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EXPONENTIAL FORMULA FOR C REGULARIZED SEMIGROUPS

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Abstract. In this paper, we show that C-resolvent of generator can be represented by Laplace transform and establish an exponential formula for C regularized semigroups whose antiderivatives are exponentially bounded.

1. Introduction

Let X be a Banach space. Consider the following abstract Cauchy problem

$$\text{(ACP)} \qquad \frac{du}{dt} = Au, \quad u(0) = x,$$

where A is a linear operator in X.

Let A be the generator of a C_0 semigroup $\{T(t) : t \geq 0\}$ on X. Then the solution of (ACP) is given by $u(t) = T(t)x$ for all $x \in X$ and the C_0 semigroup $T(t)$ is given by the exponential formula $T(t)x = \lim_{n\to\infty}(I - tA/n)^{-n}x$ for all $x \in X$. Moreover, $u_n(t) = (I - tA/n)^{-n}x$ is the solution of an implicit difference approximation of (ACP) and is an approximation of the solution of (ACP) (see [4, 5]). To establish the exponential formula, the operator A must be a generator of a C_0 semigroup, in particular, A is densely defined and has nonempty resolvent set. But C_0 semigroup theory is not always sufficient for the application

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to (ACP) and so several generalizations of C_0 semigroup have been introduced and developed, e.g., existence families and C regularized semigroup, etc (see [3, 4]). In particular, operators with empty resolvent set occur in Petrovskii correct system of partial differential equation and the C regularized semigroup theory is useful to treat systems of this type.

In this paper, we establish the exponential formula for C regularized semigroup, which may not be exponentially bounded. In the case of exponentially bounded C regularized semigroup, the exponential formula can be established by using C_0 semigroup on a Hille-Yosida space for the generator A. In the next section, we give a direct proof of the exponential formula for C regularized semigroup whose antiderivative is exponentially bounded.

Throughout this paper, X is always a Banach space, $B(X)$ is the space of all bounded linear operators on X, C is a bounded linear injective operator on X and M, ω are constants. For an operator A, $D(A)$ and $R(A)$ are the domain and range of A, respectively.

2. C regularized semigroups and exponential formula

First, we recall the definition and basic properties of C regularized semigroup. For more information, see [3].

Definition 1. The strong continuous family $\{S(t) : t \geq 0\}$ of $B(X)$ is a C-regularized semigroup if $S(0) = C$ and $S(t)S(s) = CS(t+s)$ for all $s,\ t \geq 0$.

An operator A is called the generator of $\{S(t) : t \geq 0\}$ if

$$Ax = C^{-1}\left(\lim_{h \to 0} \frac{1}{h}(S(h)x - Cx)\right)$$

with the maximal domain $D(A)$.

The complex number λ is in $\rho_C(A)$, the C-resolvent set of A, if $\lambda - A$ is injective and $R(C) \subset R(\lambda - A)$.

Lemma 2. Let A be the generator of a C-regularized semigroup $\{S(t) : t \geq 0\}$. Then

(1) A is closed and $R(C) \subset \overline{D(A)}$

(2) if $f : [0, \infty) \to X$ is continuously differentiable, then $\int_0^t S(s)f(s)ds \in D(A)$ and

$$A\left(\int_0^t S(s)f(s)ds\right) = S(t)f(t) - Cf(0) - \int_0^t S(s)f'(s)ds.$$

By Lemma 2, we know $u(t) = S(t)x$ is a mild solution of (ACP) with $u(0) = Cx$ for all $x \in X$, that is,

$$A\left(\int_0^t S(s)xds\right) = S(t)x - Cx$$

for all $x \in X$ and $t \geq 0$.

Theorem 3. Let A be the generator of a C-regularized semigroup $\{S(t) : t \geq 0\}$ satisfying $|| \int_0^t S(s)ds|| \leq Me^{\omega t}$ for all $t \geq 0$. Then

(1) $(\omega, \infty) \subset \rho_C(A)$

(2) $S(t)x \in R(\lambda - A)$ and $(\lambda - A)^{-1}S(t)x = \int_0^\infty e^{-\lambda s}S(t+s)xds$ for all $x \in X$ and $\lambda > \omega$.

(3) $R(C) \subset R((\lambda - A)^n)$ and

$$(\lambda - A)^{-n}Cx = \frac{1}{(n-1)!}\int_0^\infty e^{-\lambda t}t^{n-1}S(t)xdt$$

for all $n \in N$, $\lambda > \omega$ and $x \in X$.

Proof. Let $x \in X$ and let $\lambda > \omega$. By Lemma 2 and integration by part, we have

$$\begin{aligned} A(\int_0^\infty e^{-\lambda t}S(t)xdt) &= \lambda\int_0^\infty e^{-\lambda t}A(\int_0^t S(s)xds)dt \\ &= \lambda\int_0^\infty e^{-\lambda t}S(t)xdt - Cx. \end{aligned}$$

Thus we have $(\lambda - A)\int_0^\infty e^{-\lambda t}S(t)xdt = Cx$. Since C is injective, $\lambda \in \rho_C(A)$ and $(\lambda - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}S(t)xdt$.

Note that

$$\begin{aligned}\int_0^\infty e^{-\lambda s}S(t+s)xds &= e^{\lambda t}\int_t^\infty e^{-\lambda s}S(s)xds \\ &= e^{\lambda t}(\lambda - A)^{-1}Cx - e^{\lambda t}\int_0^t e^{-\lambda s}S(s)xds \in D(A).\end{aligned}$$

By Lemma 2,

$$A\left(\int_0^t e^{-\lambda s}S(s)xds\right) = e^{-\lambda t}S(t)x - Cx + \lambda\int_0^t e^{-\lambda s}S(s)xds$$

and we have

$$\begin{aligned}&A\int_0^\infty e^{-\lambda s}S(t+s)xds \\ &= e^{\lambda t}A(\lambda - A)^{-1}Cx - e^{\lambda t}A\int_0^t e^{-\lambda s}S(s)xds \\ &= e^{\lambda t}(\lambda(\lambda - A)^{-1}Cx - Cx) \\ &\quad - e^{\lambda t}(e^{-\lambda t}S(t)x - Cx + \lambda\int_0^t e^{-\lambda s}S(s)xds) \\ &= \lambda e^{\lambda t}\int_t^\infty e^{-\lambda s}S(s)xds - S(t)x.\end{aligned}$$

Therefore, we have

$$(\lambda - A)\int_0^\infty e^{-\lambda s}S(t+s)xds = S(t)x$$

and (2) follows.

To prove (3), we use induction. Assume that

$$(\lambda - A)^{-k}Cx = \frac{1}{(k-1)!}\int_0^\infty e^{-\lambda t}t^{k-1}S(t)xdt.$$

By closedness of $(\lambda - A)^{-1}$ and Fubini's theorem

$$\begin{aligned}
(\lambda - A)^{-(k+1)}Cx &= \frac{1}{(k-1)!}(\lambda - A)^{-1}\int_0^\infty e^{-\lambda t}t^{k-1}S(t)xdt \\
&= \frac{1}{(k-1)!}\int_0^\infty e^{-\lambda t}t^{k-1}\int_0^\infty e^{-\lambda s}S(t+s)xdsdt \\
&= \frac{1}{(k-1)!}\int_0^\infty t^{k-1}\int_t^\infty e^{-\lambda u}S(u)xdudt \\
&= \frac{1}{(k-1)!}\int_0^\infty \int_0^u t^{k-1}e^{-\lambda u}S(u)xdtdu \\
&= \frac{1}{k!}\int_0^\infty e^{-\lambda u}u^k S(u)xdu.
\end{aligned}$$

By Theorem 3, we know that $(\lambda - A)^{-1}Cx$ is the Laplace transform of $S(t)x$, and so we have

$$\frac{d^n}{d\lambda^n}(\lambda - A)^{-1}Cx = (-1)^n \int_0^\infty e^{-\lambda t}t^n S(t)xdt.$$

Thus we have

$$(\lambda - A)^{-(n+1)}Cx = (-1)^n \frac{1}{n!}\frac{d^n}{d\lambda^n}(\lambda - A)^{-1}Cx.$$

By the Post-Widder inversion theorem

$$\begin{aligned}
S(t)x &= \lim_{n\to\infty}(-1)^n \frac{1}{n!}\left(\frac{n}{t}\right)^{n+1}\frac{d^n}{d\lambda^n}(\lambda - A)^{-1}Cx\,|_{\lambda=n/t} \\
&= \lim_{n\to\infty}(I - \frac{t}{n}A)^{-(n+1)}Cx
\end{aligned}$$

for $x \in X$. In the C_0 semigroup theory ($C = I$), n/t is in the resolvent set of A and $\lim_{n\to\infty} n/t(n/t - A)^{-1}x = x$. So we have the exponential formula for C_0 semigroup. Since the resolvent set of the generator of C regularized semigroup may be empty, this argument is not valid.

Let A is the generator of bounded strongly uniformly continuous C regularized semigroup. In [3], deLaubenfels have introduced Hille-Yosida space Z_0 for A and showed that $A|_{Z_0}$ is the generator of the contraction C_0 semigroup $\{T(t) : t \geq 0\}$ on Z_0 and $S(t)x = T(t)Cx$ for all $x \in X$. This argument can be extended to exponentially bounded C regularized semigroup. So we can establish the exponential formula for C regularized semigroups which are exponentially bounded by using C_0 semigroup on a Hille-Yosida space for A.

Next, we present exponential formula for C regularized semigroup whose antiderivative is exponentially bounded. Our result includes the exponential formula for exponentially bounded C regularized semigroup. We give a direct proof of our exponential formula without using a Hille-Yosida space and its proof is a modification of that of Post-Widder inversion theorem in [1].

Theorem 4. Let A be the generator of C regularized semigroup $\{S(t) : t \geq 0\}$ satisfying $|| \int_0^t S(s)ds|| \leq Me^{\omega t}$ for all $t \geq 0$. Then

$$\lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} Cx = S(t)x$$

for all $x \in X$.

Proof. By Theorem 3 and integration by part we have

$$\begin{aligned}
&\lambda^n(\lambda - A)^{-n}Cx - S(t)x \\
&= \frac{\lambda^n}{(n-1)!}\int_0^\infty e^{-\lambda s}s^{n-1}S(s)xds - S(t)x \\
&= \frac{\lambda^n}{(n-1)!}\int_0^\infty e^{-\lambda s}s^{n-1}(S(s)x - S(t)x)ds \\
&= \frac{\lambda^n}{(n-1)!}\int_0^\infty (\lambda s - (n-1))e^{-\lambda s}s^{n-2}\int_t^s (S(r)x - S(t)x)drds \\
&= \frac{\lambda^n}{(n-1)!}\int_0^\infty (\lambda s - (n-1))e^{-\lambda s}s^{n-2}F(s)ds,
\end{aligned}$$

where $F(s) = \int_t^s (S(r)x - S(t)x)dr$.

Let $t > 0$ and let $\lambda = n/t$. By change of variable, we have

$$\begin{aligned}
&\left(I - \frac{n}{t}A\right)^{-n} Cx - S(t)x \\
&= \frac{n^n}{(n-1)!}\frac{1}{t}\int_0^\infty (n(u-1)+1)e^{-nu}u^{n-2}F(tu)du.
\end{aligned}$$

Let $\varepsilon > 0$ be given. Since $S(t)x$ is continuous in $t \geq 0$, there exists $0 < \delta < 1$ such that $|u-1| < \delta$ implies

$$||F(tu)|| \leq \int_t^{tu} ||S(r)x - S(t)x||\, dr \leq t|u-1|\varepsilon.$$

By the exponential boundedness of the antiderivative of $S(t)$, there exists a constant M_1 such that $||F(s)|| \leq M_1 e^{\omega s}$.

Therefore,

$$\begin{aligned}
&|| \left(I - \frac{n}{t}A\right)^{-n} Cx - S(t)x|| \\
\leq\ & \frac{n^n}{(n-1)!}\frac{1}{t}\int_0^{\infty} |n(u-1)+1| e^{-nu}u^{n-2}||F(tu)||\, du \\
=\ & \frac{n^n}{(n-1)!}\frac{1}{t}\int_0^{1-\delta} |n(u-1)+1| e^{-nu}u^{n-2}||F(tu)||\, du \\
& + \frac{n^n}{(n-1)!}\frac{1}{t}\int_{1-\delta}^{1+\delta} |n(u-1)+1| e^{-nu}u^{n-2}||F(tu)||\, du \\
& + \frac{n^n}{(n-1)!}\frac{1}{t}\int_{1+\delta}^{\infty} |n(u-1)+1| e^{-nu}u^{n-2}||F(tu)||\, du \\
:=\ & I_1 + I_2 + I_3.
\end{aligned}$$

Since $u < 1-\delta < 1$ and $n(u-1)+1 < 0$ for $n > 1/\delta$, we have

$$\begin{aligned}
I_1 \leq\ & \frac{n^n}{(n-1)!}\frac{1}{t}\int_0^{1-\delta} (n - nu - 1)e^{-nu}u^{n-2}M_1 e^{\omega tu} du \\
\leq\ & \frac{n^n}{(n-1)!}\frac{M_1 e^{\omega t}}{t}\int_0^{1-\delta} ((n-1)u^{n-2} - nu^{n-1})e^{-nu} du \\
=\ & \frac{M_1 e^{\omega t}}{t}\frac{n^n}{(n-1)!}\left((n-1)\int_0^{1-\delta} u^{n-2}e^{-nu}du - \int_0^{1-\delta} nu^{n-1}e^{-nu}du\right) \\
=\ & \frac{M_1 e^{\omega t}}{t}\frac{n^n}{(n-1)!}\left((n-1)\int_0^{1-\delta} u^{n-2}e^{-nu}du \right.\\
& \left. + \left[e^{-nu}u^{n-1}\right]_0^{1-\delta} - (n-1)\int_0^{1-\delta} u^{n-2}e^{-nu}du\right) \\
=\ & \frac{M_1 e^{\omega t}}{t}\frac{n^n}{(n-1)!}e^{-n(1-\delta)}(1-\delta)^{n-1}.
\end{aligned}$$

Let $a_n = (M_1 e^{\omega t}/t)(n^n/(n-1)!)e^{-n(1-\delta)}(1-\delta)^{n-1}$. Then $\lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty}(1+1/n)^n e^{-(1-\delta)}(1-\delta) = e^\delta(1-\delta) < 1$. Thus $\lim_{n\to\infty} a_n = 0$.

Since $\|F(tu)\| \le t|u-1|\varepsilon$ for $|u-1| < \delta$ and $|u-1| < \delta < 1$,

$$\begin{aligned}
I_2 &\le \frac{n^n}{(n-1)!}\int_{1-\delta}^{1+\delta} |n(u-1)+1|e^{-nu}u^{n-2}|u-1|du\varepsilon \\
&\le \frac{n^n}{(n-1)!}\left(\int_{1-\delta}^{1+\delta} n(u-1)^2 e^{-nu}u^{n-2}du + \int_{1-\delta}^{1+\delta} e^{-nu}u^{n-2}|u-1|du\right)\varepsilon \\
&\le \frac{n^n}{(n-1)!}\left(\int_{1-\delta}^{1+\delta} (n(u-1)^2+1)e^{-nu}u^{n-2}du\right)\varepsilon \\
&\le \frac{n^n}{(n-1)!}\left(\int_0^\infty (nu^n - 2nu^{n-1} + (n+1)u^{n-2})e^{-nu}du\right)\varepsilon \\
&= \frac{n^n}{(n-1)!}\left(n\frac{n!}{n^{n+1}} - 2n\frac{(n-1)!}{n^n} + (n+1)\frac{(n-2)!}{n^{n-1}}\right)\varepsilon \\
&= \frac{2n}{n-1}\varepsilon
\end{aligned}$$

Note that $e^{-mu}u^m$ is decreasing on $u > 1+\delta$ for all m. Choose $n_0 > \omega t$. Since $n(u-1)+1 > 0$ for $u > 1+\delta$, for all $n > n_0$ we have

$$\begin{aligned}
I_3 &\le \frac{n^n}{(n-1)!}\frac{1}{t}\int_{1+\delta}^\infty |n(u-1)+1|e^{-nu}u^{n-2}M_1 e^{\omega tu}du \\
&= \frac{n^{n+1}}{(n-1)!}\frac{M_1}{t}\int_{1+\delta}^\infty (u-1)e^{-nu}u^{n-2}e^{\omega tu}du \\
&\quad + \frac{n^n}{(n-1)!}\frac{M_1}{t}\int_{1+\delta}^\infty e^{-nu}u^{n-2}e^{\omega tu}du \\
&= \frac{n^{n+1}}{(n-1)!}\frac{M_1}{t}\int_{1+\delta}^\infty (u-1)u^{n-n_0}e^{-(n-n_0)u}e^{-n_0 u}u^{n_0-2}e^{\omega tu}du \\
&\quad + \frac{n^n}{(n-1)!}\frac{M_1}{t}\int_{1+\delta}^\infty u^{n-n_0}e^{-(n-n_0)u}e^{-n_0 u}u^{n_0-2}e^{\omega tu}du \\
&\le \frac{n^{n+1}}{(n-1)!}\frac{M_1}{t}e^{-(n-n_0)(1+\delta)}(1+\delta)^{n-n_0}\int_{1+\delta}^\infty (u-1)e^{-n_0 u}u^{n_0-2}e^{\omega tu}du \\
&\quad + \frac{n^n}{(n-1)!}\frac{M_1}{t}e^{-(n-n_0)(1+\delta)}(1+\delta)^{n-n_0}\int_{1+\delta}^\infty e^{-n_0 u}u^{n_0-2}e^{\omega tu}du.
\end{aligned}$$

Let b_n and c_n be the first and second term in the last equation. Then

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n+2} e^{-(1+\delta)}(1+\delta) = e^{-\delta}(1+\delta) < 1$$

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n+1} e^{-(1+\delta)}(1+\delta) = e^{-\delta}(1+\delta) < 1.$$

So $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 0$.

Therefore we have

$$\lim_{n\to\infty} \left|\left| \left(I - \frac{t}{n}A\right)^{-n} Cx - S(t)x \right|\right| \leq 2\varepsilon.$$

Since ε is arbitrary, the result follows.

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