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EIGHT-DIMENSIONAL EINSTEIN'S CONNECTION FOR THE SECOND CLASS I. THE RECURRENCE RELATIONS IN 8-g-UFT

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Abstract. Lower dimensional cases of Einstein's connection were already investigated by many authors for $n = 2, 3, 4, 5, 6, 7$. This paper is the first part of the following series of two papers, in which we obtain a surveyable tensorial representation of 8-dimensional Einstein's connection in terms of the unified field tensor, with main emphasis on the derivation of powerful and useful recurrence relations which hold in 8-dimensional Einstein's unified field theory(i.e., 8-g-UFT):

I. The recurrence relations in 8-g-UFT

II. The Einstein's connection in 8-g-UFT

All considerations in these papers are restricted to the second class only, since the case of the first class are done in [1], [2] and the case of the third class, the simplest case, was already studied by many authors.

1. INTRODUCTION.

In Appendix II to his last book Einstein ([14],1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. Characterizing Einstein's unified field theory as a

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set of geometrical postulates in the space-time X_4, Hlavatý([15],1957) gave its mathematical foundation for the first time. Since then Hlavatý and number of mathematicians contributed for the development of this theory and obtained many geometrical consequences of these postulates.

Generalizing X_4 to n-dimensional generalized Riemannian manifold X_n, n-dimensional generalization of this theory, so called *Einstein's n-dimensional unified field theory* (n-g-UFT hereafter), had been attempted by Wrede([18],1958) and Mishra([17],1959). On the other hand, corresponding to n-g-UFT, Chung([1], 1963) introduced a new unified field theory, called *the Einstein's n-dimensional* $^*g$*-unified field theory*(n-$^*g$-UFT hereafter). This theory is more useful than n-g-UFT in some physical aspects. Chung and et al obtained many results concerning this theory ([2],1969; [6],1981; [9],1988; [10]-[11],1998), particularly proving that n-$^*g$-UFT is equivalent to n-g-UFT so far as the classes and indices of inertia are concerned ([8],1985). The case of the *third class*, which is the simplest case of both unified field theories, was completely studied for a general n by many authors([17], [18], [9], etc.). However, in the cases of the first and second class of both n-dimensional generalizations, it has been unable yet to represent the general n-dimensional Einstein's connection in a surveyable tensorial form in terms of the unified field tensor $g_{\lambda\mu}$. This is probably due to the complexity of the higher dimensions.

However, the lower dimensional cases of the Einstein's connection in n-g-UFT were investigated by many authors: 2-dimensional case by Jakubowicz([16], 1969) and Chung et al([7], 1983), 3-dimensional case by Chung et al([3]-[5], 1979-1981), and 4-dimensional case by Hlavatý([15], 1957) and many other geometricians. Recently, Chung et al also studied the Einstein's connection in 4-$^*g$-UFT([1], 1963), in 3-and 5-$^*g$-UFT([9], 1988), and in 6-g-UFT([12]-[13], 1999), and obtained respective Einstein's connection in a surveyable tensorial form.

The purpose of the present paper, the first part of a series of two papers, is to derive powerful recurrence relations which hold in 8-g-UFT. In the second part, we prove a necessary and sufficient condition for the existence and uniqueness of the Einstein's connection in 8-g-UFT and establish a linear system of equations for the solution of 8-dimensional Einstein's connection for the second class, employing the powerful recurrence relations obtained in Part I.

All considerations in this and subsequent papers are dealt for the second class only.

2. PRELIMINARIES.

This section is a brief collection of basic concepts, notations, and results, which are needed in our subsequent considerations. They are due to Chung([1],1963; [9],1988), Hlavatý([15],1957) and Mishra([17],1959). All considerations in this section are dealt for a general $n > 1$.

2.1. n-dimensional g-unified field theory.

The Einstein's n-dimensional unified field theory, denoted by n-g-UFT, is an n-dimensional generalization of the usual Einstein's 4-dimen sional unified field theory in the space-time X_4. It is based on the following three principles as indicated by Hlavatý([15]).

Principle A. Let X_n be an n-dimensional generalized Riemannian manifold referred to a real coordinate system x^ν, which obeys the coordinate transformation $x^\nu \rightarrow x^{\nu'}$ [1] for which

$$det(\frac{\partial x'}{\partial x}) \neq 0 \tag{2.1}$$

[1]Throughout the present paper, Greek indices are used for the holonomic components of tensors, while Roman indices are used for the nonholonomic components of a tensor in X_n. All indices take the values $1, 2, \cdots, n$, and follow the summation convention with the exception of nonholonomic indices x, y, z, t.

In n-g-UFT the manifold X_n is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, called *the unified field tensor of* X_n. This tensor may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2a}$$

where

$$g = det(g_{\lambda\mu}) \neq 0, h = det(h_{\lambda\mu}) \neq 0, k = det(k_{\lambda\mu}) \tag{2.2b}$$

We may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu \tag{2.3}$$

In n-g-UFT the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of tensors in X_n in the usual manner.

Principle B. The differential geometric structure on X_n is imposed by the tensor $g_{\lambda\mu}$ by means of a connection $\Gamma^\nu_{\lambda\mu}$ defined by a system of equations

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha} \tag{2.4}$$

Here D_ω denotes the symbolic vector of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and $S_{\lambda\mu}{}^\nu$ is the torsion tensor of $\Gamma^\nu_{\lambda\mu}$. The connection $\Gamma^\nu_{\lambda\mu}$ satisfying (2.4) is called *the Einstein's connection.* Under certain conditions the system (2.4) admits a unique solution $\Gamma^\nu_{\lambda\mu}$.

Principle C. In order to obtain $g_{\lambda\mu}$ involved in the solution for $\Gamma^\nu_{\lambda\mu}$ certain conditions are imposed. These conditions may be condensed to

$$S_\lambda = S_{\lambda\alpha}{}^\alpha = 0, R_{[\mu\lambda]} = \partial_{[\mu} X_{\lambda]} \tag{2.5}$$

where X_λ is an arbitrary non-zero vector, and $R_{\omega\mu\lambda}{}^\nu$ and $R_{\mu\lambda}$ are the curvature tensors of $\Gamma^\nu_{\lambda\mu}$ defined by

$$R_{\omega\mu\lambda}{}^\nu = 2(\partial_{[\mu}\Gamma_{|\lambda|}{}^\nu{}_{\omega]} + \Gamma_\alpha{}^\nu{}_{[\mu}\Gamma_{|\lambda|}{}^\alpha{}_{\omega]}), R_{\mu\lambda} = R_{\alpha\mu\lambda}{}^\alpha \tag{2.6}$$

2.2. Algebraic preliminaries.

In this subsection, notations, concepts, and several algebraic results in n-g-UFT are introduced.

(i) Notations. The following scalars, tensors, and notations are frequently used in our further considerations:

$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, k = \frac{\mathfrak{k}}{\mathfrak{h}} \tag{2.7a}$$

$$K_p = k_{[\alpha_1}{}^{\alpha_1} k_{\alpha_2}{}^{\alpha_2} \cdots k_{\alpha_p]}{}^{\alpha_p}, (p = 0, 1, 2, \cdots) \tag{2.7b}$$

$$^{(0)}k_\lambda{}^\nu = \delta_\lambda^\nu, {}^{(1)}k_\lambda{}^\nu = k_\lambda{}^\nu, {}^{(p)}k_\lambda{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha \ k_\alpha{}^\nu, (p = 1, 2, \cdots) \tag{2.7c}$$

$$K_{\omega\mu\nu} = \nabla_\nu k_{\omega\mu} + \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu} \tag{2.7d}$$

$$\sigma = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases} \tag{2.7e}$$

where ∇_ω is the symbolic vector of covariant derivative with respect to the Christoffel symbols $\{ {}^{\,\nu}_{\lambda\mu} \}$ defined by $h_{\lambda\mu}$. It has been shown that the scalars and tensors intorduced in (2.7) satisfy

$$K_0 = 1, K_n = k \text{ if n is even, and } K_p = 0 \text{ if p is odd} \tag{2.8a}$$

$$g = 1 + K_2 + \cdots + K_{n-\sigma} \tag{2.8b}$$

$$^{(p)}k_{\lambda\mu} = (-1)^{p\,(p)}k_{\mu\lambda}, {}^{(p)}k^{\lambda\nu} = (-1)^{p\,(p)}k^{\nu\lambda} \tag{2.8c}$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\lambda}$, skew-symmetric in the first two indices, by T:

$$\overset{pqr}{T} = \overset{pqr}{T}{}_{\omega\mu\lambda} = T_{\alpha\beta\gamma}\, {}^{(p)}k_\omega{}^{\alpha\,(q)}k_\mu{}^{\beta\,(r)}k_\lambda{}^\gamma \tag{2.9a}$$

$$T = T_{\omega\mu\lambda} = \overset{000}{T} \tag{2.9b}$$

$$2\overset{pqr}{T}{}_{\omega[\lambda\mu]} = \overset{pqr}{T}{}_{\omega\lambda\mu} - \overset{pqr}{T}{}_{\omega\mu\lambda}, 2\overset{(pq)r}{T}{}_{\omega\lambda\mu} = \overset{pqr}{T}{}_{\omega\lambda\mu} + \overset{qpr}{T}{}_{\omega\lambda\mu}, etc \tag{2.9c}$$

We then have

(2.10) $$\overset{pqr}{T}{}_{\omega\lambda\mu} = -\overset{qpr}{T}{}_{\lambda\omega\mu}$$

(ii) Classification, basic vectors, and basic scalars.

DEFINITION 2.1 The tensor $g_{\lambda\mu}$(or $k_{\lambda\mu}$) is said to be :

1. *of the first class* if $K_{n-\sigma} \neq 0$
2. *of the second class with the jth category* $(j \geq 1)$ if

$$K_{2j} \neq 0, K_{2j+2} = K_{2j+4} = \cdots = K_{n-\sigma} = 0$$

3. *of the third class* if $K_2 = K_4 = \cdots = K_{n-\sigma} = 0$

The solution of the system of equations (2.4) is most conveniently brought about in a nonholonomic frame of reference, which may be introduced by the projectivity

(2.11) $$MA^{\nu} = k_{\mu}{}^{\nu} A^{\mu}, \text{(M a scalar)}$$

DEFINITION 2.2 An eigenvector A^{ν} of $k_{\lambda\mu}$ that satisfies (2.11) is called a *basic vector* in X_n, and the corresponding eigenvalue M is termed a *basic scalar.*

It has been shown that the basic scalars M are solutions of the characteristic equation

(2.12) $$M^{\sigma}(M^{n-\sigma} + K_2 M^{n-2-\sigma} + \cdots + K_{n-2-\sigma} M^2 + K_{n-\sigma}) = 0$$

(iii) Nonholonomic frame of reference.

In the first and second class, we have a set of n linearly independent basic vectors $\underset{i}{\rightarrow} A^{\nu} (i = 1, \cdots, n)$ and a unique reciprocal set $\overset{i}{\rightarrow} A_{\lambda} (i =$

$1, \cdots, n)$, satisfying

$$\overset{j}{A}_\lambda \underset{i}{A}^\lambda = \delta_i^j, \overset{i}{A}_\lambda \underset{i}{A}^\nu = \delta_\lambda^\nu \tag{2.13}$$

With these two sets of vectors, we may construct a nonholonomic frame of reference as follows:

DEFINITION 2.3 If $T_{\lambda\cdots}^{\nu\cdots}$ are holonomic components of a tensor, then its *nonholonomic components* $T_{j\cdots}^{i\cdots}$ are defined by

$$T_{j\cdots}^{i\cdots} = T_{\lambda\cdots}^{\nu\cdots} \overset{i}{A}_\nu \cdots \underset{j}{A}^\lambda \cdots \tag{2.14a}$$

An easy inspection shows that

$$T_{\lambda\cdots}^{\nu\cdots} = T_{j\cdots}^{i\cdots} \underset{i}{A}^\nu \cdots \overset{j}{A}_\lambda \cdots \tag{2.14b}$$

Furthermore, if $\underset{x}{M}$ is the basic scalar corresponding to $\underset{x}{A}^\nu$, then the nonholonomic components of ${}^{(p)}k_\lambda{}^\nu$ are given by

$$ {}^{(p)}k_x{}^i = \underset{x}{M}^p \delta_x^i, {}^{(p)}k_{xi} = \underset{x}{M}^p h_{xi}, {}^{(p)}k^{xi} =_{\overrightarrow{x}} M^p h^{xi} \tag{2.15}$$

Without loss of generality we may choose the nonholonomic components of $h_{\lambda\mu}$ as

$$\begin{aligned} & h_{12} = h_{34} = \cdots = h_{n-1-\sigma, n-\sigma} = 1 \\ & \sigma h_{ni_0} = \delta_\sigma^1, \quad \text{the remaining } h_{ij} = 0 \end{aligned} \tag{2.16}$$

where the index i_0 is taken so that $Det(h_{ij}) \neq 0$ when n is odd.

2.3. Differential geometric preliminaries.

In this subsection, we present several useful results involving Einstein's connection. These results are needed in our subsequent considerations for the solution of (2.4).

If the system (2.4) admits a solution $\Gamma^{\nu}_{\lambda\mu}$, it must be of the form

$$\Gamma^{\nu}_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu} \tag{2.17}$$

where

$$U_{\nu\lambda\mu} = 2 \overset{001}{S}{}_{\nu(\lambda\mu)} \tag{2.18}$$

The above two relations show that *our problem of determining* $\Gamma^{\nu}_{\lambda\mu}$ *in terms of* $g_{\lambda\mu}$ *is reduced to that of studying the tensor* $S_{\lambda\mu}{}^{\nu}$. On the other hand, it has been shown that the tensor $S_{\lambda\mu}{}^{\nu}$ satisfies

$$S = B - 3 \overset{(110)}{S} \tag{2.19}$$

where

$$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{[\alpha\beta\nu]} k_{\omega}{}^{\alpha} k_{\mu}{}^{\beta} \tag{2.20}$$

Therefore, the Einstein's connection $\Gamma^{\nu}_{\lambda\mu}$ satisfying (2.4) may be determined if the solution $S_{\lambda\mu}{}^{\nu}$ of the system (2.19) is found. *The main purpose of the present paper is to find a device to solve the system (2.19) when* $n = 8$.

Furthermore, for the first two classes, the nonholonomic solution of (2.19) is given by

$$\underset{xyz}{M} S_{xyz} = B_{xyz} \tag{2.21a}$$

or equivalently

$$2\underset{xyz}{M} S_{xyz} = K_{xyz} + 3K_{[xyz]} \underset{x}{M}\underset{y}{M} \tag{2.21b}$$

where

$$\underset{xyz}{M} = 1 + \underset{x}{M}\underset{y}{M} + \underset{y}{M}\underset{z}{M} + \underset{z}{M}\underset{x}{M} \tag{2.22}$$

Therefore, in virtue of (2.21), we see that *a necessary and sufficient nonholonomic conditions for the system* (2.4) *to have a unique solution* *in the first two classes is*

2.23 $$\underset{xyz}{M} \neq 0 \text{ for all x,y,z}$$

3. THE RECURRENCE RELATIONS OF THE FIRST KIND IN n-g-UFT.

This section is devoted to the derivation of the recurrence relations of the first kind and two other useful relations which hold in n-g-UFT. *All considerations in this section are also dealt for a general* $n > 1$.

The recurrence relations of the first kind in n-g-UFT are those which are satisfied by the tensors ${}^{(p)}k_\lambda{}^\nu$. These relations will be proved in the following theorem.

THEOREM 3.1 (The recurrence relations of the first kind in n-g-UFT). The tensors ${}^{(p)}k_\lambda{}^\nu$ satisfy the following recurrence relations:

(For the second class with the jth category).

(3.1a) $${}^{(2j+p)}k_\lambda{}^\nu + K_2{}^{(2j+p-2)}k_\lambda{}^\nu + \cdots + K_{2j}{}^{(p)}k_\lambda{}^\nu = 0$$

which may be condensed to

(3.1b) $$\sum_{f=0}^{2j} K_f{}^{(2j+p-f)}k_\lambda{}^\nu = 0, (p = 1, 2, \cdots)$$

Proof. The case of the second class with the jth category. When $g_{\lambda\mu}$ belongs to the second class with the jth category, the characteristic equation (2.12) is reduced to

(3.2a) $$\sum_{f=0}^{2j} K_f M^{n-f} = M^{n-2j} \sum_{f=0}^{2j} K_f M^{2j-f} = 0$$

Hence, if $\underset{x}{\rightarrow} M$ is a root of (3.2a), it satisfies

$$\underset{x}{M}\sum_{f=0}^{2j} K_f \underset{x}{M}^{2j-f} = \sum_{f=0}^{2j} K_f \underset{x}{M}^{2j-f+1} = 0 \tag{3.2b}$$

In virtue of (2.15), multiplication of δ_x^i to both sides of (3.2b) gives

$$\sum_{f=0}^{2j} K_f {}^{(2j-f+1)}k_x{}^i = 0 \tag{3.3a}$$

The holonomic form of (3.3a) is

$$\sum_{f=0}^{2j} K_f {}^{(2j-f+1)}k_\lambda{}^\alpha = 0 \tag{3.3b}$$

Consequently, the relation (3.2) follows by multiplying ${}^{(p-1)}k_\alpha{}^\nu$ to both sides of (3.3b).

REMARK 3.2 When $g_{\lambda\mu}$ belongs to the second class with the first category, the relation (3.1) is reduced to

$$ {}^{(p+2)}k_\lambda{}^\nu + K_2 {}^{(p)}k_\lambda{}^\nu = 0, (p = 1, 2, \cdots) \tag{3.4}$$

In the following two theorems we prove two useful relations.

THEOREM 3.3 (For the second class). In the second class, a tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, satisfies

$$\overset{(pq)r}{T}{}_{\omega\mu\nu} = \sum_{x,y,z} T_{xyz} \underset{x}{M}^{(p} \underset{y}{M}^{q)} \underset{z}{M}^r \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \tag{3.5a}$$

$$\overset{r(pq)}{T}{}_{\nu[\omega\mu]} = \sum_{x,y,z} T_{x[yz]} \underset{y}{M}^{(p} \underset{z}{M}^{q)} \underset{x}{M}^r \overset{x}{A}_\nu \overset{y}{A}_\omega \overset{z}{A}_\mu \tag{3.5b}$$

Proof Making use of (2.14b) and (2.16), the relation (3.5a) may be proved as in the following way:

$$\begin{aligned}\overset{(pq)r}{T}{}_{\omega\mu\nu} &= \sum_{x,y,z} \overset{(pq)r}{T}{}_{xyz} \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\ &= \frac{1}{2}\sum_{x,y,z} T_{ijk}[{}^{(p)}k_x{}^{i}\,{}^{(q)}k_y{}^{j} + {}^{(q)}k_x{}^{i}\,{}^{(p)}k_y{}^{j}]\,{}^{(r)}k_z{}^{k} \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\ &= \frac{1}{2}\sum_{x,y,z} T_{xyz}(\underset{x}{M^p}\underset{y}{M^q} + \underset{x}{M^q}\underset{y}{M^p})\underset{z}{M^r} \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu\end{aligned}$$

The second relation (3.5b) can be proved similarly.

THEOREM 3.4 (For all classes). The tensor $B_{\omega\mu\nu}$, given by (2.20), satisfies

$$\overset{(pq)r}{B} = \overset{(pq)r}{S} + \overset{(p'q')r}{S} + \overset{(p'q)r'}{S} + \overset{(pq')r'}{S} \tag{3.6}$$

$$2\overset{(pq)r}{B}{}_{\omega\mu\nu} = \overset{(pq)r}{K}{}_{\omega\mu\nu} + \overset{(p'q')r}{K}{}_{\omega\mu\nu} + \overset{(p'q)r'}{K}{}_{\nu[\omega\mu]} + \overset{(pq')r'}{K}{}_{\nu[\omega\mu]} \tag{3.7}$$

where

$$p' = p+1, q' = q+1, r' = r+1 \tag{3.8}$$

Proof. In virtue of (2.9) and (2.19), the relation (3.6) may be shown as in the following way:

$$\begin{aligned}\overset{(pq)r}{B} &= \overset{(pq)r}{B}{}_{\omega\mu\nu} = \frac{1}{2}B_{\alpha\beta\gamma}\,[{}^{(p)}k_\omega{}^{\alpha}\,{}^{(q)}k_\mu{}^{\beta} + {}^{(q)}k_\omega{}^{\alpha}\,{}^{(p)}k_\mu{}^{\beta}]\,{}^{(r)}k_\nu{}^{\gamma} \\ &= \frac{1}{2}[S_{\alpha\beta\gamma} + S_{\epsilon\eta\gamma}\,k_\alpha{}^\epsilon\,k_\beta{}^\eta + S_{\epsilon\beta\eta}\,k_\alpha{}^\epsilon\,k_\gamma{}^\eta + S_{\alpha\epsilon\eta}\,k_\beta{}^\epsilon\,k_\gamma{}^\eta] \times \\ &\quad \times [{}^{(p)}k_\omega{}^{\alpha}\,{}^{(q)}k_\mu{}^{\beta} + {}^{(q)}k_\omega{}^{\alpha}\,{}^{(p)}k_\mu{}^{\beta}]\,{}^{(r)}k_\nu{}^{\gamma}\end{aligned}$$

After a lengthy calculation, we note that the right-hand side of the above equation is equal to (3.6). The relation (3.7) may be proved similarly.

4. THE RECURRENCE RELATIONS OF THE SECOND AND THIRD KIND IN 8-g-UFT.

This section is particularly concerned with the 8-dimensional case; that is with 8-g-UFT. In this section, we first investigate the basic scalars and some relations satisfied by them. In order to obtain a tensorial representation of the 8-dimensional Einstein's connection $\Gamma^{\nu}_{\lambda\mu}$ in terms of $g_{\lambda\mu}$, we need powerful *recurrence relations of the third kind* which are satisfied by an arbitrary tensor $T_{\omega\lambda\mu}$, skew-symmetric in the first two indices. Therefore, we finally derive these relations, after introducing the *recurrence relations of the second kind* which are satisfied by the basic scalars. *All considerations in this section are restricted to* $n = 8$.

In 8-g-UFT there are five cases; that is, the unified field tensor $g_{\lambda\mu}$ belongs to

(1) *the first class*, if $K_8 \neq 0$

(2) *the second class with the first category*, if $K_2 \neq 0, K_4 = K_6 = K_8 = 0$

(3) *the second class with the second category*, if $K_4 \neq 0, K_6 = K_8 = 0$

(4) *the second class with the third category*, if$K_6 \neq 0, K_8 = 0$

(5)*the third class*, if $K_2 = K_4 = K_6 = K_8 = 0$

In this section we investigate the cases of (2), (3) and (4).

Before we start investigations about the basic scalars, we first note that in 8-g-UFT the relation (2.8b) is reduced to

$$g = 1 + K_2 + K_4 + K_6 + K_8 \tag{4.1}$$

and formally state in the following theorem the recurrence relations of the first kind when $n = 8$, which are direct consequences of (3.1) :

THEOREM 4.1 (The recurrence relations of the first kind in 8-g-UFT). The tensors ${}^{(p)}k_\lambda{}^\nu$ satisfy the following recurrence relations in 8-g-UFT for $p = 0, 1, 2, \cdots$:

	class and category	Recurrence relations of the first kind in 8-g-UFT
(4.2a)	The second class with the third category	${}^{(p+6)}k_\lambda{}^\nu = -K_2\,{}^{(p+4)}k_\lambda{}^\nu - K_4\,{}^{(p+2)}k_\lambda{}^\nu - -K_6\,{}^{(p)}k_\lambda{}^\nu$
(4.2b)	The second class with the second category	${}^{(p+4)}k_\lambda{}^\nu = -K_2\,{}^{(p+2)}k_\lambda{}^\nu - K_4\,{}^{(p)}k_\lambda{}^\nu$
(4.2c)	The second class with the first category	${}^{(p+2)}k_\lambda\nu = -K_2\,{}^{(p)}k_\lambda{}^\nu$

Proof The relations (4.2) are direct consequences of (3.1).

THEOREM 4.2 The basic scalars in 8-g-UFT are given by

	class and category	The basic scalars $\underset{x}{M}$
(4.3a)	The second class with the third category	$\underset{1}{M} = -\underset{2}{M} = \sqrt{-\frac{K_2}{3} + \alpha + \beta}$ $\underset{3}{M} = -\underset{4}{M} = \sqrt{-\frac{K_2}{3} + \omega\alpha + \omega^2\beta}$ $\underset{5}{M} = -\underset{6}{M} = \sqrt{-\frac{K_2}{3} + \omega^2\alpha + \omega\beta}$ $\underset{7}{M} = \underset{8}{M} = 0$
(4.3b)	The second class with the second category	$\underset{1}{M} = -\underset{2}{M} = \sqrt{-L-K}$ $\underset{3}{M} = -\underset{4}{M} = \sqrt{L-K}$ $\underset{5}{M} = \underset{6}{M} = \underset{7}{M} = \underset{8}{M} = 0$
(4.3c)	The second class with the first category	$\underset{1}{M} = -\underset{2}{M} = \sqrt{-K_2} \neq 0$ $\underset{3}{M} = \underset{4}{M} = \underset{5}{M} = \underset{6}{M} = \underset{7}{M} = \underset{8}{M} = 0$

where

(4.4a) $$\omega = \frac{-1+\sqrt{3}i}{2}$$

(4.4b) $$\alpha = \left[-\frac{\phi}{2} + \sqrt{(\frac{\phi}{2})^2 + (\frac{\theta}{3})^3}\right]^{\frac{1}{3}}, \beta = \left[-\frac{\phi}{2} - \sqrt{(\frac{\phi}{2})^2 + (\frac{\theta}{3})^3}\right]^{\frac{1}{3}}$$

(4.4c) $$\theta = K_4 - \frac{(K_2)^2}{3}, \phi = K_6 - \frac{K_2K_4}{3} + \frac{2}{27}(K_2)^3$$

(4.4d) $$L = \sqrt{(\frac{K_2}{2})^2 - K_4}, K = \frac{K_2}{2}$$

Proof. The relations (4.3) may be obtained from (4.3) and (4.4) in [1] by substituting the corresponding condition of each case.

THEOREM 4.3 The basic scalars $\rightarrow \underset{x}{M}$ in 8-g-UFT satisfy the following relations:

	class and category	The basic scalars $\underset{x}{M}$
(4.5a)	The second class with the third category	$\underset{1}{M} + \underset{2}{M} = \underset{3}{M} + \underset{4}{M} = \underset{5}{M} + \underset{6}{M} = \underset{7}{M} + \underset{8}{M} = 0$ $\underset{x}{M}\underset{d}{M} = 0, \quad \underset{x}{M}\underset{y}{M}\underset{d}{M} = 0$ $\underset{a}{M}^2 + \underset{b}{M}^2 + \underset{c}{M}^2 = -K_2$ $\underset{a}{M}^2\underset{b}{M}^2 + \underset{a}{M}^2\underset{c}{M}^2 + \underset{b}{M}^2\underset{c}{M}^2 = K_4$ $\underset{a}{M}^2\underset{b}{M}^2\underset{c}{M}^2 = -K_6$
(4.5b)	The second class with the second category	$\underset{1}{M} + \underset{2}{M} = \underset{3}{M} + \underset{4}{M} = \underset{5}{M} + \underset{6}{M} = \underset{7}{M} + \underset{8}{M} = 0$ $\underset{a}{M}\underset{x}{M} = \underset{b}{M}\underset{x}{M} = \underset{x}{M}\underset{y}{M} = 0$ $\underset{a}{M}^2 + \underset{b}{M}^2 = -K_2, \quad \underset{a}{M}^2\underset{b}{M}^2 = K_4$
(4.5c)	The second class with the first category	$\underset{1}{M} + \underset{2}{M} = \underset{3}{M} + \underset{4}{M} = \underset{5}{M} + \underset{6}{M} = \underset{7}{M} + \underset{8}{M} = 0$ $\underset{1}{M}\underset{2}{M} = K_2, \underset{1}{M}\underset{x}{M} = \underset{2}{M}\underset{x}{M} = \underset{x}{M}\underset{y}{M} = 0$

Here, the indices a, b, c, d are assumed to take values as $a = 1, 2;\ \ b = 3, 4;\ \ c = 5, 6;\ \ d = 7, 8$.

Proof. The relations (4.5a), (4.5b) and (4.5c) follow from (4.3a), (4.3b) and (4.3c), respectively.

Using the relations given in Theorem (4.3), we may prove the recurrence relations of the second kind in the following theorem.

THEOREM 4.4 (The recurrence relations of the second kind in 8-g-UFT). In 8-g-UFT the basic scalars $\underset{x}{M}$ satisfy the following recurrence relations which hold for all values of x and y when $x \neq y$:

(For the second class with the third category)

$$(4.6a)\quad \underset{x}{M}^{(5}\underset{y}{M}^{0)} = -\underset{x}{M}^{(4}\underset{y}{M}^{1)} - \underset{x}{M}^{(3}\underset{y}{M}^{2)} - K_2\underset{x}{M}^{(3}\underset{y}{M}^{0)} - K_2\underset{x}{M}^{(2}\underset{y}{M}^{1)} - K_4\underset{x}{M}^{(1}\underset{y}{M}^{0)}$$

$$(4.6b)\quad 2\underset{x}{M}^{(5}\underset{y}{M}^{1)} = -\underset{x}{M}^{3}\underset{y}{M}^{3} - 2K_2\underset{x}{M}^{(3}\underset{y}{M}^{1)} - K_4\underset{x}{M}\underset{y}{M}$$

$$(4.6c)\quad \underset{x}{M}^{(5}\underset{y}{M}^{2)} = -\underset{x}{M}^{(4}\underset{y}{M}^{3)} - K_2\underset{x}{M}^{(3}\underset{y}{M}^{2)} + K_6\underset{x}{M}^{(1}\underset{y}{M}^{0)}$$

$$(4.6d)\quad 2\underset{x}{M}^{(5}\underset{y}{M}^{3)} = -\underset{x}{M}^{4}\underset{y}{M}^{4} - K_2\underset{x}{M}^{3}\underset{y}{M}^{3} + K_4\underset{x}{M}^{2}\underset{y}{M}^{2} + +2K_6\underset{x}{M}^{(2}\underset{y}{M}^{0)} + K_6\underset{x}{M}\underset{y}{M}$$

$$(4.6e)\quad \underset{x}{M}^{(5}\underset{y}{M}^{4)} = K_4\underset{x}{M}^{(3}\underset{y}{M}^{2)} + K_6\underset{x}{M}^{(3}\underset{y}{M}^{0)} + K_6\underset{x}{M}^{(2}\underset{y}{M}^{1)}$$

$$(4.6f)\quad \underset{x}{M}^{5}\underset{y}{M}^{5} = K_2\underset{x}{M}^{4}\underset{y}{M}^{4} + 2K_4\underset{x}{M}^{(4}\underset{y}{M}^{2)} + 2K_6\underset{x}{M}^{(4}\underset{y}{M}^{0)} + K_4\underset{x}{M}^{3}\underset{y}{M}^{3} + 2K_6\underset{x}{M}^{(3}\underset{y}{M}^{1)} + K_6\underset{x}{M}^{2}\underset{y}{M}^{2}$$

Furthermore, we also have

$$
\begin{aligned}
(4.7a)\qquad & K_2\underset{x}{M}^{(5}\underset{y}{M}^{0)} - \underset{x}{M}^{(5}\underset{y}{M}^{2)} \\
&= \underset{x}{M}^{(4}\underset{y}{M}^{3)} - K_2\underset{x}{M}^{(4}\underset{y}{M}^{1)} - (K_2)^2\underset{x}{M}^{(3}\underset{y}{M}^{0)} \\
&\quad -(K_2)^2\underset{x}{M}^{(2}\underset{y}{M}^{1)} - (K_2K_4 + K_6)\underset{x}{M}^{(1}\underset{y}{M}^{0)}
\end{aligned}
$$

$$
\begin{aligned}
(4.7b)\qquad & K_4\underset{x}{M}^{(5}\underset{y}{M}^{0)} + \underset{x}{M}^{(5}\underset{y}{M}^{4)} \\
&= (K_6 - K_2K_4)\underset{x}{M}^{(3}\underset{y}{M}^{0)} - K_4\underset{x}{M}^{(4}\underset{y}{M}^{1)} - (K_4)^2 \\
&\quad \underset{x}{M}^{(1}\underset{y}{M}^{0)} + (K_6 - K_2K_4)\underset{x}{M}^{(2} \underset{y}{\to} M^{1)}
\end{aligned}
$$

$$
\begin{aligned}
(4.7c)\qquad & K_6\underset{x}{M}^{(5}\underset{y}{M}^{0)} + K_2\underset{x}{M}^{(5}\underset{y}{M}^{4)} \\
&= -K_6\underset{x}{M}^{(4}\underset{y}{M}^{1)} - (K_6 - K_2K_4)\underset{x}{M}^{(3}\underset{y}{M}^{2)} - K_4K_6\underset{x}{M}^{(1}\underset{y}{M}^{0)}
\end{aligned}
$$

$$
\begin{aligned}
(4.7d)\qquad & K_6\underset{x}{M}^{(5}\underset{y}{M}^{0)} + K_4\underset{x}{M}^{(5}\underset{y}{M}^{2)} \\
&= -K_4\underset{x}{M}^{(4}\underset{y}{M}^{3)} - K_6\underset{x}{M}^{(4}\underset{y}{M}^{1)} - (K_2K_4 + K_6) \\
&\quad \underset{x}{M}^{(3}\underset{y}{M}^{2)} - K_2K_6\underset{x}{M}^{(3}\underset{y}{M}^{0)} - K_2K_6\underset{x}{M}^{(2}\underset{y}{M}^{1)}
\end{aligned}
$$

$$
\begin{aligned}
(4.7e)\qquad & 2K_2\underset{x}{M}^{(5}\underset{y}{M}^{1)} - 2\underset{x}{M}^{(5}\underset{y}{M}^{3)} \\
&= \underset{x}{M}^{4}\underset{y}{M}^{4} - 2(K_2)^2\underset{x}{M}^{(3}\underset{y}{M}^{1)} - K_4\underset{x}{M}^{2}\underset{y}{M}^{2} \\
&\quad -(K_2K_4 + K_6)\underset{x}{M}\underset{y}{M} - 2K_6\underset{x}{M}^{(2}\underset{y}{M}^{0)}
\end{aligned}
$$

$$
\begin{aligned}
(4.7f)\qquad & 2K_6\underset{x}{M}^{(5}\underset{y}{M}^{1)} + 2K_4\underset{x}{M}^{(5}\underset{y}{M}^{3)} \\
&= -K_4\underset{x}{M}^{4}\underset{y}{M}^{4} - (K_2K_4 + K_6)\underset{x}{M}^{3}\underset{y}{M}^{3} - 2K_2K_6 \\
&\quad \underset{x}{M}^{(3}\underset{y}{M}^{1)} + (K_4)^2\underset{x}{M}^{2}\underset{y}{M}^{2} + 2K_4K_6\underset{x}{M}^{(2}\underset{y}{M}^{0)}
\end{aligned}
$$

$$
\begin{aligned}
(4.7g)\qquad & 2K_6\underset{x}{M}^{(5}\underset{y}{M}^{1)} + K_2\underset{x}{M}^{5}\underset{y}{M}^{5} \\
&= 2K_2K_4\underset{x}{M}^{(4}\underset{y}{M}^{2)} + 2K_2K_6\underset{x}{M}^{(4}\underset{y}{M}^{0)} + (K_2K_4 - K_6)\underset{x}{M}^{3}\underset{y}{M}^{3} \\
&\quad +K_2K_6\underset{x}{M}^{2}\underset{y}{M}^{2} - K_4K_6\underset{x}{M}\underset{y}{M} + (K_2)^2\underset{x}{M}^{4}\underset{y}{M}^{4}
\end{aligned}
$$

(4.7h)

$$\begin{aligned}&\underset{x}{M}^{5}\underset{y}{M}^{5}+2K_4\underset{x}{M}^{(5}\underset{y}{M}^{1)}\\&\quad=2K_4\underset{x}{M}^{(4}\underset{y}{M}^{2)}+2(K_6-K_2K_4)\underset{x}{M}^{(3}\underset{y}{M}^{1)}+K_2\underset{x}{M}^{4}\underset{y}{M}^{4}\\&\qquad+K_6\underset{x}{M}^{2}\underset{y}{M}^{2}-(K_4)^2\underset{x}{M}\underset{y}{M}+2K_6\underset{x}{M}^{(4}\underset{y}{M}^{0)}\end{aligned}$$

(4.7i)

$$\begin{aligned}&K_4\underset{x}{M}^{(5}\underset{y}{M}^{2)}+K_2\underset{x}{M}^{(5}\underset{y}{M}^{4)}\\&\quad=-K_4\underset{x}{M}^{(4}\underset{y}{M}^{3)}+K_2K_6\underset{x}{M}^{(2}\underset{y}{M}^{1)}+K_2K_6\underset{x}{M}^{(2}\underset{y}{M}^{1)}\\&\qquad+K_2K_6\underset{x}{M}^{(3}\underset{y}{M}^{0)}+K_4K_6\underset{x}{M}^{(1}\underset{y}{M}^{0)}\end{aligned}$$

(4.7j)

$$\begin{aligned}&2K_4\underset{x}{M}^{(5}\underset{y}{M}^{3)}+K_2\underset{x}{M}^{5}\underset{y}{M}^{5}\\&\quad=((K_2)^2-K_4)\underset{x}{M}^{4}\underset{y}{M}^{4}+((K_4)^2+K_2K_6)\underset{x}{M}^{2}\underset{y}{M}^{2}\\&\qquad+2K_2K_6\underset{x}{M}^{(4}\underset{y}{M}^{0)}\\&2K_2K_4\underset{x}{M}^{(4}\underset{y}{M}^{2)}+2K_2K_6\underset{x}{M}^{(3}\underset{y}{M}^{1)}+2K_4K_6\underset{x}{M}^{(2}\underset{y}{M}^{0)}\\&\qquad+K_4K_6\underset{x}{M}\underset{y}{M}\end{aligned}$$

(4.7k)

$$\begin{aligned}&2K_2\underset{x}{M}^{(5}\underset{y}{M}^{3)}+\underset{x}{M}^{5}\underset{y}{M}^{5}\\&\quad=2K_4\underset{x}{M}^{(4}\underset{y}{M}^{2)}+2K_6\underset{x}{M}^{(3}\underset{y}{M}^{1)}+(K_4-(K_2)^2)\underset{x}{M}^{3}\underset{y}{M}^{3}\\&\qquad+(K_6+K_2K_4)\underset{x}{M}^{2}\underset{y}{M}^{2}+2K_6\underset{x}{M}^{(4}\underset{y}{M}^{0)}\\&\qquad+2K_2K_6\underset{x}{M}^{(2}\underset{y}{M}^{0)}+K_2K_6\underset{x}{M}\underset{y}{M}\end{aligned}$$

(4.7l)

$$\begin{aligned}&2K_6\underset{x}{M}^{(5}\underset{y}{M}^{3)}-K_4\underset{x}{M}^{5}\underset{y}{M}^{5}\\&\quad=-(K_2K_4+K_6)\underset{x}{M}^{4}\underset{y}{M}^{4}-((K_4)^2+K_2K_6)\underset{x}{M}^{3}\underset{y}{M}^{3}\\&\qquad-2K_4K_6\underset{x}{M}^{(3}\underset{y}{M}^{1)}\\&-2((K_4)^2)\underset{x}{M}^{(4}\underset{y}{M}^{2)}-2K_4K_6\underset{x}{M}^{(4}\underset{y}{M}^{0)}+2(K_6)^2\underset{x}{M}^{(2}\underset{y}{M}^{0)}\\&\qquad+(K_6)^2\underset{x}{M}\underset{y}{M}\end{aligned}$$

(For the second class with the second category).

$$\underset{x}{M}^{(3}\underset{y}{M}^{0)} = -\underset{x}{M}^{(2}\underset{y}{M}^{1)} - K_2\underset{x}{M}^{(1}\underset{y}{M}^{0)} \tag{4.8a}$$

$$2\underset{x}{M}^{(3}\underset{y}{M}^{1)} = -\underset{x}{M}^{2}\underset{y}{M}^{2} + K_4\underset{x}{M}^{0}\underset{y}{M}^{0} - K_2\underset{x}{M}\underset{y}{M} \tag{4.8b}$$

$$\underset{x}{M}^{(3}\underset{y}{M}^{2)} = K_4\underset{x}{M}^{(1}\underset{y}{M}^{0)} \tag{4.8c}$$

$$\underset{x}{M}^{3}\underset{y}{M}^{3} = K_2\underset{x}{M}^{2}\underset{y}{M}^{2} + 2K_4\underset{x}{M}^{(2}\underset{y}{M}^{0)} + K_4\underset{x}{M}\underset{y}{M} \tag{4.8d}$$

Furthermore, we have

$$K_4\underset{x}{M}^{(3}\underset{y}{M}^{0)} + K_2\underset{x}{M}^{(3}\underset{y}{M}^{2)} = -K_4\underset{x}{M}^{(2}\underset{y}{M}^{1)} \tag{4.9a}$$

$$\begin{aligned} &2K_4\underset{x}{M}^{(3}\underset{y}{M}^{1)} + K_2\underset{x}{M}^{3}\underset{y}{M}^{3} \\ &= ((K_2)^2 - K_4)\underset{x}{M}^{2}\underset{y}{M}^{2} + 2K_2K_4\underset{x}{M}^{(2}\underset{y}{M}^{0)} + (K_4)^2\underset{x}{M}^{0}\underset{y}{M}^{0} \end{aligned} \tag{4.9b}$$

$$\begin{aligned} &2K_2\underset{x}{M}^{(3}\underset{y}{M}^{1)} + \underset{x}{M}^{3}\underset{y}{M}^{3} \\ &= -((K_2)^2 - K_4)\underset{x}{M}\underset{y}{M} + 2K_4\underset{x}{M}^{(2}\underset{y}{M}^{0)} + K_2K_4\underset{x}{M}^{0}\underset{y}{M}^{0} \end{aligned} \tag{4.9c}$$

(For the second class with the first category).

$$\underset{1}{M}\underset{2}{M} = K_2 \tag{4.10}$$

Proof. The proof of the relations in (4.6), (4.7), (4.8), (4.9) and (4.10). These relations may be obtained from (4.7) and (4.8) in [1] by substituting the corresponding conditions of each case.

Now, we are ready to prove the recurrence relations of the third kind in the following theorem. These relations are very important for the solution of (2.4) or (2.19) in 8-g-UFT. We use these relations in our subsequent paper to establish a linear system equivalent to (2.4) and to find a precise and surveyable tensorial representation of 8-dimensional Einstein's connection in terms of the unified field tensor $g_{\lambda\mu}$.

THEOREM 4.5 (The recurrence relations of the third kind in 8-g-UFT). If T is a skew-symmetric tensor in the first two indices, the following recurrence relations hold in 8-g-UFT:

(For the second class with the third category).

$$\overset{(50)r}{T} = -\overset{(41)r}{T} - \overset{(32)r}{T} - K_2 \overset{(30)r}{T} - K_2 \overset{(21)r}{T} - K_4 \overset{(10)r}{T} \tag{4.11a}$$

$$2\overset{(51)r}{T} = -\overset{33r}{T} - 2K_2 \overset{(31)r}{T} - K_4 \overset{11r}{T} \tag{4.11b}$$

$$\overset{(52)r}{T} = -\overset{(43)r}{T} - K_2 \overset{(32)r}{T} + K_6 \overset{(10)r}{T} \tag{4.11c}$$

$$2\overset{(53)r}{T} = -\overset{44r}{T} - K_2 \overset{33r}{T} + K_4 \overset{22r}{T} + 2K_6 \overset{(20)r}{T} + K_6 \overset{11r}{T} \tag{4.11d}$$

$$\overset{(54)r}{T} = K_4 \overset{(32)r}{T} + K_6 \overset{(30)r}{T} + K_6 \overset{(21)r}{T} \tag{4.11e}$$

$$\overset{55r}{T} = K_2 \overset{44r}{T} + 2K_4 \overset{(42)r}{T} + 2K_6 \overset{(40)r}{T} + K_4 \overset{33r}{T} + \\ + 2K_6 \overset{(31)r}{T} + K_6 \overset{22r}{T} \tag{4.11f}$$

Furthermore, the following identities also hold in the second class with the third category:

$$K_2 \overset{(50)r}{T} - \overset{(52)r}{T} \\ = \overset{(43)r}{T} - K_2 \overset{(41)r}{T} - (K_2)^2 \overset{(30)r}{T} - (K_2)^2 \overset{(21)r}{T} - (K_2K_4 + K_6) \overset{(10)r}{T} \tag{4.12a}$$

$$K_4 \overset{(50)r}{T} + \overset{(54)r}{T} \\ = (K_6 - K_2K_4) \overset{(30)r}{T} - K_4 \overset{(41)r}{T} - (K_4)^2 \overset{(10)r}{T} - (K_2K_4 - K_6) \overset{(21)r}{T} \tag{4.12b}$$

$$K_6 \overset{(50)r}{T} + K_2 \overset{(54)r}{T} \\ = -K_6 \overset{(41)r}{T} - K_4K_6 \overset{(10)r}{T} - (K_6 - K_2K_4) \overset{(32)r}{T} \tag{4.12c}$$

$$K_6 \overset{(50)r}{T} + K_4 \overset{(52)r}{T} \\ = -K_4 \overset{(43)r}{T} - K_6 \overset{(41)r}{T} - K_2K_6 \overset{(21)r}{T} - K_2K_6 \overset{(30)r}{T} - (K_2K_4 + K_6) \overset{(32)r}{T} \tag{4.12d}$$

(4.12e) $$2\overset{(53)r}{T} - 2K_2\overset{(51)r}{T}$$
$$= -\overset{44r}{T} + 2(K_2)^2\overset{(31)r}{T} + K_4\overset{22r}{T} + (K_6 + K_2K_4)\overset{11r}{T} + 2K_6\overset{(20r)}{T}$$

(4.12f) $$K_2\overset{55r}{T} + 2K_6\overset{(51)r}{T}$$
$$= 2K_2K_4\overset{(42)r}{T} + (K_2)^2\overset{44r}{T} + K_2K_6\overset{22r}{T} + (K_2K_4 - K_6)\overset{33r}{T}$$
$$-K_4K_6\overset{11r}{T} + 2K_2K_6\overset{(40)r}{T}$$

(4.12g) $$\overset{55r}{T} + 2K_4\overset{(51)r}{T}$$
$$= 2K_4\overset{(42)r}{T} + 2(K_6 - K_2K_4)\overset{(31)r}{T} + K_2\overset{44r}{T} + K_6\overset{22r}{T}$$
$$-(K_4)^2\overset{11r}{T} + 2K_6\overset{(40)r}{T}$$

(4.12h) $$2K_4\overset{(53)r}{T} + K_2\overset{55r}{T}$$
$$= ((K_2)^2 - K_4)\overset{44r}{T} + 2K_2K_4\overset{(42)r}{T} + 2K_2K_6\overset{(31)r}{T}$$
$$+((K_4)^2 + K_2K_6)\overset{22r}{T} + 2K_2K_6\overset{(40)r}{T} + 2K_4K_6\overset{(20)r}{T} + K_4K_6\overset{11r}{T}$$

(4.12i) $$2K_2\overset{(53)r}{T} + \overset{55r}{T}$$
$$= 2K_4\overset{(42)r}{T} + 2K_6\overset{(31)r}{T} + (K_4 - (K_2)^2)\overset{33r}{T}$$
$$+(K_6 + K_2K_4)\overset{22r}{T} + 2K_6\overset{(40)r}{T} + 2K_2K_6\overset{(20)r}{T} + K_2K_6\overset{11r}{T}$$

(4.12j) $$K_4\overset{55r}{T} - 2K_6\overset{(53)r}{T}$$
$$= (K_2K_4 + K_6)\overset{44r}{T} + 2(K_4)^2\overset{(42)r}{T} + ((K_4)^2 + K_2K_6)$$
$$\overset{33r}{T} + 2K_4K_6\overset{(31)r}{T} + 2K_4K_6\overset{(40)r}{T} - 2(K_6)^2\overset{(20)r}{T} - (K_6)^2\overset{11r}{T}$$

(4.12k) $$K_2\overset{(54)r}{T} + K_4\overset{(52)r}{T}$$
$$= -K_4\overset{(43)r}{T} + K_2K_6\overset{(21)r}{T} + K_2K_6\overset{(30)r}{T} + K_4K_6\overset{(10)r}{T}$$

(4.12l) $$2K_4\overset{(53)r}{T} + 2K_6\overset{(51)r}{T}$$
$$= -K_4\overset{44r}{T} - (K_2K_4 + K_6)\overset{33r}{T} - 2K_2K_6\overset{(31)r}{T} + (K_4)^2\overset{22r}{T}$$
$$+2K_4k_6\overset{(20)r}{T}$$

(For the second class with the second category).

$$(4.13a) \qquad \overset{(30)r}{T} = -\overset{(21)r}{T} - K_2 \overset{(10)r}{\to} T$$

$$(4.13b) \qquad 2\overset{(31)r}{T} = -\overset{22r}{T} - K_2 \overset{11r}{T} + K_4 \overset{00r}{T}$$

$$(4.13c) \qquad \overset{(32)r}{T} = K_4 \overset{(10)r}{\to} T$$

$$(4.13d) \qquad \overset{33r}{T} = K_2 \overset{22r}{T} + 2K_4 \overset{(20)r}{T} + K_4 \overset{11r}{T}$$

Furthermore, the following identities also hold in the second class with the second category:

$$(4.14a) \qquad K_2 \overset{(32)r}{T} + K_4 \overset{(30)r}{T} = -K_4 \overset{(21)r}{T}$$

$$(4.14b) \qquad K_2 \overset{33r}{T} + 2K_4 \overset{(31)r}{T} = ((K_2)^2 - K_4) \overset{22r}{T} + 2K_2K_4 \overset{(20)r}{T} + (K_4)^2 \overset{00r}{T}$$

$$(4.14c) \qquad 2K_2 \overset{(31)r}{T} + \overset{33r}{T} = -((K_2)^2 - K_4) \overset{11r}{T} + 2K_4 \overset{(20)r}{T} + K_2K_4 \overset{00r}{T}$$

(For the second class with the first category).

$$(4.15) \qquad \overset{11r}{T} = K_2 \overset{00r}{T}$$

Proof. We first note that the terms in the right-hand side of (3.5a) vanishes identically when $x = y$. Therefore, whenever we use (3.5a), it suffices to consider the terms corresponding to the cases $x \neq y$ only. The proof of the above relations follow from (3.5a), using (4.6) for the proof of (4.11), (4.7) for the proof of (4.12), (4.8) for the proof of (4.13), (4.9) for the proof of (4.14) and (4.10) for the proof of (4.15), respectively. For example, the relation (4.11a) may be proved as in the following way:

$$
\begin{aligned}
\overset{(50)r}{T} &= \overset{(50)r}{T}{}_{\omega\mu\nu} \\
&= \sum_{x,y,z} T_{xyz} \underset{x}{M}{}^{(5}\underset{y}{M}{}^{0)}\underset{z}{M}{}^{r} \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\
&= \sum_{x,y,z} T_{xyz} [-\underset{x}{M}{}^{(4}\underset{y}{M}{}^{1)} - \underset{x}{M}{}^{(3}\underset{y}{M}{}^{2)} - K_2 \underset{x}{M}{}^{(3}\underset{y}{M}{}^{0)} \\
&\quad -K_2 \underset{x}{M}{}^{(2}\underset{y}{M}{}^{1)} - K_4 \underset{x}{M}{}^{(1}\underset{y}{M}{}^{0)}] \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\
&= -\overset{(41)r}{T}{}_{\omega\mu\nu} - \overset{(32)r}{T}{}_{\omega\mu\nu} - K_2 \overset{(30)r}{T}{}_{\omega\mu\nu} - K_2 \overset{(21)r}{T}{}_{\omega\mu\nu} - K_4 \overset{(10)r}{T}{}_{\omega\mu\nu} \\
&= -\overset{(41)r}{T} - \overset{(32)r}{T} - K_2 \overset{(30)r}{T} - K_2 \overset{(21)r}{T} - K_4 \overset{(10)r}{T}
\end{aligned}
$$

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