Honam Mathematical J. **26**(2004), No. 4, pp. 411–421

DIGITAL TOPOLOGICAL PROPERTY OF THE DIGITAL 8-PSEUDOTORI

SIK LEE, SAM-TAE KIM AND SANG-EON HAN

Abstract. A digital (k_0, k_1)-homotopy is induced from digital (k_0, k_1)-continuity with the n kinds of k_i-adjacency relations in $\mathbb{Z}^n$, $i \in \{0,1\}$. The k-fundamental group, $\pi_1^k(X, x_0)$, is derived from the pointed digital k-homotopy, $k \in \{3^n - 1(n \geq 2), 3^n - \sum_{k=0}^{r-2} C_k^n 2^{n-k} - 1(2 \leq r \leq n-1(n \geq 3)), 2n(n \geq 1)\}$. In this paper two kinds of digital 8-pseudotori stemmed from the minimal simple closed 4-curve and the minimal simple closed 8-curve with 8-contractibility or without 8-contractibility, e.g., DT_8 and DT_8', are introduced and their digital topological properties are studied by the calculation of the k-fundamental groups, $k \in \{8, 32, 64, 80\}$.

1. Introduction

The concept of digital continuity and its various properties have been studied in wide variety practical applications to the research of digital curves and digital surfaces, which are related to the k-fundamental group constructed for the study of the digital topological properties of discrete spaces [1, 2, 4, 6, 8, 9, 10, 11, 12, 13, 14]. Further, a digital homeomorphism was studied for the classification of digital images(discrete spaces)

Received September 7,2004; Revised October 6,2004.

2000 Mathematics Subject Classification : 55Q70, 57M10.

Key words and phrases :digital k-neighborhood, k-contractibility, digital (k_0, k_1)-homotopy, k-fundamental group, digital (k_0, k_1)-homeomorphism, digital covering space, product space, digital 8-pseudotorus.

This study is supported by Yosu National University through research fund.

and was investigated with relation to the digital retract and the extension [1, 2]. Recently, n kinds of k-adjacency relations of digital images in $\mathbb{Z}^n$ were shown [6, 7]. Furthermore, the notion of computer continuity, which is different from digital continuity [5], was introduced. Basically, we follow the notion of the digital (k_0, k_1)-homotopy of [3] with the general k_i-adjacency relations, $i \in \{0,1\}$. A pointed digital k-homotopy leads to the k-fundamental group [1, 6, 9, 10]. And further, digital images can be classified in terms of their k-fundamental groups. Moreover, the reasonable k_3-adjacency relation of the product space $X \times Y$ is given, which is compatible with the given spaces (X, k_1) and (Y, k_2) [10]. Thus three types of minimal simple closed curves in $\mathbb{Z}^2$, MSC_4, MSC_8 and MSC_8' [1, 2, 3, 6, 8, 9, 10], give rise to many types of minimal 8-pseudotori with 8-adjacency in $\mathbb{Z}^4$, such as $DT_8 = MSC_4 \times MSC_8$ and $DT_8' = MSC_4 \times MSC_8'$. Actually, the digital topological properties of $MSC_8' \times MSC_8'$ and $MSC_8 \times MSC_8'$ was studied [8]. Further, the concept of a digital (k_0, k_1)-covering was originally established [7, 10] which plays an important role in calculating the k_i-fundamental group, $i \in \{0,1\}$. And product properties of digital covering maps were studied [10]. Further, in the current paper we find digital topological properties of the spaces DT_8 and DT_8' with relation to each of their k-fundamental groups, $k \in \{8, 32, 64, 80\}$.

2. Preliminary and notation

Let $\mathbb{Z}$ be the set of integers. In $\mathbb{Z}^2$, each digital image is considered with k-adjacency, $k \in \{4,8\}$ and in $\mathbb{Z}^3$ a digital image is also assumed with k-adjacency, $k \in \{26, 18, 6\}$ [1, 2, 8, 9, 10, 11, 12, 13, 14]. For $\{a,b\} \subset \mathbb{Z}$ with $a \leq b$, $[a,b]_{\mathbb{Z}} = \{a \leq n \leq b\}$ is called a *digital interval* [1, 9]. As a generalization of the commonly used k-adjacency relations of $\mathbb{Z}^2$, and $\mathbb{Z}^3$, the n types of adjacency relations in $\mathbb{Z}^n$ can be established as follows [6, 9, 10]:

Proposition 2.1[7, 9, 10]. There are n kinds of k-adjacency relations in $\mathbb{Z}^n$, $k \in \{3^n - 1(n \geq 2), 3^n - \sum_{k=0}^{r-2} C_k^n 2^{n-k} - 1(2 \leq r \leq n-1(n \geq 3)), 2n(n \geq 1)\}$, where C_t^n stands for the combination of n objects taken t at a time.

Hereafter, the digital image X with k-adjacency is considered in a digital picture $(\mathbb{Z}^n, k, 2n, X) \in \{(\mathbb{Z}^n, k, 2n, X), (\mathbb{Z}^n, 2n, 3^n - 1, X)\}$ with one of the general k-adjacency relations from Proposition 2.1. For example, in $\mathbb{Z}^4$, a digital picture $(\mathbb{Z}^4, k, 2n, X)$ is considered with one of the following cases: $k \in \{80, 64, 32, 8\}$ [3, 9].

For a digital image $X \subset \mathbb{Z}^n$, two points $x, y \in X$ with $x \neq y$ are called k-connected [1, 10] if there is a k-path $f : [0, m]_{\mathbb{Z}} \to X$ which the image is a sequence $(x_0, x_1, \cdots, x_m)$ from the set of points $\{f(0) = x_0 = x, f(1) = x_1, \cdots, f(m) = x_m = y\}$ such that x_i and x_{i+1} are k-adjacent, $i \in \{0, 1, \cdots, m-1\}$ and $m \geq 1$. The number m above is called the length of a k-path [6, 7, 9]. And a *simple k-curve* is considered as a sequence $(x_0, x_1, \cdots, x_m)$ of the image of the k-path which x_i and x_j are k-adjacent if and only if $j = i + 1(\text{mod}m)$ or $i = j - 1(\text{mod}m)$ [1, 2, 3, 9].

Definition 2.2[7, 9, 10]. Let (X, k) be a digital image in $\mathbb{Z}^n$ and $\varepsilon \in \mathbb{N}$. The digital k-neighborhood of $x_0 \in X$ with radius ε is the set

$$N_k(x_0, \varepsilon) = \{x \in X | \quad l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\},$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x in X. □

Now we need to restate the digital (k_0, k_1)-continuity of [4, 5, 6, 7, 8] with relation to its local property with the general k_i-adjacency relations, $i \in \{0, 1\}$. The following definition characterizes digital continuity in a fashion used later in the paper.

Proposition 2.3[7, 9, 10]. Let (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$ be digital images. A function $f : X \to Y$ is said to be digitally (k_0, k_1)-continuous if and only if for every $x_0 \in X, \varepsilon \in \mathbb{N}$, and $N_{k_1}(f(x_0), \varepsilon) \subset Y$, there is $\delta \in \mathbb{N}$ such that the corresponding $N_{k_0}(x_0, \delta) \subset X$ satisfies $f(N_{k_0}(x_0, \delta)) \subset N_{k_1}(f(x_0), \varepsilon)$. □

In this paper we use Boxer's digital fundamental group [1] to find some digital topological properties of digital images with the general k-adjacency relations.

Thus we need the following.

Definition 2.4[6, 7, 9, 10]. For digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, let $f, g : X \to Y$ be digitally (k_0, k_1)-continuous functions. And further, suppose there is a positive integer m and a function, $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

- for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$,
- for all $x \in X$, the induced map $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is digitally $(2, k_1)$-continuous, and
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced map F_t which is defined by $F_t(x) = F(x, t) : X \to Y$ is digitally (k_0, k_1)-continuous for all $x \in X$.

Then F is called a digital (k_0, k_1)-homotopy between f and g and we use the notation $f \simeq_{(k_0, k_1)} g$. Especially, for the case of a digital (k, k)-homotopy, we call it a digital k-homotopy and use the notation: $f \simeq_k g$ instead of $f \simeq_{(k,k)} g$.

- If further, $F(x_0, t) = y_0$ for some $(x_0, y_0) \in X \times Y$ and all $t \in [0, m]_{\mathbb{Z}}$, we say that F is a pointed digital (k_0, k_1)-homotopy. □

If $X = [0, m_X]_{\mathbb{Z}}$ and for all $t \in [0, m]_{\mathbb{Z}}$ we have $F(0, t) = F(0, 0)$ and $F(m_X, t) = F(m_X, 0)$, we say that F holds the endpoints fixed.

Furthermore, we say that the space X is *k-contractible* if $1_X \simeq_k c_{\{x_0\}}$, where $c_{\{x_0\}}$ is a constant map for some point $x_0 \in X$ [1, 6, 9]. In the following, we use the pointed digital (k_0, k_1)-homotopy [1]. Thus, from the pointed digital homotopy for a pointed digital image (X, x_0), the k-fundamental group $\pi_1^k(X, x_0)$ was established as follows [1].

For a pointed image (X, x_0), a k-loop based at x_0 is a $(2, k)$-continuous function $f : [0, m]_{\mathbb{Z}} \to X$ with $f(0) = x_0 = f(m)$. The number m depends on the k-loop; different loops are allowed to have different digital interval domains. In the point (k, k)-homotopy, via the notion of a *trivial extension* [1], we do not need to restrict homotopy classes to k-loops defined on the same interval. Precisely, for $f \in F_1^k(X, x_0)$, if f' is a trivial extension of f if f' follows the same path as f, but it stays with pause for rest subinterval of the domain on which f' is constant [1]. For example, if $m_f \leq m_{f'}$, we can extend a k-loop $f : [0, m_f]_{\mathbb{Z}} \to X$ to a k-loop $f' : [0, m_{f'}]_{\mathbb{Z}} \to X$ via

$$f'(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq m_f; \\ f(0) & \text{if } m_f \leq t \leq m_{f'}. \end{cases}$$

Consequently, different k-loops are allowed to have different digital interval domains.

We have $g \in [f]$ if and only if there is a homotopy, holding the endpoint fixed, between *trivial extension* F, G of f, g, respectively, where a trivial extension F of f is a map. Let $F_1^k(X, x_0) = \{f \mid f \text{is a } \ k - \text{loop based at } \ x_0\}$.

For members $f : [0, m_1]_{\mathbb{Z}} \to X$, $g : [0, m_2]_{\mathbb{Z}} \to X$ of $F_1^k(X, x_0)$, we get a map [11] $f * g : [0, m_1 + m_2]_{\mathbb{Z}} \to X$ defined by

$$f * g(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq m_1; \\ g(t - m_1) & \text{if } m_1 \leq t \leq m_1 + m_2. \end{cases}$$

The k-homotopy class of a pointed k-loop f defined in [1] is denoted by $[f]$. The star operation preserves homotopy classes in the sense that if $f_1, f_2, g_1, g_2 \in F_1^k(X, x_0)$, $f_1 \in [f_2]$, and $g_1 \in [g_2]$, then $f_1 * g_1 \in [f_2 * g_2]$, *i.e.*, $[f_1 * g_1] = [f_2 * g_2]$ [1, 12]. Then

$$\pi_1^k(X, x_0) = \{[f] | f \in F_1^k(X, x_0)\}$$

is a group [1] with the operation $[f] \cdot [g] = [f * g]$ [1, 11], the *k-fundamental group* of (X, x_0).

If x_0 and x_1 belong to the same k-connected component of X, then $\pi_1^k(X, x_0)$ and $\pi_1^k(X, x_1)$ are isomorphic to each other [1]. Further, if X is k-contractible, then $\pi_1^k(X, x_0)$ is trivial [10]. We say that a digitally (k_0, k_1)-continuous function $f : X \to Y$ is *k_1-nullhomotopic* in Y if f is digitally k_1-homotopic in Y to a constant function $c_{\{y_0\}}$, $y_0 \in Y$ [1, 3, 6, 7].

For digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$, a map $h : X \to Y$ is called a *digital (k_0, k_1)-homeomorphism* if h is bijective and digitally (k_0, k_1)-continuous and further, $h^{-1} : Y \to X$ is digitally (k_1, k_0)-continuous. Then we denote it by $X \approx_{d\cdot(k_0,k_1)\cdot h} Y$ [1, 3]. If $k_0 = k_1$, then we call it a *digital k_0-homeomorphism* [1, 3, 5, 6, 7, 10]. Moreover, we get that a digital (k_0, k_1)-homeomorphism preserves the pointed k_0-contractibility to the pointed k_1-contractibility.

Theorem 2.5[3, 10]. If $h : (X, x_0) \to (Y, y_0)$ is a digital (k_0, k_1)-homeomorphism, then the induced map $h_* : \pi_1^{k_0}(X, x_0) \to \pi_1^{k_1}(Y, y_0)$ by the equation $h_*([f]) = [h \circ f]$ is a digital fundamental group isomorphism, $[f] \in \pi_1^{k_0}(X, x_0)$.

3. Digital 8-pseudotori, DT_8 and DT_8'

The three types of minimal simple closed curves were shown in $\mathbb{Z}^2$, MSC_8, MSC_4 and MSC_8' [2, 3, 4], which are not digitally k-homeomorphic to each other, $k \in \{4, 8\}$. Precisely, let MSC_8 be any set which is digitally 8-homeomorphic to the set $\{(0,0), (-1,1), (-2,0), (-2,-1), (-1,-2), (0,-1)\}$, let MSC_4 be any set which is digitally 4-homeomorphic to the set $N_8^* = \{q|p \text{ and } q \text{ are 8-adjacent}\}$ and let MSC_8' be any set which is digitally 8-homeomorphic to the set $N_4^* = \{q|p \text{ and } q \text{ are 4-adjacent}\}$.

For the digital images (X, k_1) in $\mathbb{Z}^{n_1}$ and (Y, k_2) in $\mathbb{Z}^{n_2}$, the Cartesian product image $X \times Y = \{(x, y)|x \in X, y \in Y\}$ is defined with some k_3-adjacency in $\mathbb{Z}^{n_1+n_2}$ so that each of the natural projection maps $p_i : X \times Y \to X$ and Y is digitally (k_3, k_i)-continuous, respectively, $i \in \{1, 2\}$.

Definition 3.1[10]. For two digital images (X, k_1) in $\mathbb{Z}^{n_1}$ and (Y, k_2) in $\mathbb{Z}^{n_2}$, the k_3-adjacency on the product image $X \times Y \subset \mathbb{Z}^{n_1+n_2}$ is given as follows: $(x, y) \in X \times Y$ is k_3-adjacent to $(x', y') \in X \times Y$ if and only if

(1) x is k_1-adjacent to x' and y is equal to y';

(2) x is equal to x' and y is k_2-adjacent to y'; or

(3) x is k_1-adjacent to x' and y is k_2-adjacent to y'. □

Then we say that the k_3-adjacency is *compatible* with the k_1- and k_2-adjacency.

Example 3.2. The two spaces $MSC_4 \times MSC_8$ and $MSC_4 \times MSC_8'$ are considered with 8-adjacency, respectively, in $\mathbb{Z}^4$. And we use the denotations $MSC_4 \times MSC_8 = DT_8$, and $MSC_4 \times MSC_8' = DT_8'$. □

The following definition was originally introduced [7, 10, 11] to use for a calculation of digital k-fundamental group of digital image [10].

Definition 3.3. [7, 10] Let (E, k_0) and (B, k_1) be digital images. Let $p : E \to B$ be a (k_0, k_1)-continuous surjection. Suppose for any $b \in B$ there exists $\varepsilon \in \mathbb{N}$ such that

- (DC 1) for some $\delta \in \mathbb{N}$ and some index set M,
  $p^{-1}(N_{k_1}(b, \varepsilon)) = \cup_{i \in M} N_{k_0}(e_i, \delta)$ with $e_i \in p^{-1}(b)$;
- (DC 2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i, \delta) \cap N_{k_0}(e_j, \delta) = \phi$; and

• (DC 3) the restriction map $p|_{N_{k_0}(e_i,\delta)} : N_{k_0}(e_i,\delta) \to N_{k_1}(b,\varepsilon)$ is a (k_0,k_1)-homeomorphism for all $i \in M$.

Then the map p is called a (k_0,k_1)-covering map and (E,p,B) is called a (k_0,k_1)-covering. The collection $\{N_{k_0}(e_i,\delta) | i \in M\}$ is a partition of $p^{-1}(N_{k_1}(b,\varepsilon))$ into slices. $N_{k_1}(b,\varepsilon)$ is called an elementary k_1-neighborhood of b. □

Let $SC_k^{n,l}$ be a simple closed k-curve with $l \geq 4$ points in $\mathbb{Z}^n$. We may assume that $SC_k^{n,l} = \{c_i\}_{i=0}^{l-1} \subset \mathbb{Z}^n$ such that c_i and c_j are adjacent if and only if $j = i \pm 1(\text{mod}l)$. Let $p_1 : \mathbb{Z} \to SC_k^{n,l}$ be defined for all $t \in \mathbb{Z}$ by $p_1(t) = c_{t(\text{mod}l)}$. Then $(\mathbb{Z}, p_1, SC_k^{n,l})$ is a $(2,k)$-covering in $\mathbb{Z}^n, n \geq 2$ [7, 10, 11].

For digital pictures $(\mathbb{Z}^{n_0}, k_0, \overline{k}_0, E)$, $(\mathbb{Z}^{n_1}, k_1, \overline{k}_1, B)$ and $(\mathbb{Z}^{n_2}, k_2, \overline{k}_2, X)$, let $p : E \to B$ be a digitally (k_0,k_1)-continuous map and let f be a digitally (k_2,k_1)-continuous map from X into B. We say that a *digital lifting* of f is a digitally (k_2,k_0)-continuous map $\tilde{f} : X \to E$ if $p \circ \tilde{f} = f$ [10].

Lemma 3.4[10]. $\pi_1^4(MSC_4, s_0) \simeq 8\mathbb{Z}$, and $\pi_1^8(MSC_8, x_0) \simeq 6\mathbb{Z}$.

Due to Definition 3.1 and Lemma 3.4, we find some digital topological properties of DT_8 and DT_8' with relation to the digital k-fundamental group, $k \in \{8, 32, 64, 80\}$

Theorem 3.5. $DT_8 = MSC_4 \times MSC_8$ is not 32-contractible.

Proof. We assume $DT_8 = MSC_4 \times MSC_8 = \{(t_1,t_2) | t_1 \in MSC_4, t_2 \in MSC_8\}$. There are 32-loops on $\{t_1\} \times MSC_8 \subset DT_8$ which can not be 32-nullhomotopic. Thus the proof is completed, as required. □

Theorem 3.6. $DT_8' = MSC_4 \times MSC_8'$ is 32-contractible.

Proof. (Step 1) We assume that $DT'_8 = MSC_4 \times MSC'_8 = \cup_{i \in M}\{T_i\}$, where $T_i = \{t_i | i \in M = [1,8]_{\mathbb{Z}}\} \times MSC'_8 \subset DT'_8$. Then we check the digital 32-homotopy on $DT'_8 \subset \mathbb{Z}^4$ as follows:

$H : \cup_{i \in M}\{T_i\} \times [0,6]_{\mathbb{Z}} \to \cup_{i \in M}\{T_i\}$.

First, let us proceed a digital 32-homotopy on each T_i, $i \in M$, simultaneously.

Especially, let us check the digital 32-homotopy on T_1 as follows:

(1) $H(r_i, 0) = r_i$, for any $r_i \in T_1$, where $T_1 = \{(r_0, r_1, r_2, r_3) | r_i$ and r_{i+1} are 32-adjacent (mod 4)$\}$;

(2) $H(r_1, 1) = H(r_0, 1) = r_0, H(r_2, 1) = r_3 = H(r_3, 1)$; and

(3) $H(r_i, 2) = r_0$, $i \in [0,3]_{\mathbb{Z}}$.

Therefore $T_1 \simeq_{d \cdot 32 \cdot h} c_{\{r_0\}}$.

Next, we proceed a digital 32-homotopy on each of $\{T_i\}_{i \in M_1}$ with the same method that in T_1 above, $M_1 = M - \{1\}$. Consequently, we get that each T_i is 32-contractible, $i \in M$.

(Step 2) We return to the following minimal simple closed 32-curve in DT'_8,

$MSC_4 \times \{t_2\} = \{(s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7) | s_i$ and s_{i+1} are 32-adjacent (mod 8)$\} \subset DT'_8$, where $t_2 \in MSC'_8$. Then we get the digital 32-homotopy on $MSC_4 \times \{t_2\}$ as follows.

(4) $H(s_i, 3) = s_i$, for any $s_i \in MSC_4 \times \{t_2\}$, $t_2 \in MSC'_8$;

(5) $H(s_{2i+1}, 4) = s_{2i}, H(s_{2i}, 4) = s_{2i}, i \in [0,3]_{\mathbb{Z}}$;

(6) $H(s_i, 5) = s_2, i \in \{2,3,4,5\}$ and $H(s_j, 2) = s_0, j \in \{0,1,6,7\}$; and

(7) $H(s_i, 6) = s_0$, $i \in [0,7]_{\mathbb{Z}}$.

The digital homotopy via Steps 1 and 2 has been proceeded under the digital 32-homotopy on DT'_8. Consequently, $1_{\{DT'_8\}} \simeq_{d \cdot 32 \cdot h} 1_{\{(t_1, t_2)\}}$ for some $(t_1, t_2) \in DT'_8$.

Since a digital 32-homotopy is also a digital k-homotopy, $1_{\cup_{i \in M}\{T_i\}} \simeq_{d \cdot k \cdot h} 1_{\{(t_1, t_2)\}}$, where $k \in \{80, 64, 32\}$. Thus we get trivial $\pi_1^k(DT'_8)$ for $k \in \{80, 64, 32\}$.

Meanwhile, any nontrivial homotopy classes of 8-loop on each of $\{t_1\} \times MSC'_8 \subset DT'_8$ are 8-nullhomotopic, any nontrivial homotopy classes of 8-loop on $MSC_4 \times \{t_2\} \subset DT'_8$ is not 6-nullhomotopic. Thus $\pi_1^8(DT'_8, (t_1, t_2))$ is not trivial, where $(t_1, t_2) \in MSC_4 \times \{t_2\}$. □

From Theorem 3.6, we get $\pi_1^k(DT'_8)$ is trivial, $k \in \{32, 64, 80\}$.

Theorem 3.7. $\pi_1^8(DT_8, (t_0, t_1)) \cong 8\mathbb{Z} \cong \pi_1^8(DT'_8, (t_2, t_3))$, where $(t_0, t_1) \in MSC_4 \times \{t_1\} \subset MSC_4 \times MSC_8$ and $(t_2, t_3) \in MSC_4 \times \{t_3\} \subset MSC_4 \times MSC'_8$.

Proof. Since there are 8-homotopy classes based at $(t_0, t_1) \in DT_8$ and 8-homotopy classes based at $(t_2, t_3) \in DT'_8$ which are not 8-nullhomotopic, the assertion is completed from Lemma 3.4.

References

[1] L. Boxer, A classical construction for the digital fundamental group, Jour. of Mathematical Imaging and Vision,10(1999) 51-62

[2] S. E. Han, Computer Topology and Its Applications, Honam Math. Jour. 25(2003) 153-162

[3] S. E. Han, A generalized digital (k_0, k_1)-homeomorphism, Note di Matematica, to appear, 22(2)(2004)

[4] S. E. Han, An extended digital (k_0, k_1)-continuity, JAMC,16(1-2)(2004) 445-452

[5] S. E. Han , Comparison between digital continuity and computer continuity, Honam Math. Jour. to appear,26(3)(2004)

[6] S. E. Han, Connected sum of digital simple closed surfaces, Information Sciences, to appear(2004)

[7] S. E. Han, Digital (k_0, k_1)-covering map and its properties, Honam Math. Jour. 26(2004) 107-117

[8] S. E. Han, Minimal digital pseudotorus with k-adjacency, $k \in \{6, 18, 26\}$, Honam Math. Jour.26(2)(2004) 237-246

[9] S. E. Han, Minimal simple closed 18-surfaces and a topological preservation of 3D-surfaces, Information Sciences, to appear(2004)

[10] S. E. Han, Non-product property of the digital fundamental group, Information Sciences, Available online at *www.sciencedirect.com(Articles in press)*. (2004)

[11] S. E. Han, Product properties of digital covering maps, JAMC, to appear, 17(1-2)(2004)

[12] E. Khalimsky, Motion, deformation, and homotopy in finite spaces *Proceedings IEEE International Conferences on Systems, Man, and Cybernetics*(1987)227-234

[13] T. Y. Kong, A. W. Roscoe and A. Rosenfeld, Concepts of digital topology, Topology and Its Applications,46(1992)219- 262

[14] ———, and A. Rosenfeld, (Digital topology - A brief introduction and bibliography) Topological Algorithms for the Digital Image Processing,(1996), Elsevier Science, Amsterdam

Sik Lee
Department of Applied Mathematics
College of Natural Science, Yosu National University
Yosu, 550-749, Korea
E-mail : slee@yosu.ac.kr

Sam-Tae Kim
Department of Computer and Applied Mathematics
College of Natural Science, Honam University
Gwangju, 506-714, Korea

Sang-Eon Han
Department of Computer and Applied Mathematics
College of Natural Science, Honam University
Gwangju, 506-714, Korea
E-mail :sehan@honam.ac.kr