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COMPLETION OF FUNDAMENTAL TOPOLOGICAL VECTOR SPACES

E. ANSARI-PIRI

Abstract. A class of topological algebras, which we call it a fundamental one, has already been introduced generalizing the famous Cohen factorization theorem to more general topological algebras. To prove the generalized versions of Cohen's theorem to locally multilplicatively convex algebras, and finally to fundamental topological algebras, the completness of the background spaces is one of the main conditions. The local convexity of the completion of a locally convex space is a well known fact and here we have a discussion on the completness of fundamental metrizable topological vector spaces.

1. Introduction

In 1989, the author introduced the notion of fundamental topological algebras generalizing both local boundedness and local convexity. A natural question is to ask for generalizing the basic results on this new class. This is of course a wide question and the first successful step answering it, has already been done in [1], where the Cohen factorization theorem is proved for complete metrizable fundamental topological algebras. In [2] and [3], we have also proved some more results on fundamental topological algebras. In fact fundamentality is a topological vector space property rather than an algebra property, and a necessary condition is

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given in [1] to check the fundamentality of a class of topological vector spaces.

Here, we have a discussion on the completion of metrizable fundamental topological vector spaces and give a necessary and sufficient condition for a metrizable fundamental topological vector space to be complete, extending the famous normed version in the absence of local convexity and local boundedness.

2. Definitions and related results

In this section we recall some definitions concerning fundamental topological vector spaces and some related propositions, which are all proved in [1].

Definition 2.1. Let A be a topological vector space. We say A is a fundamental topological vector space if there exists $b > 1$ such that for every sequence (a_n) of A, the convergence of $b^n(a_n - a_{n-1})$ to zero in A implies that (a_n) is Cauchy sequence.

It is easy to chech that every locally convex and every locally bounded topological vector space is fundamental. On the other hand, if A is a locally bounded but not locally convex, and B is a locally convex but not locally bounded space, then $A \bigoplus B$ with product topology, componentwise definitions for addition; and scaler multiplication is a fundamental topological vector space which is not locally bounded and not locally convex.

Definition 2.2. Let E be any set. We say A is uniformly fundamental on E if there exists $b > 1$ such that for every sequence (f_n) of functions from E into A, the uniform convergence of $b^n(f_n - f_{n-1})$ to zero on E implies that (f_n) is uniformly Cauchy on E. If A is uniformly fundamental on every set E, then we call it a uniformly fundamental one.

Proposition 2.3. Let A be a topological vector space. A is uniformly fundamental if and only if it is uniformly fundamental on the set of nutural numbers : $\mathbf{N}$.

Proposition 2.4. Let A be a fundamental topological vector space. Then, for every $c > 1$ and every sequence (a_n) of A, the convergence of $c^n(a_n - a_{n-1})$ to zero in A implies that (a_n) is a Cauchy sequence. For the uniformly fundamental case the corresponding result holds.

Proposition 2.5. If A is complete metrizable and fundamental, then it is uniformly fundamental.

Proposition 2.6. Let A be a metrizable uniformly fundamental topological vector space and $M \subseteq A$ a closed vector subspace. Then the quotient space A/M is uniformly fundamental.

Theorem 2.7. Let X be a complete metrizable fundamental topological algebra with a uniformly bounded left approximate identity. Then every element $a \in X$ factors.

Proof. This is a simple form of the last generalized versions of Cohen's theorem which is proved in [1].

3. Completeness in fundamental topological vector spaces

As is well-known, the completion of a normed vector space A is a Banach space X such that A is dense in X and X is unique up to isomorphism. When A is a locally bounded, or a metrizable locally convex topological vector space, we can easily obtain the similar results. Obviously, every locally bounded or locally convex topological vector space is a uniformly fundamental one. What can we say about the completion of A when it is a uniformly fundamental topological vector space? Let us recall the definition of the completion of a metric space.

Definition 3.1. Let $(A; d)$ be a metric space. We say two Cauchy sequences (x_n) and (y_n) of A are equivalent if $d(x_n, y_n) \to 0$.

Theorem 3.2. Suppose X is the set of all equivalent classes of Cauchy sequences of A. Define $\rho : X \times X \to R$ by $\rho([(x_n)], [(y_n)]) = \lim d(x_n, y_n)$. Then $(X; \rho)$ is a complete metric space and A is dense in X.

This X is unique up to isomorphism and is called the completion of A.

Theorem 3.3. The completion of a metrizable topological vector space is a complete metrizable one.

Proof. The proof can be found in general references of topological vector spaces. □

Theorem 3.4. The completion of a metrizable uniformly fundamental topological vector space $(A; \tau)$ is uniformly fundamental.

Proof. Let $(X; \rho)$ be the completion of $(A; \tau)$ and $b > 1$. It suffices to show that the completion is uniformly fundamental on $\mathbf{N}$.

Let (f_n) be a Cauchy sequence of functions from $\mathbf{N}$ into X, such that $b^n(f_n - f_{n-1})$ converges uniformly on $\mathbf{N}$. For each $n \in \mathbf{N}$, the map $f_n : \mathbf{N} \to X$ is itself a sequence such that for all $p \in \mathbf{N}, f_n(p) \in X$. Put $f_n^p = f_n(p)$ and let f_n^p be the equivalent class $[(f_n^p(k))_{k \in \mathbf{N}}]$. Suppose $\epsilon > 0$. Since $b^n(f_n - f_{n-1})$ converges uniformly to zero on $\mathbf{N}$, there exists $M \in \mathbf{N}$ such that, for all $n \geq M$ and all $p \in \mathbf{N}$ we have $\rho(b^n(f_n^p - f_{n-1}^p), 0) < \epsilon$. Now, since

$$B_\rho(0, \epsilon) \subseteq \{[(x_n)] \in X : d(x_n, 0) < \epsilon, \forall n \in \mathbf{N}\},$$

we suppose $\forall n \geq M, \forall p \in \mathbf{N}$, and $\forall k \in \mathbf{N}, d(b^n(f_n^p(k) - f_{n-1}^p(k)), 0) < \epsilon$.

Now, define the sequence $g_n : \mathbf{N} \times \mathbf{N} \to A$ by

$$g_n(p, k) = f_n^p(k).$$

We have proved then, $b^n(g_n - g_{n-1})$ converges uniformly on $\mathbf{N} \times \mathbf{N}$ in $(A; \tau)$. Thus (g_n) is uniformly Cauchy on $\mathbf{N} \times \mathbf{N}$. Therefore, there exists

M such that $m, n \geq M; p, k \in \mathbf{N}$ implies that $d(f_m^p(k), f_n^p(k)) < \epsilon$, and by taking the limit with respect to k we have

$$\rho(f_m(p), f_n(p)) \leq \epsilon, \forall \epsilon > 0, \forall p \in \mathbf{N},$$

i.e. (f_n) is uniformly Cauchy on $\mathbf{N}$. □

Theorem 3.5. Suppose $b > 1$ and A is a fundamental metrizable topological vector space. Then A is complete if and only if for every sequence (x_n) with the condition $b^n x_n \to 0$ in A, the series $\sum(x_n)$ is summable.

Proof. Let A be complete and (x_n) be any sequence with $b^n x_n \to 0$ in A. Put $S_n = \sum_{k=1}^n x_k$. Then $b^n(S_n - S_{n-1}) = b^n x_n \to 0$, so (S_n) is Cauchy and the series $\sum x_n$ is summable.

Conversly, suppose the condition holds and let (x_n) be any Cauchy sequence of A and let d be a transelation invariant metric compatiable with the topology on A. Put $\alpha = [b] + 1$. Choose the increasing sequence (n_k) such that $d(x_{n_{k+1}}, x_{n_k}) < \alpha^{-2k}$. Define $y_1 = x_{n_1}, y_k = x_{n_{k+1}} - x_{n_k}$. Then

$$\begin{aligned} d(\alpha^k y_k, 0) &\leq \alpha^k d(y_k, 0) = \alpha^k d(x_{n_{k+1}} - x_{n_k}, 0) \\ &= \alpha^k d(x_{n_{k+1}}, x_{n_k}) < \alpha^k \alpha^{-2k} = \alpha^{-k}, \end{aligned}$$

i.e. $\alpha^k y_k \to 0$. Since $(\frac{b}{\alpha})^k \to 0$, so $b^k y_k = (\frac{b}{\alpha})^k \alpha^k y_k \to 0$, thus $\sum y_k$ is summable. Now we have $x_{n_k} = x_{n_1} + \sum_{i=1}^{k-1}(x_{n_{i+1}} - x_{n_i}) = y_1 + \sum_{i=1}^{k-1} y_i$. Therefore, (x_{n_k}) converges and hence (x_n) converges. □

Theorem 3.6. Suppose X is a metrizable topological vector space and $A \subseteq X$ is a uniformly fundamental subspace. Then $\bar{A}$ is uniformly fundamental.

Proof. First of all we see that if X is complete, $\bar{A}$ is a complete metrizable topological vector space, being the competion of A, is uniformly fundamental one.

Now, let X^* be the completion of X and let $f : X \to X^*$ be the linear isometric embeding map. Put $Y = f(X)$. Then, $f : X \to Y$ is a linear homeomorphism, and $f(\bar{A}) = \overline{f(A)}$ where the linear space $\overline{f(A)}$ is the closure of the linear subspace $f(A)$ with respect to the topology of Y. Let Z be the closure of $f(A)$ in X^*. Since $f(A)$ is uniformly fundamental and X^* is complete, Z is uniformly fundamental. Now, since $\overline{f(A)}$ is a subspace of Z, it is uniformly fundamental, and then so is $\bar{A}$. □

Corollary 3.7. Suppose X is a metrizable topological vector space. Then X is uniformly fundamental if and only if X has a uniformly fundamental dense subspace.

We close the section with two more corollaries which reduce the conditions of the last version of the Cohen factorization theorem (3.4).

Corollary 3.8. Let X be a complete metrizable topological algebra with a uniformly bounded left approximate identity. If X has a dense uniformly fundamental subspace, then X facrors.

Corollary 3.9. Let A be a uniformly fundamental metrizable topological algebra with a uniformly bounded left approximate identity. For each $a \in A$, there exist Cauchy sequences $(x_n), (y_n)$ from A such that $a = \lim x_n y_n$.

Proof. Let X be the completion of A. Since A is dense in X, the approximate identity for A is an approximate identity for X with the same sort of boundedness. Now, the rest follows from theorems (3.4) and (2.7). □

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Department of Mathematics,
Faculty of science,
Guilan University, P.O.Box 1914,
RASHT,IRAN.
E-mail: eansari@guilan.ac.ir