(0)=A(0)=A(r)$. Thus in any event, it is shown that $\overline{A}(r\oplus(x^m)')=A(r)$ for all $r\in R$. Since $\overline{A}(r\oplus(x^m)')=(x^m+A)(r)$, $(x^m+A)(r)=A(r)$ for all $r\in R$. Hence $x+A$ is nilpotent. In case $y^n\in A_R$ for some $n\in N$, then a similar argument will confirm that $y+A$ is nilpotent.

Theorem 3-7. Let A be as in Theorem 2 17. and R/A the semiring of all fuzzy cosets of A in R. If every zero divisor of R/A is nilpotent, then the ideal A_R is a semiprimary k-ideal of R.

Proof. Let $xy\in A_R$. Then $A(xy)=A(0)$. By Theorem 2-20, $xy+A=A$. By hypothesis, $(x+A)^m=A$ or $(y+A)^n=A$ for some $m,n\in N$. Thus $A(x^m)=A(0)$ or $A(y^n)=A(0)$ for some $m,n\in N$. Hence A_R is semiprimary and A_R is k-ideal by the condition of A. This completes the proof.

From Theorem 2-14, Theorem 2-22, Theorem 3-3, Theorem 3-6 and Theorem 3-7, we have the following result.

Corollary 3-8. Let A be a semiprimary fuzzy k-ideal as in Theorem 2-17.

Then the following statements are equivalent.

(1) A_R is a semiprimary k-ideal of R.

(2) every zero divisor of R/A is nilpotent.

(3) every zero divisor of R/A_R is nilpotent.

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