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GENERALIZED VECTOR QUASIVARIATIONAL-LIKE INEQUALITIES

MEE-KWANG KANG AND BYUNG-SOO LEE*

Abstract. In this paper, we introduce two kinds of generalized vector quasivariational-like inequalities for multivalued mappings and show the existence of solutions to those variational inequalities under compact and non-compact assumptions, respectively.

1. Introduction and Preliminaries

A vector variational inequality problem was firstly introduced in a finite dimensional Euclidean space with its applications by Giannessi [9]. Later, many authors [1-6, 9, 10, 13-17, 21-25] have extensively studied the problem in infinite dimensional spaces under different assumptions. In particular, vector variational-like inequalities were considered in [1-2, 10, 13, 15] and vector quasivariational inequalities were considered in [3-6, 10, 13, 14, 17, 23-25].

In this paper we introduce two kinds of generalized vector quasivar iational-like inequality problems for multivalued mappings and show the existence of solutions to our inequality problems.

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* Corresponding author.

Let X and Y be topological spaces, and $F : X \to 2^Y$ a multivalued mapping.

Definition 1.1. F is called upper semi-continuous (in short, u.s.c.) at $x \in X$ if for each open set V in Y containing $F(x)$, there is an open set U containing x such that $F(u) \subseteq V$ for all $u \in U$; F is called u.s.c. on X if F is u.s.c. at every point of X. F is called lower semi-continuous (in short, l.s.c.) at $x \in X$ if for each open set V in Y with $F(x) \cap V \neq \emptyset$, there is an open set U containing x such that $F(u) \cap V \neq \emptyset$ for all $u \in U$; F is called l.s.c. on X if F is l.s.c. at every point of X. F is called continuous at $x \in X$ if F is both u.s.c. and l.s.c. at $x \in X$.

Lemma 1.1. F is l.s.c. at $x \in X$ if and only if for any $y \in F(x)$ and for any net $\{x_\alpha\}$ in X converging to x, there is a net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ for each α, and $\{y_\alpha\}$ converges to y.

Definition 1.2. F is called closed if the graph $G_rF = \{(x, y) \in X \times Y : y \in F(x)\}$ of F is closed in $X \times Y$, i.e., for each $x \in X$, $\{x_\alpha\} \subset X$ with $x_\alpha \to x$ and each $\{y_\alpha\} \subset Y$ with $y_\alpha \in F(x_\alpha)$ and $y_\alpha \to y$, then we have $y \in F(x)$.

Definition 1.3. F is called compact if $F(X)$ is contained in some compact subset of Y.

Definition 1.4. Let $F^- : Y \to 2^X$ be a multivalued mapping defined by

$$x \in F^-(y) \quad \text{if and only if} \quad y \in F(x).$$

F is said to have *open lower sections* if for each $y \in Y$, $F^-(y)$ is open in X.

In an ordered Hausdorff topological vector space Z, usually a closed convex pointed solid proper cone P in Z defines partial orders $<$ and $\leq$ as

$$x <_P y \quad \text{iff} \quad x - y \in -intP$$

$$x \leq_P y \quad \text{iff} \quad x - y \in -P$$

for x, $y \in Z$. To an arbitrary subset C of Z, the orders can be extended by setting

$$C <_P 0 \quad \text{iff} \quad C \subseteq -intP$$

$$C \leq_P 0 \quad \text{iff} \quad C \subseteq -P.$$

A point z_0 in a nonempty subset C of Z is called a *vector maximal point of* C [27] if the set $\{z \in C : z_0 \leq_P z, z \neq z_0\} = \emptyset$, which is equivalent to

$$C \cap (z_0 + P) = \{z_0\}.$$

The following simple fact needed in our research was first introduced by Luc;

Lemma 1.2 [18] Let C be a nonempty compact subset of an ordered Banach space Z. Then $\max C \neq \emptyset$, where $\max C$ denotes the set of all vector maximal points of C.

2. Main results

Now we introduce P-convexity of a two variable function, which is an essential concept to our results.

Definition 2.1. Let K be a nonempty convex subset of a vector space X, and P a pointed, closed convex cone in a topological vector space Z, which has an apex at the origin and a nonempty interior $intP$. A multivalued mapping $H : K \times K \to 2^Z$ is said to be P-convex with respect to the first variable if for x_1, x_2, $y \in K$, $u_1 \in H(x_1, y)$, $u_2 \in$

$H(x_2, y)$ and $\lambda \in [0,1]$, there exists $u \in H(\lambda x_1 + (1-\lambda)x_2, y)$ such that

$$\lambda u_1 + (1-\lambda)u_2 \in u + P.$$

Throughout this section, X, Y denote two Hausdorff topological vector spaces, and Z denotes an ordered Hausdorff topological vector space. Let K be a nonempty convex subset of X, D a nonempty subset of Y and $\{C(x)|x \in K\}$ a family of solid convex cones in Z, that is, for each $x \in K$, $intC(x)$ is nonempty and $C(x) \neq Z$. $L(X, Z)$ denotes the space of all continuous linear operators from X to Z. Let $F : K \to 2^D$, $G : K \to 2^K$, $M : K \times D \to 2^{L(X,Z)}$ and $H : K \times K \to 2^Z$ be multivalued mappings, and $\eta : X \times X \to X$ a mapping.

We consider the following two kinds of generalized vector quasivaria tional-like inequalities for multivalued mappings;

(VQVLI)$_1$ Find $\bar{x} \in K$ such that for each $x \in K$ there exists $\bar{s} \in F(\bar{x})$ satisfying the following inequality;

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle + u \notin -intC(\bar{x})$$

for any $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$, where

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle >= \max_{s \in M(\bar{x}, \bar{s})} \langle s, \eta(x, z)\rangle$$

and $\langle s, \eta(x, z)\rangle$ denotes the evaluation of a continuous linear operator s from X into Z at $\eta(x, z)$,

(VQVLI)$_2$ Find $\bar{x} \in K$ and $\bar{s} \in F(\bar{x})$ such that

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle + u \notin -intC(\bar{x})$$

for $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$.

Putting $H \equiv \bar{0}$ in **(VQVLI)$_1$** and **(VQVLI)$_2$**, we obtain the following vector quasivariational-like inequalities;

(VQVLI)$'_1$ Find $\bar{x} \in K$ such that for any $x \in K$ there exists $\bar{s} \in F(\bar{x})$ satisfying the following inequality;

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle \notin -intC(\bar{x})$$

for $z \in G(\bar{x})$ and

(VQVLI)$'_2$ Find $\bar{x} \in K$ and $\bar{s} \in F(\bar{x})$ such that

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle \notin -intC(\bar{x})$$

for $x \in K$ and $z \in G(\bar{x})$.

By replacing Y, $H : K \times K \to 2^Z$ and $M : K \times D \to 2^{L(X,Z)}$ with Z, $H : K \times K \to Z$ and $S : K \to 2^{L(X,Z)}$, respectively in **(VQVLI)$_1$** and **(VQVLI)$_2$**, we obtain the following vector variational-like inequalities for multivalued mappings;

(VVLI) Find $\bar{x} \in K$ satisfying the following inequality;

$$\max\langle S(\bar{x}), \eta(x, z)\rangle + H(x, \bar{x}) \notin -intC(\bar{x})$$

for $x \in K$ and $z \in G(\bar{x})$.

Putting $H \equiv \bar{0}$ and $G(\bar{x}) = K$ in **(VVLI)**, we obtain the following vector variational-like inequalities for multivalued mappings, introduced and studied by Chang, Thompson and Yuan [2];

(VVLI)$'$ Find $\bar{x} \in K$ satisfying the following inequality;

$$\max\langle S(\bar{x}), \eta(x, \bar{x})\rangle \notin -intC(\bar{x}) \quad \text{for} \quad x \in K.$$

Putting $Z = Y$, $\eta(x, z) = x - z$ and $H = \bar{0}$, and replacing $M : K \times D \to 2^{L(X,Z)}$ with $S : K \to L(X, Y)$ in **(VQVLI)$_1$** and **(VQVLI)$_2$**, we have the following variational inequality;

(VVI) Find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - z\rangle \notin -intC(\bar{x}) \quad \text{for } x \in K \text{ and } z \in G(\bar{x}).$$

Putting $C(x) \equiv C$ for $x \in K$ and $\eta(x,y) = x - y$ in **(VVLI)$'$**, we obtain the following vector-valued variational inequality considered by Lee et al. [16];

Find $\bar{x} \in K$ such that for each $x \in K$, there exists $\bar{s} \in S(\bar{x})$ such that

$$\langle \bar{s}, x - \bar{x}\rangle \not\geq_{-intC} 0,$$

where $x \not\geq_P y$ means $x - y \notin P$.

Putting $Z = \mathbb{R}$, $L(X,Z) = X^*$, the dual of X and $C(x) \equiv \mathbb{R}^+$, the positive orchant for $x \in K$ in **(VVLI)$'$**, we obtain the following scalar-valued variational inequality considered by Cottle and Yao [7], Isac [12], and Noor [19];

Find $\bar{x} \in K$ such that

$$\sup_{u \in S(\bar{x})} \langle u, \eta(x, \bar{x})\rangle \geq 0, \quad \text{for } x \in K.$$

Replacing $S : K \to 2^{L(X,Z)}$ with $S : X \to L(X,Z)$ and putting $\eta(x,z) = x - g(z)$, where $g : K \to K$ is a mapping, then **(VVLI)$'$** reduces to the following vector variational inequality **(VVI)** considered by Siddiqi et al. [22];

(VVI)$'$ Find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - g(\bar{x})\rangle \not\geq_{-intC(\bar{x})} 0, \quad \text{for } x \in K.$$

Putting $G(x) = \{x\}$ for $x \in K$ in **(VVI)** or $g(x) = x$ for $x \in K$ in **(VVI)$'$**, we obtain the following vector-valued variational inequality considered by Chen [3];

Find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - \bar{x} \rangle \not\geq_{-intC(\bar{x})} 0, \quad \text{for } x \in K.$$

Putting $C(x) \equiv C$ and $g(x) = x$ for $x \in K$ in **(VVI)**$'$, we obtain the following vector-valued variational inequality considered by Chen et al. [3-5];

Find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - \bar{x} \rangle \not\geq_{-intC} 0, \quad \text{for } x \in K.$$

Putting $Z = \mathbb{R}$, $X = \mathbb{R}^n$, $C(x) \equiv \mathbb{R}^+$ for $x \in K \subseteq \mathbb{R}^n$, $L(X, Z) = \mathbb{R}^n$ and $\eta(x, y) = x - y$, we obtain the following scalar-valued variational inequality considered by Hartman and Stampacchia [11]; find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - \bar{x} \rangle \geq 0 \quad \text{for } x \in K.$$

2.1. Compact set case

When we consider the existence of solutions to $\mathbf{(VQVLI)}_1$ for the compact set case, Ky Fan's Section Theorem in [8] is very useful and indispensable.

Theorem 2.1 [8]. Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let A be a subset of $K \times K$ having the following properties

(i) $(x, x) \in A$ for all $x \in K$;

(ii) for any $x \in K$, the set $A_x := \{y \in K : (x, y) \in A\}$ is closed in K;

(iii) for any $y \in K$, the set $A^y := \{x \in K : (x, y) \notin A\}$ is convex or empty in K.

Then there exists $\bar{y} \in K$ such that $K \times \{\bar{y}\} \subset A$.

The following main theorem for the existence of solutions to **(VQVLI)**$_1$ is for the compact set case.

Theorem 2.2. Let K be a nonempty compact convex subset of X and D a nonempty subset of Y. Let $F : K \to 2^D$ be closed, $G : K \to 2^K$ be l.s.c. and nonempty convex-valued, $M : K \times D \to 2^{L(X,Z)}$ be nonempty compact-valued, and a multivalued mapping $W : K \to 2^Z$ defined by $W(x) = Z \backslash \{-intC(x)\}$, $x \in K$, closed. Let $\eta : X \times X \to X$ be linear, and $H : K \times K \to 2^Z$ be P-convex with respect to the first variable and l.s.c. with respect to the second, where $P := \bigcap_{x \in K} C(x)$.

Suppose further that

(1) $\langle M(x, \cdot), \eta(x, \cdot)\rangle = 0$ and $H(x, x) = \{0\}$ for all $x \in K$;

(2) F is compact; and

(3) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha)\rangle$ converges to $\max\langle M(y, s), \eta(x, z)\rangle$ provided that $y_\alpha \to y$, $s_\alpha \to s$ and $z_\alpha \to z$.

Then **(VQVLI)**$_1$ is solvable.

Proof. By the assumption that M is nonempty compact-valued, from the continuity of $\langle \cdot, \cdot \rangle$, $\langle M(y, s), \eta(x, z)\rangle$ is compact in Z. So we can define $A = \{(x, y) \in K \times K$: there exists $s \in F(y)$ such that $\max\langle M(y, s), \eta(x, z)\rangle + u \notin -intC(y)$ for any $z \in G(y)$ and $u \in H(x, y)\}$. By the condition (1), it is easily shown that $(x, x) \in A$ for all $x \in K$. Next, $A_x = \{y \in K : (x, y) \in A\}$, $x \in K$ is closed. In fact, let $\{y_\alpha\}$ be a net in A_x such that $y_\alpha \to y$. Then by Lemma 1.1, for any $z \in G(y)$ there exists a net $\{z_\alpha\}$ converging to z such that $z_\alpha \in G(y_\alpha)$ for each α. Also by the lower semi-continuity of H with respect to the second variable, for any $u \in H(x, y)$ there exists a net $\{u_\alpha\}$ converging to u such that $u_\alpha \in H(x, y_\alpha)$ for each α. Since $y_\alpha \in A_x$ we can choose $s_\alpha \in F(y_\alpha)$ such that

$$\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha)\rangle + u_\alpha \in W(y_\alpha)$$

for $z_\alpha \in G(y_\alpha)$ and $u_\alpha \in H(x, y_\alpha)$. By the condition (2) and the closedness of F, we can assure the existence of limit s of $\{s_\alpha\}$ such that $s \in F(y)$. Hence by the condition (3) and the closedness of W, we have

$$\max\langle M(y, s), \eta(x, z)\rangle + u \in W(y)$$

for any $z \in G(y)$ and $u \in H(x, y)$. Finally, $A^y = \{x \in K : (x, y) \notin A\}$, $y \in K$ is convex. Indeed, let x_1, $x_2 \in A^y$ and $\lambda \in [0, 1]$. Then from the fact that $(x_1, y) \notin A$, for any $s \in F(y)$ there exist $z_1 \in G(y)$ and $u_1 \in H(x_1, y)$ such that

$$\max\langle M(y, s), \eta(x_1, z_1)\rangle + u_1 \in -intC(y)$$

and from the fact that $(x_2, y) \notin A$, for any $s \in F(y)$ there exist $z_2 \in G(y)$ and $u_2 \in H(x_2, y)$ such that

$$\max\langle M(y, s), \eta(x_2, z_2)\rangle + u_2 \in -intC(y).$$

Hence, for any $s \in F(y)$ there exist $u \in H(\lambda x_1 + (1 - \lambda)x_2, y)$ and $z := \lambda z_1 + (1 - \lambda)z_2 \in G(y)$ for $\lambda \in [0, 1]$ such that

$$\begin{aligned}
&\max\langle M(y, s), \eta(\lambda x_1 + (1 - \lambda)x_2, z)\rangle + u \\
&= \max\langle M(y, s), \eta(\lambda x_1 + (1 - \lambda)x_2, \lambda z_1 + (1 - \lambda)z_2)\rangle + u \\
&= \max\langle M(y, s), \lambda\eta(x_1, z_1) + (1 - \lambda)\eta(x_2, z_2)\rangle + u \\
&\leq \lambda \max\langle M(y, s), \eta(x_1, z_1)\rangle + (1 - \lambda) \max\langle M(y, s), \eta(x_2, z_2)\rangle + u \\
&\in \lambda \max\langle M(y, s), \eta(x_1, z_1)\rangle + (1 - \lambda) \max\langle M(y, s), \eta(x_2, z_2)\rangle + \lambda u_1 \\
&\quad + (1 - \lambda)u_2 - P \\
&= \lambda(\max\langle M(y, s), \eta(x_1, z_1)\rangle + u_1) + (1 - \lambda)(\max\langle M(y, s), \eta(x_2, z_2)\rangle \\
&\quad + u_2) - P \\
&\subseteq -intC(y) - intC(y) - C(y) \\
&= -intC(y).
\end{aligned}$$

Thus $\lambda x_1 + (1-\lambda)x_2 \in A^y$, which shows that A^y is convex. Hence by Ky Fan's Section Theorem there exists $\bar{x} \in K$ such that

$$K \times \{\bar{x}\} \subset A,$$

which implies that for any $x \in K$, there exists $\bar{s} \in F(\bar{x})$ such that

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle + u \notin -intC(\bar{x})$$

for $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$. This completes the proof.

2.2. Noncompact set case

For considering the existence of solutions to **(VQVLI)**$_2$ for noncompact set case, we use the following particular form of the generalized Ky Fan's Section Theorem due to Park [20].

Theorem 2.3. Let K be a nonempty convex subset of X and $A \subset K \times K$ satisfy the following conditions;

(i) $(x, x) \in A$, $x \in K$;

(ii) $A_x = \{y \in K : (x, y) \in A\}$, $x \in K$, is closed;

(iii) $A^y = \{x \in K : (x, y) \notin A\}$, $y \in K$, is convex or empty;

(iv) there exists a nonempty compact subset B of K such that for each finite subset N of K there exists a nonempty compact convex subset L_N of K containing N such that

$$L_N \cap \{y \in K : (x, y) \in A \text{ for any } x \in L_N\} \subset B.$$

Then there exists a $y_0 \in B$ such that $K \times \{y_0\} \subset A$.

In particular, if $K = B$, that is, K is a compact convex subset of X, then the condition (iv) is obviously true, thus the three conditons of Ky Fan's Section Theorem are sufficient to show the existence of $y_0 \in K$ such that $K \times \{y_0\} \subset A$.

To show the existence of solutions to **(VQVLI)**$_2$ for the noncompact set case, the following lemmas are essential .

Lemma 2.4. Let K be a nonempty convex subset of X and D be a nonemtpy subset of Y. Let $f : K \to D$ be a continuous function, $M : K \times D \to 2^{L(X,Z)}$ be nonempty compact-valued, and $G : K \to 2^K$ a l.s.c. mapping with nonempty convex-values. Let a multivalued mapping $W : K \to 2^Z$ defined by $W(x) = Z\backslash\{-intC(x)\}$, $x \in K$, be closed. Let $\eta : X \times X \to X$ be linear and $H : K \times K \to 2^Z$ P-convex with respect to the first variable and l.s.c. with respect to the second, where $P = \bigcap_{x\in K} C(x)$. Suppose further that

(1) $\langle M(x,\cdot), \eta(x,\cdot)\rangle = 0$ and $H(x,x) = \{0\}$ for all $x \in K$,

(2) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha)\rangle$ converges to $\max\langle M(y,s), \eta(x,z)\rangle$ provided that $y_\alpha \to y$, $s_\alpha \to s$ and $z_\alpha \to z$;

(3) there is a nonempty compact subset B of K such that for each nonempty finite subset N of K, there is a nonempty compact convex subset L_N of K containing N such that for $y \in L_N\backslash B$, there exist $x \in L_N$, $z \in G(y)$ and $u \in H(x,y)$ such that

$$\max\langle M(y, f(y)), \eta(x,z)\rangle + u \in -intC(y).$$

Then there exists $\bar{x} \in K$ such that

$$\max\langle M(\bar{x}, f(\bar{x})), \eta(x,z)\rangle + u \notin -intC(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x,\bar{x})$.

Proof. Let $A = \{(x,y) \in K \times K : \max\langle M(y, f(y)), \eta(x,z)\rangle + u \notin -intC(y)$ for any $z \in G(y)$ and $u \in H(x,y)\}$. It is easily shown that $(x,x) \in A$ for $x \in K$ from the condition (2). And $A_x = \{y \in K : (x,y) \in A\}$, $x \in K$, is closed. In fact, for any net $\{y_\alpha\}$ in A_x converging to y, we have $\max\langle M(y_\alpha, f(y_\alpha)), \eta(x, z_\alpha)\rangle + u_\alpha \notin -intC(y_\alpha)$ for any $z_\alpha \in G(y_\alpha)$ and $u_\alpha \in H(x, y_\alpha)$. From Lemma 2.1 and the condition

(1), $\max\langle M(y,f(y)),\eta(x,z)\rangle + u \notin -intC(y)$ for any $z \in G(y)$ and $u \in H(x,y)$, we have $y \in A_x$, showing the closedness of A_x for $x \in K$. By a similar method shown in the proof of Theorem 2.2, we can show that the set $A^y = \{x \in K | (x,y) \notin A\}$, $y \in K$, is convex. Further note that the assumption (3) implies that for $y \in L_N \backslash B$ there exists $x \in L_N$ such that $y \notin A_x$. Hence the condition (iv) of Theorem 2.3 is satisfied. Hence there exists $\bar{x} \in K$ such that

$$\max\langle M(\bar{x},f(\bar{x})),\eta(x,z)\rangle + u \notin -intC(\bar{x})$$

for $x \in K$, $z \in G(\bar{x})$ and $u \in H(x,\bar{x})$. This completes the proof.

Lemma 2.5 [26]. Let X be a paracompact Hausdorff topological space and Y a topological vector space. Let $F : X \to 2^Y$ be a multivalued mapping with nonempty convex-values. If F has open lower sections, then there exists a continuous function $f : X \to Y$ such that $f(x) \in F(x)$ for $x \in X$.

Now we consider the existence of solution to $\mathbf{(VQVLI)}_2$.

Theorem 2.6. Let K be a nonempty paracompact convex subset of X and D a nonemtpy convex subset of Y. Let $F : K \to 2^D$ have nonempty convex-values and open lower sections, $G : K \to 2^K$ be a l.s.c. mapping with nonempty convex-values, $M : K \times D \to 2^{L(X,Z)}$ be nonempty compact-valued, and $W : K \to 2^Z$ defined by $W(x) = Z \backslash \{-intC(x)\}$, $x \in K$, closed. Let $\eta : X \times X \to X$ be linear and $H : K \times K \to 2^Z$ be P-convex with respect to the first variable and l.s.c. with respect to the second, where $P = \bigcap_{x \in K} C(x)$.

Suppose further that

(1) $\langle M(x,\cdot),\eta(x,\cdot)\rangle = 0$ and $H(x,x) = \{0\}$ for all $x \in K$,

(2) $\max\langle M(y_\alpha,s_\alpha),\eta(x,z_\alpha)\rangle \to \max\langle M(y,s),\eta(x,z)\rangle$ provided that $y_\alpha \to y$, $s_\alpha \to s$ and $z_\alpha \to z$,

(3) F is compact,

(4) there is a nonempty compact subset B of K such that for any nonempty finite subset N of K, there is a nonempty compact convex subset L_N of K containing N such that for any $y \in L_N \setminus B$, there exist $x \in L_N$, $z \in G(y)$ and $u \in H(x, y)$ such that

$$\max\langle M(y, s), \eta(x, z)\rangle + u \in -intC(y)$$

for any $s \in F(y)$.
Then **(VQVLI)**$_2$ is solvable, i.e., there exist $\bar{x} \in K$ and $\bar{s} \in F(\bar{x})$ such that

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle + u \notin -intC(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$.

Proof. Since $F^{-}(y)$ is open in X for $y \in D$, by Lemma 2.5 there exists a continuous function $f : K \to D$ such that $f(x) \in F(x)$ for $x \in K$. So, by Lemma 2.4 there exists $\bar{x} \in K$ such that

$$\max\langle M(\bar{x}, f(\bar{x})), \eta(x, z)\rangle + u \notin -intC(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$. This completes the proof.

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Mee-Kwang Kang
Department of Mathematics,
Dongeui University,
Busan 614-714, Korea
E-mail:mee@deu.ac.kr

Byung-Soo Lee
Department of Mathematics,
Kyungsung University,
Busan 608-736, Korea
E-mail:bslee@ks.ac.kr