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GENERALIZATION OF WHIPPLE'S THEOREM FOR DOUBLE SERIES

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Abstract. In 1965, Bhatt and Pandey have obtained an analogue of the Whipple's theorem for double series by using Watson's theorem on the sum of a ${}_3F_2$. The aim of this paper is to derive twenty five results for double series closely related to the analogue of the Whipple's theorem for double series obtained by Bhatt and Pandey. The results are derived with the help of twenty five summation formulas closely related to the Watson's theorem on the sum of a ${}_3F_2$ obtained recently by Lavoie, Grondin, and Rathie.

1. Introduction

The generalized Kampé de Fériet function introduced by Srivastava and Panda [5] is defined and represented in the following manner:

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k)\\(\alpha_l):(\beta_m);(\gamma_n)\end{matrix}\middle| x,y\right] \tag{1.1}$$
$$=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}$$

where, for convergence,

(i) $p+q<l+m+1$, $p+k<l+n+1$, $|x|<\infty$, $|y|<\infty$, or

(ii) $p+q=l+m+1$, $p+q=l+n+1$, and

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$$\begin{cases} |x|^{\frac{1}{p-l}} + |y|^{\frac{1}{p-l}} < 1, & \text{if } p > l \\ \max\{|x|, |y|\} < 1, & \text{if } p \leq l. \end{cases}$$

(1.1) reduces to the Kampé de Fériet function [3] in the special case $q = k$ and $m = n$.

In 1965, Bhatt and Pandey [2] obtained the Whipple's theorem for double series by using Watson's theorem on the sum of a ${}_3F_2$ with unit argument. Their result is given by

$$\begin{aligned} & F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, 1-a_1 : b_1; b_1' \\ c_1, 1+2b_1+2b_1'-c_1 : -; - \end{matrix}\middle| 1, 1\right] \\ (1.2) \qquad & = \frac{2^{1-2b_1-2b_1'}\Gamma(c_1)\Gamma(1+2b_1+2b_1'-c_1)}{\Gamma(\frac{1}{2}+\frac{1}{2}c_1-a_1)\Gamma(\frac{1}{2}a_1+\frac{1}{2}c_1)\Gamma(1+b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1)} \\ & \times \frac{1}{\Gamma(\frac{1}{2}-\frac{1}{2}c_1+\frac{1}{2}a_1+b_1+b_1')}. \end{aligned}$$

In 1992, Lavoie, Grondin and Rathie [4] generalized the Watson's theorem which is given in the next section and they have obtained twenty five results closely related to the Watson's theorem. In the same paper, they have also obtained a large number of special cases of their results.

The aim of this paper is to derive twenty five results for the double series closely related to the Whipple's theorem for the double series obtained by Bhatt and Pandey (See (1.2)). A few interesting special cases have also been considered.

2. Results required

The following result will be required in our present investigation.

Transformation formula [1]:

$$\begin{aligned} & {}_3F_2\left[\begin{matrix} a, b, c \\ e, f \end{matrix}\middle| 1\right] \\ (2.1) \qquad & = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)} {}_3F_2\left[\begin{matrix} e-a, f-a, s \\ s+b, s+c \end{matrix}\middle| 1\right], \end{aligned}$$

where $s = e + f - a - b - c$ and $\Re e(s) > 0$.

Generalized Watson's theorem [4]:

(2.2)

$$
{}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1}{2}(a+b+i+1), 2c+j \end{matrix}\middle| 1\right]
= A_{i,j}\frac{2^{a+b+i-2}\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2})\Gamma(c+[\frac{j}{2}]+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(a)}
\times\frac{\Gamma(c-\frac{1}{2}a-\frac{1}{2}b-\frac{|i+j|}{2}+\frac{j}{2}+\frac{1}{2})}{\Gamma(b)}
$$
$$
\times\left\{B_{i,j}\frac{\Gamma\left(\frac{1}{2}a+\frac{1-(-1)^i}{4}\right)\Gamma(\frac{1}{2}b)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}+[\frac{j}{2}]-\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma(c-\frac{1}{2}b+\frac{1}{2}+[\frac{j}{2}])}
+C_{i,j}\frac{\Gamma(\frac{1}{2}a+\frac{1+(-1)^i}{4})\Gamma(\frac{1}{2}b+\frac{1}{2})}{\Gamma\left(c-\frac{1}{2}a+[\frac{j+1}{2}]+\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma(c-\frac{1}{2}b+[\frac{j+1}{2}])}\right\}
$$

for $i, j = 0, \pm1, \pm2$. $[x]$ is the greatest integer less than or equal to x and its absolute value is denoted by $|x|$. The coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are recorded in the tables at the end of the paper.

3. Main summation formulae

The results to be established are

$$
F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, 1-a_1+i+j : b_1; b'_1 \\ c_1, 1+2b_1+2b'_1-c_1+i : -; - \end{matrix}\middle| 1,1\right]
= A_{i,j}\frac{2^{b_1+2b'_1-2a_1+2i-1}\Gamma(c_1)\Gamma(b_1+b'_1-j)\Gamma(1+2b_1+2b'_1-c_1+i)}{\Gamma(a_1)\Gamma(\frac{1}{2})\Gamma(2b_1+2b'_1-j)\Gamma(c_1-a_1)}
$$
$$
\times\frac{\Gamma(b_1+b'_1-j+[\frac{j}{2}]+\frac{1}{2})\Gamma(a_1-\frac{1}{2}i-\frac{1}{2}j-\frac{|i+j|}{2})}{\Gamma(1+2b_1+2b'_1-c_1-a_1+i)}
$$
$$
\times\left\{B_{i,j}\frac{\Gamma\left(\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1-(-1)^i}{4}\right)}{\Gamma\left(b_1+b'_1-j-\frac{1}{2}c_1+\frac{1}{2}a_1+\frac{1}{2}+[\frac{j}{2}]-\frac{(-1)^j}{4}(1-(-1)^i)\right)}\right. \tag{3.1}
$$
$$
\times\frac{\Gamma(\frac{1}{2}+b_1+b'_1-\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2}i)}{\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1-\frac{1}{2}i-j+[\frac{j}{2}])}
$$

$$+ C_{i,j} \frac{\Gamma(\frac{1}{2}c_1 - \frac{1}{2}a_1 + \frac{1+(-1)^i}{4})}{\Gamma\left(b_1 + b_1' - j - \frac{c_1}{2} + \frac{a_1}{2} + [\frac{j+1}{2}] + \frac{(-1)^j}{4}(1-(-1)^i)\right)}$$

$$\left. \times \frac{\Gamma(b_1 + b_1' - \frac{1}{2}c_1 - \frac{1}{2}a_1 + \frac{1}{2}i + 1)}{\Gamma(\frac{c_1}{2} + \frac{a_1}{2} - \frac{i}{2} - j - \frac{1}{2} + [\frac{j+1}{2}])} \right\}$$

for $i, j = 0, \pm 1, \pm 2$.

The coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ can be obtained from the tables of A_{ij}, B_{ij} and $C_{i,j}$ given at the end of this paper by simply changing a to $c_1 - a_1$, b to $1 + 2b_1 + 2b_1' - c_1 - a_1 + i$ and c to $b_1 + b_1' - j$, respectively.

4. Proof

In order to prove (3.1), by noting $(\alpha)_{m+n} = (\alpha + m)_n (\alpha)_m$ and starting with (1.1), we have

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, a_2; b_1, b_1' \\ c_1, c_2; -, - \end{matrix}\middle| 1, 1\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n}(b_1)_m(b_1')_n}{(c_1)_{m+n}(c_2)_{m+n}m!n!}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1+m)_n(a_1)_m(a_2+m)_n(a_2)_m(b_1)_m(b_1')_n}{(c_1+m)_n(c_1)_m(c_2+m)_n(c_2)_m m!n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m(b_1)_m}{(c_1)_m(c_2)_m m!} \sum_{n=0}^{\infty} \frac{(a_1+m)_n(a_2+m)_n(b_1')_n}{(c_1+m)_n(c_2+m)_n n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m(b_1)_m}{(c_1)_m(c_2)_m m!} {}_3F_2\left[\begin{matrix} a_1+m, a_2+m, b_1' \\ c_1+m, c_2+m \end{matrix}\middle| 1\right].$$

Now by using (2.1) and taking $a = a_1 + m$, $b = a_2 + m$, $c = b_1'$, $e = c_1 + m$ and $f = c_2 + m$ and observing that $s = c_1 + c_2 - a_1 - a_2 - b_1'$, $s + b = c_1 + c_2 - a_1 - b_1' + m$, $s + c = c_1 + c_2 - a_1 - a_2$, $e - a = c_1 - a_1$ and $f - a = c_2 - a_1$, we get

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m(b_1)_m}{(c_1)_m(c_2)_m m!} \cdot \frac{\Gamma(c_1+m)\Gamma(c_2+m)}{\Gamma(a_1+m)\Gamma(c_1+c_2-a_1-b_1'+m)}$$

$$\times \frac{\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(c_1+c_2-a_1-a_2)}$$

$$
\begin{aligned}
&\times {}_3F_2\left[\begin{matrix} c_1-a_1, c_2-a_1, c_1+c_2-a_1-a_2-b_1' \\ c_1+c_2-a_1-b_1'+m, c_1+c_2-a_1-a_2 \end{matrix}\Big| 1\right] \\
&= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b_1')\Gamma(c_1+c_2-a_1-a_2)} \\
&\times \sum_{m=0}^{\infty} \frac{(a_2)_m (b_1)_m}{m!(c_1+c_2-a_1-b_1')_m} {}_3F_2\left[\begin{matrix} c_1-a_1, c_2-a_1, c_1+c_2-a_1-a_2-b_1' \\ c_1+c_2-a_1-b_1'+m, c_1+c_2-a_1-a_2 \end{matrix}\Big| 1\right] \\
&= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b_1')\Gamma(c_1+c_2-a_1-a_2)} \\
&\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_2)_m (b_1)_m}{m!(c_1+c_2-a_1-b_1')_m} \cdot \frac{(c_1-a_1)_n (c_2-a_1)_n}{(c_1+c_2-a_1-b_1'+m)_n} \\
&\times \frac{(c_1+c_2-a_1-a_2-b_1')_n}{(c_1+c_2-a_1-a_2)_n n!}.
\end{aligned}
$$

But it is easy to see that

$$
\begin{aligned}
&(c_1+c_2-a_1-b_1'+m)_n (c_1+c_2-a_1-b_1')_m \\
=&(c_1+c_2-a_1-b_1'+n)_m (c_1+c_2-a_1-b_1')_n.
\end{aligned}
$$

$$
\begin{aligned}
&F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, a_2 & : b_1, b_1' \\ c_1, c_2 & : -, - \end{matrix}\Big| 1, 1\right] \\
&= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b_1')\Gamma(c_1+c_2-a_1-a_2)} \\
&\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_2)_m (b_1)_m}{m!(c_1+c_2-a_1-b_1'+n)_m} \cdot \frac{(c_1-a_1)_n (c_2-a_1)_n}{(c_1+c_2-a_1-b_1')_n} \\
&\times \frac{(c_1+c_2-a_1-a_2-b_1')_n}{(c_1+c_2-a_1-a_2)_n n!} \\
&= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b_1')\Gamma(c_1+c_2-a_1-a_2)} \\
&\times \sum_{n=0}^{\infty} \frac{(c_1-a_1)_n (c_2-a_1)_n (c_1+c_2-a_1-a_2-b_1')_n}{(c_1+c_2-a_1-b_1')_n (c_1+c_2-a_1-a_2)_n n!} \\
&\sum_{m=0}^{\infty} \frac{(a_2)_m (b_1)_m}{(c_1+c_2-a_1-b_1'+n)_m m!}
\end{aligned}
$$

$$= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b'_1)}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b'_1)\Gamma(c_1+c_2-a_1-a_2)}$$
$$\times \sum_{n=0}^{\infty} \frac{(c_1-a_1)_n(c_2-a_1)_n(c_1+c_2-a_1-a_2-b'_1)_n}{(c_1+c_2-a_1-b'_1)_n(c_1+c_2-a_1-a_2)_n n!}$$
$$\times {}_2F_1\left[\begin{matrix} a_2, b_1 \\ c_1+c_2-a_1-b'_1+n \end{matrix}\middle| 1\right].$$

Using the well-known Gauss's theorem [1], we get after a little simplification

$$= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b'_1)}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b'_1)\Gamma(c_1+c_2-a_1-a_2)}$$
$$\times \sum_{n=0}^{\infty} \frac{(c_1-a_1)_n(c_2-a_1)_n(c_1+c_2-a_1-a_2-b'_1)_n}{(c_1+c_2-a_1-b'_1)_n(c_1+c_2-a_1-a_2)_n n!}$$
$$\times \frac{\Gamma(c_1+c_2-a_1-b'_1+n)\Gamma(c_1+c_2-a_1-a_2-b_1-b'_1+n)}{\Gamma(c_1+c_2-a_1-b_1-b'_1+n)\Gamma(c_1+c_2-a_1-a_2-b'_1+n)}$$
$$= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b'_1)}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-a_2)\Gamma(c_1+c_2-a_1-b_1-b'_1)}$$
$$\times \sum_{n=0}^{\infty} \frac{(c_1-a_1)_n(c_2-a_1)_n(c_1+c_2-a_1-a_2-b_1-b'_1)_n}{(c_1+c_2-a_1-a_2)_n(c_1+c_2-a_1-b_1-b'_1)_n n!}$$
$$= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1-b'_1)}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-a_2)\Gamma(c_1+c_2-a_1-b_1-b'_1)}$$
$$\times {}_3F_2\left[\begin{matrix} c_1-a_1, c_2-a_1, c_1+c_2-a_1-a_2-b_1-b'_1 \\ c_1+c_2-a_1-a_2, c_1+c_2-a_1-b_1-b'_1 \end{matrix}\middle| 1\right].$$

Now, here, if we take $a_2 = 1-a_1+i+j$ and $c_2 = 1+2b_1+2b'_1-c_1+i$, then for $i, j = 0, \pm1, \pm2$, we have

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, 1+a_1+i+j : b_1; b'_1 \\ c_1, 1+2b_1+2b'_1-c_1+i : -; - \end{matrix}\middle| 1, 1\right]$$
$$= \frac{\Gamma(c_1)\Gamma(1+2b_1+2b'_1-c_1+i)\Gamma(b_1+b'_1-j)}{\Gamma(a_1)\Gamma(2b_1+2b'_1-j)\Gamma(1+b_1+b'_1-a_1+i)}$$
$$\times {}_3F_2\left[\begin{matrix} c_1-a_1, 1+2b_1+2b'_1-c_1-a_1+i, b_1+b'_1-j \\ 1+b_1+b'_1-a_1+i, 2b_1+2b'_1-j \end{matrix}\middle| 1\right].$$

The right hand side can now be summed up by generalized Watson's theorem (2.2) by taking $a = c_1 - a_1$, $b = 1 + 2b_1 + 2b_1' - c_1 - a_1 + i$ and $c = b_1 + b_1' - j$, and we get after some simplification

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, 1 - a_1 + i + j : b_1; b_1' \\ c_1, 1 + 2b_1 + 2b_1' - c_1 + i : -; - \end{matrix}\middle| 1, 1\right]$$

$$= A_{i,j}\frac{2^{2b_1+2b_1'-2a_1+2i-1}\Gamma(c_1)\Gamma(b_1+b_1'-j)\Gamma(1+2b_1+2b_1'-c_1+i)}{\Gamma(a_1)\Gamma(\frac{1}{2})\Gamma(2b_1+2b_1'-j)\Gamma(c_1-a_1)}$$

$$\times\frac{\Gamma(b_1+b_1'-j+\left[\frac{j}{2}\right]+\frac{1}{2})\Gamma(a_1-\frac{1}{2}i-\frac{1}{2}j-\frac{|i+j|}{2})}{\Gamma(1+2b_1+2b_1'-c_1-a_1+i)}$$

$$\times\left\{B_{i,j}\frac{\Gamma\left(\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1-(-1)^i}{4}\right)}{\Gamma\left(b_1+b_1'-j-\frac{1}{2}c_1+\frac{1}{2}a_1+\frac{1}{2}+\left[\frac{j}{2}\right]-\frac{(-1)^j}{4}(1-(-1)^i)\right)}\right.$$

$$\times\frac{\Gamma(\frac{1}{2}+b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2}i)}{\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1-\frac{1}{2}i-j+\left[\frac{j}{2}\right])}$$

$$+C_{i,j}\frac{\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1+(-1)^i}{4})}{\Gamma\left(b_1+b_1'-j-\frac{c_1}{2}+\frac{a_1}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j}{4}(1-(-1)^i)\right)}$$

$$\left.\times\frac{\Gamma(b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2}i+1)}{\Gamma(\frac{c_1}{2}+\frac{a_1}{2}-\frac{i}{2}-j-\frac{1}{2}+\left[\frac{j+1}{2}\right])}\right\}$$

for $i, j = 0, \pm1, \pm2$. This completes the proof of (3.1).

5. Special cases

Some interesting special cases of our summation formula (3.1) are given below:

1. For $i = 0$, $j = 1$, we have

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, 2 - a_1 : b_1; b_1' \\ c_1, 1 + 2b_1 + 2b_1' - c_1 : -; - \end{matrix}\middle| 1, 1\right]$$

$$= \frac{2^{2b_1+2b_1'-2a_1-1}\Gamma(c_1)\Gamma(b_1+b_1'-1)\Gamma(1+2b_1+2b_1'-c_1)}{\Gamma(a_1)\Gamma(\frac{1}{2})\Gamma(2b_1+2b_1'-1)\Gamma(c_1-a_1)}$$

$$\times \frac{\Gamma(b_1+b_1'-\frac{1}{2})\Gamma(a_1-1)}{\Gamma(1+2b_1+2b_1'-c_1-a_1)} \tag{5.1}$$
$$\times \left\{ \frac{\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1)\Gamma(b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2})}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+\frac{1}{2}a_1-\frac{1}{2})\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1-1)} - \frac{\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2})\Gamma(b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1+1)}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+\frac{1}{2}a_1)\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1-\frac{1}{2})} \right\}.$$

2. For $i=1$, $j=0$, we have

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, 2-a_1 : b_1; b_1' \\ c_1, 2+2b_1+2b_1'-c_1 : -; - \end{matrix} \middle| 1,1\right]$$
$$= \frac{2^{2b_1+2b_1'-2a_1+1}\Gamma(c_1)\Gamma(b_1+b_1')\Gamma(2+2b_1+2b_1'-c_1)}{(a-b)\Gamma(a_1)\Gamma(\frac{1}{2})\Gamma(2b_1+2b_1')\Gamma(c_1-a_1)}$$
$$\times \frac{\Gamma(b_1+b_1'+\frac{1}{2})\Gamma(a_1-1)}{\Gamma(2+2b_1+2b_1'-c_1-a_1)} \tag{5.2}$$
$$\times \left\{ \frac{\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2})\Gamma(1+b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1)}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+\frac{1}{2}a_1)\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1-\frac{1}{2})} - \frac{\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1)\Gamma(b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{3}{2})}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+\frac{1}{2}a_1+\frac{1}{2})\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1-1)} \right\}.$$

3. For $i=1$, $j=-1$, we have

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, 1-a_1 : b_1; b_1' \\ c_1, 2+2b_1+2b_1'-c_1 : -; - \end{matrix} \middle| 1,1\right]$$
$$= \frac{2^{2b_1+2b_1'-2a_1+1}\Gamma(c_1)\Gamma(b_1+b_1'+1)\Gamma(2+2b_1+2b_1'-c_1)\Gamma(b_1+b_1'+\frac{1}{2})}{(a-b)\Gamma(\frac{1}{2})\Gamma(2b_1+2b_1'+1)\Gamma(c_1-a_1)\Gamma(2+2b_1+2b_1'-c_1-a_1)}$$
$$\times \left\{ \frac{\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2})\Gamma(1+b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1)}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+1+\frac{1}{2}a_1)\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1-\frac{1}{2})} - \frac{\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1)\Gamma(b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{3}{2})}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+\frac{1}{2}a_1+\frac{1}{2})\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1)} \right\}. \tag{5.3}$$

4. For $i=-1$, $j=1$, we have

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, 1-a_1 : b_1; b_1' \\ c_1, 2+2b_1+2b_1'-c_1 : -; - \end{matrix} \middle| 1,1\right]$$

$$= \frac{2^{2b_1+2b_1'-2a_1-3}\Gamma(c_1)\Gamma(b_1+b_1'-1)\Gamma(2b_1+2b_1'-c_1)\Gamma(b_1+b_1'-\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2b_1+2b_1'-1)\Gamma(c_1-a_1)\Gamma(2b_1+2b_1'-c_1-a_1)}$$

$$(5.4)\qquad \times\left\{\frac{\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2})\Gamma(b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1)}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+\frac{1}{2}a_1)\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1-\frac{1}{2})}\right.$$

$$\left.+\frac{\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1)\Gamma(b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2})}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+\frac{1}{2}a_1-\frac{1}{2})\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1)}\right\}.$$

5. For $i=0$, $j=-1$, we have

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1,-a_1: b_1; b_1' \\ c_1, 1+2b_1+2b_1'-c_1:-;-\end{matrix}\middle| 1,1\right]$$

$$= \frac{2^{2b_1+2b_1'-2a_1-1}\Gamma(c_1)\Gamma(b_1+b_1'+1)\Gamma(1+2b_1+2b_1'-c_1)\Gamma(b_1+b_1'+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2b_1+2b_1'+1)\Gamma(c_1-a_1)\Gamma(1+2b_1+2b_1'-c_1-a_1)}$$

(5.5)

$$\times\left\{\frac{\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1)\Gamma(\frac{1}{2}+b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1)}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+\frac{1}{2}a_1+\frac{1}{2})\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1)}\right.$$

$$\left.+\frac{\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2})\Gamma(b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1+1)}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+\frac{1}{2}a_1+1)\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1+\frac{1}{2})}\right\}.$$

6. For $i=-1$, $j=-1$, we have

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1,-1-a_1: b_1; b_1' \\ c_1, 2b_1+2b_1'-c_1:-;-\end{matrix}\middle| 1,1\right]$$

$$= \frac{2^{2b_1+2b_1'-2a_1-2}\Gamma(c_1)\Gamma(b_1+b_1'+1)\Gamma(2b_1+2b_1'-c_1)\Gamma(b_1+b_1'+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2b_1+2b_1'+1)\Gamma(c_1-a_1)\Gamma(2b_1+2b_1'-c_1-a_1)}$$

(5.6)

$$\times\left\{\frac{(2b_1+2b_1'-c_1)\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2})\Gamma(b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1)}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+\frac{1}{2}a_1+1)\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1+\frac{1}{2})}\right.$$

$$\left.+\frac{(c_1)\Gamma(\frac{1}{2}c_1-\frac{1}{2}a_1)\Gamma(b_1+b_1'-\frac{1}{2}c_1-\frac{1}{2}a_1+\frac{1}{2})}{\Gamma(b_1+b_1'-\frac{1}{2}c_1+\frac{1}{2}a_1+\frac{1}{2})\Gamma(\frac{1}{2}c_1+\frac{1}{2}a_1+1)}\right\}.$$

Similarly, other results can also be obtained.

Table for $A_{i,j}$

i \ j	-2	-1	0	1	2
2	$\frac{1}{2(c-1)(a-b-1)(a-b+1)}$	$\frac{1}{2(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{8(c+1)(a-b-1)(a-b+1)}$
1	$\frac{1}{(c-1)(a-b)}$	$\frac{1}{a-b}$	$\frac{1}{a-b}$	$\frac{1}{2(a-b)}$	$\frac{1}{2(c+1)(a-b)}$
0	$\frac{1}{2(c-1)}$	1	1	1	$\frac{1}{2(c+1)}$
-1	$\frac{1}{(c-1)}$	1	2	2	$\frac{2}{(c+1)}$
-2	$\frac{1}{2(c-1)}$	1	1	2	$\frac{2}{c+1}$

Table for $B_{i,j}$

i \ j	-2	-1	0	1	2
2	$c(a+b-1)$ $-(a+1)(b+1)+2$	$a+b-1$ $+b(2c-b)-2c+1$	$a(2c-a)$ $-(a-b)^2+1$	$2c(a+b-1)$	$A_{2,2}$
1	$c-a-1$	1	1	$2c-a+b$ $-(a-b)(c-b+1)$	$2c(c+1)$
0	$(c-a-1)(c-b-1)$ $+(c-1)(c-2)$	1	1	1	$(c-a+1)(c-b+1)$ $+c(c+1)$
-1	$2(c-1)(c-2)$ $-(a-b)(c-b-1)$	$2c-a+b-2$	1	1	$c-b+1$
-2	$A_{-2,-2}$	$2(c-1)(a+b-1)$ $-(a-b)^2+1$	$a(2c-a)$ $+b(2c-b)-2c+1$	$a+b-1$	$c(a+b-1)$ $-(a-1)(b-1)$

$A_{2,2}=2c(c+1)\{(2c+1)(a+b-1)-a(a-1)-b(b-1)\}-(a-b-1)(a-b+1)\{(c+1)(2c-a-b+1)+ab\}$

$A_{-2,-2}=2(c-1)(c-2)\{(2c-1)(a+b-1)-a(a+1)-b(b+1)+2\}-(a-b-1)(a-b+1)\{(c-1)(2c-a-b-3)+ab\}$

Table for $C_{i,j}$

i \ j	-2	-1	0	1	2
2	-4	$-(4c-a-b-3)$	-8	$-(8c^2-2c(a+b-1)$ $-(a-b)^2+1)$	$-4(2c+a-b+1)$ $(2c-a+b+1)$
1	$-(c-a-1)$	-1	-1	$-(2c+a-b)$	$-\{2c(c+1)$ $+(a-b)(c-a+1)\}$
0	4	1	0	-1	-4
-1	$2(c-1)(c-2)$ $+(a-b)(c-a-1)$	$2c+a-b-2$	1	1	$c-a+1$
-2	$4(2c-a+b-3)$ $(2c+a-b-3)$	$8c^2-2(c-1)(a+b+7)$ $-(a-b)^2-7$	8	$4c-a-b+1$	4

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