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RECURRENCE RELATIONS FOR QUOTIENT MOMENTS OF THE EXPONENTIAL DISTRIBUTION BY RECORD VALUES

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Abstract. In this paper we establish some recurrence relations satisfied by quotient moments of upper record values from the exponential distribution. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common continuous distribution function $F(x)$ and probability density function(pdf) $f(x)$. Let $Y_n = max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of $\{X_n, n \geq 1\}$, if $Y_j > Y_{j-1}, j > 1$. The indices at which the upper record values occur are given by the record times $\{u(n)\}, n \geq 1$, where $u(n) = \min\{j | j > u(n-1), X_j > X_{u(n-1)}, n \geq 2\}$ and $u(1) = 1$. Suppose $X \in Exp(1)$. Then $E\left(\dfrac{X^r_{u(m)}}{X^{s+1}_{u(n)}}\right) = \dfrac{1}{s}E\left(\dfrac{X^r_{u(m)}}{X^s_{u(n-1)}}\right) - \dfrac{1}{s}E\left(\dfrac{X^r_{u(m)}}{X^s_{u(n)}}\right)$ and $E\left(\dfrac{X^{r+1}_{u(m)}}{X^s_{u(n)}}\right) = \dfrac{1}{(r+2)}E\left(\dfrac{X^{r+2}_{u(m)}}{X^s_{u(n-1)}}\right) - \dfrac{1}{(r+2)}E\left(\dfrac{X^{r+2}_{u(m-1)}}{X^s_{u(n-1)}}\right)$.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common continuous distribution function $F(x)$ and probability density function $f(x)$. Suppose $Y_n = \max\{X_1,$

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$X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence if $Y_j > Y_{j-1}, j > 1$. We define the record times $u(n)$ by $u(1) = 1$ and

$$u(n) = \min\{j | j > u(n-1)\ ,\ X_j > X_{u(n-1)}\ ,\ n \geq 2\}.$$

The record times of the sequence $\{X_n, n \geq 1\}$ are random variables and are the same as those for the sequence $\{F(X_n), n \geq 1\}$. We know that the distribution of $u(n)$ does not depend on $F(x)$. Hence, the distribution of $u(n)$ can be determined by considering the uniform distribution $F(x) = x$. We will call the random variable $X \in Exp(1)$ if the corresponding cumulative distribution function $F(x)$ of X is of the form

$$F(x) = \begin{cases} 1 - e^{-x}, & x > 0 \\ 0 \quad , & otherwise \end{cases}$$

Characterizations of the exponential distribution have been extensively studied in the literature. Similar results have been established by Balakrishnan, Ahsanullah and Chan [4, 5] for Gumbel and generalized extreme value distribution. And Balakrishnan and Ahsanullah [2, 3] characterized for the exponential and generalized Pareto distributions. They mainly studied some recurrence relations satisfied by the single and product moments of record values.

In this paper, we will give some recurrence relations satisfied by the quotient moments of upper record values from the exponential distribution.

2. Main Results

Theorem 2.1. For $1 \leq m \leq n-2$, $r = 0, 1, 2, \cdots$ and $s = 1, 2, \cdots$,

$$E\left(\frac{X^r_{u(m)}}{X^{s+1}_{u(n)}}\right) = \frac{1}{s}\, E\left(\frac{X^r_{u(m)}}{X^s_{u(n-1)}}\right) - \frac{1}{s}\, E\left(\frac{X^r_{u(m)}}{X^s_{u(n)}}\right).$$

Proof. As in Chandler [6], the joint pdf of $X_{u(m)}$ and $X_{u(n)}$ is given by

$$f_{m,n}(x,y) = \frac{R^{m-1}(x)}{\Gamma(m)} r(x) \frac{[R(y)-R(x)]^{n-m-1}}{\Gamma(n-m)} f(y), \ -\infty < x < y < \infty,$$

where

$R(x) = -ln[1-F(x)]$, $0 < 1-F(x) < 1$ and $r(x) = R'(x) = \dfrac{f(x)}{1-F(x)}$.

First of all, the joint pdf of $X_{u(m)}$ and $X_{u(n)}$ is

$$f_{m,n}(x,y) = \frac{1}{\Gamma(m)\Gamma(n-m)} R^{m-1}(x)[R(y)-R(x)]^{n-m-1} f(y), \ 0 < x < y < \infty,$$

since the exponential distribution, $f(x) = 1 - F(x)$ and hence $r(x) = \dfrac{f(x)}{1-F(x)} = 1$.

Let us consider for $1 \le m \le n-2$, $r = 0,1,2,\cdots$ and $s = 1,2,\cdots$,

$$\begin{aligned}
E\left(\frac{X^r_{u(m)}}{X^{s+1}_{u(n)}}\right) &= \int\int_{0<x<y<\infty} \frac{x^r}{y^{s+1}} f_{m,n}(x,y)dydx \\
&= \frac{1}{\Gamma(m)\Gamma(n-m)} \int\int_{0<x<y<\infty} \\
&\quad \frac{x^r}{y^{s+1}} R^{m-1}(x)[R(y)-R(x)]^{n-m-1} f(y)dydx \\
&= \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^\infty x^r R^{m-1}(x) \\
&\quad \left(\int_x^\infty \frac{1}{y^{s+1}}[R(y)-R(x)]^{n-m-1} f(y)dy\right) dx \\
&= \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^\infty x^r R^{m-1}(x) \\
&\quad \left(\int_x^\infty \frac{1}{y^{s+1}}[R(y)-R(x)]^{n-m-1}[1-F(y)]dy\right) dx.
\end{aligned}$$

Using integrating by parts treating $\dfrac{1}{y^{s+1}}$ for integration and $[R(y) - R(x)]^{n-m-1}[1-F(y)]$ for differentiation on the second integration, we get

$$\begin{aligned}
&\int_x^\infty \frac{1}{y^{s+1}}[R(y)-R(x)]^{n-m-1}[1-F(y)]dy \\
&\quad = \left[-\frac{1}{s\ y^s}[R(y)-R(x)]^{n-m-1}[1-F(y)]\right]_x^\infty \\
&\qquad + \frac{(n-m-1)}{s}\int_x^\infty \frac{1}{y^s}[R(y)-R(x)]^{n-m-2}f(y)dy \\
&\qquad - \frac{1}{s}\int_x^\infty \frac{1}{y^s}[R(y)-R(x)]^{n-m-1}f(y)dy.
\end{aligned}$$

Then we have

$$\begin{aligned}
E\left(\frac{X^r_{u(m)}}{X^{s+1}_{u(n)}}\right) &= \frac{1}{s\ \Gamma(m)\Gamma(n-m-1)}\int\int_{0<x<y<\infty} \\
&\quad \frac{x^r}{y^s}R^{m-1}(x)[R(y)-R(x)]^{n-m-2}f(y)dydx \\
&\quad - \frac{1}{s\ \Gamma(m)\Gamma(n-m)}\int\int_{0<x<y<\infty} \\
&\quad \frac{x^r}{y^s}R^{m-1}(x)[R(y)-R(x)]^{n-m-1}f(y)dydx \\
&= \frac{1}{s}\int\int_{0<x<y<\infty}\frac{x^r}{y^s}f_{m,n-1}(x,y)dydx \\
&\quad - \frac{1}{s}\int\int_{0<x<y<\infty}\frac{x^r}{y^s}f_{m,n}(x,y)dydx.
\end{aligned}$$

Hence

$$E\left(\frac{X^r_{u(m)}}{X^{s+1}_{u(n)}}\right) = \frac{1}{s}\ E\left(\frac{X^r_{u(m)}}{X^s_{u(n-1)}}\right) - \frac{1}{s}\ E\left(\frac{X^r_{u(m)}}{X^s_{u(n)}}\right).$$

This completes the proof.

Corollary 2.2. For $m \geq 1$, $r = 0, 1, 2, \cdots$ and $s = 1, 2, \cdots$,

$$E\left(\frac{X^r_{u(m)}}{X^{s+1}_{u(m+1)}}\right) = \frac{1}{s}\ E(X^{r-s}_{u(m)}) - \frac{1}{s}\ E\left(\frac{X^r_{u(m)}}{X^s_{u(m+1)}}\right).$$

Proof. Upon substituting $n = m + 1$ in Theorem 2.1 and simplifying, then we have

$$E\left(\frac{X_{u(m)}^{r}}{X_{u(m+1)}^{s+1}}\right) = \frac{1}{s}\ E(X_{u(m)}^{r-s}) - \frac{1}{s}\ E\left(\frac{X_{u(m)}^{r}}{X_{u(m+1)}^{s}}\right).$$

Theorem 2.3. For $1 \leq m \leq n-2$ and $r,\ s = 0, 1, 2, \cdots$,

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^{s}}\right) = \frac{1}{(r+2)}\ E\left(\frac{X_{u(m)}^{r+2}}{X_{u(n-1)}^{s}}\right) - \frac{1}{(r+2)}\ E\left(\frac{X_{u(m-1)}^{r+2}}{X_{u(n-1)}^{s}}\right).$$

Proof. In the same manner as Theorem 2.1, let us consider for $1 \leq m \leq n-2$ and $r,\ s = 0, 1, 2, \cdots$,

$$\begin{aligned} E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^{s}}\right) &= \int\int_{0<x<y<\infty} \frac{x^{r+1}}{y^{s}} f_{m,n}(x,y)dxdy \\ &= \frac{1}{\Gamma(m)\Gamma(n-m)} \int\int_{0<x<y<\infty} \\ &\quad \frac{x^{r+1}}{y^{s}} R^{m-1}(x)[R(y)-R(x)]^{n-m-1} f(y)dxdy \\ &= \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \frac{1}{y^s} f(y) \\ &\quad \left(\int_0^y x^{r+1} R^{m-1}(x)[R(y)-R(x)]^{n-m-1}dx\right) dy. \end{aligned}$$

Using integrating by parts treating x^{r+1} for integration and $R^{m-1}(x)$ $[R(y)-R(x)]^{n-m-1}$ for differentiation on the second integration, we get

$$\begin{aligned} &\int_0^y x^{r+1} R^{m-1}(x)[R(y)-R(x)]^{n-m-1}dx \\ &= \left[\frac{1}{(r+2)} x^{r+2}\ R^{m-1}(x)[R(y)-R(x)]^{n-m-1}\right]_0^y \\ &\quad + \frac{(n-m-1)}{(r+2)} \int_0^y x^{r+2} R^{m-1}(x)[R(y)-R(x)]^{n-m-2}dx \\ &\quad - \frac{(m-1)}{(r+2)} \int_x^\infty x^{r+2} R^{m-2}(x)[R(y)-R(x)]^{n-m-1}dx. \end{aligned}$$

Then we have

$$\begin{aligned}
E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^{s}}\right) &= \frac{1}{(r+2)\ \Gamma(m)\Gamma(n-m-1)}\int\int_{0<x<y<\infty}\frac{x^{r+2}}{y^{s}}R^{m-1}(x)\\
&\quad\times[R(y)-R(x)]^{n-m-2}f(y)dxdy\\
&\quad-\frac{1}{(r+2)\ \Gamma(m-1)\Gamma(n-m)}\int\int_{0<x<y<\infty}\frac{x^{r+2}}{y^{s}}R^{m-2}(x)\\
&\quad\times[R(y)-R(x)]^{n-m-1}f(y)dxdy\\
&= \frac{1}{(r+2)}\int\int_{0<x<y<\infty}\frac{x^{r+2}}{y^{s}}f_{m,n-1}(x,y)dydx\\
&\quad-\frac{1}{(r+2)}\int\int_{0<x<y<\infty}\frac{x^{r+2}}{y^{s}}f_{m-1,n-1}(x,y)dxdy.
\end{aligned}$$

Hence

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^{s}}\right)=\frac{1}{(r+2)}\ E\left(\frac{X_{u(m)}^{r+2}}{X_{u(n-1)}^{s}}\right)-\frac{1}{(r+2)}\ E\left(\frac{X_{u(m-1)}^{r+2}}{X_{u(n-1)}^{s}}\right).$$

This completes the proof.

Corollary 2.4. For $m\geq 1$, $r=0,1,2,\cdots$ and $s=1,2,\cdots$,

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(m+1)}^{s}}\right)=\frac{1}{(r+2)}\ E(X_{u(m)}^{r-s+2})-\frac{1}{(r+2)}\ E\left(\frac{X_{u(m-1)}^{r+2}}{X_{u(m)}^{s}}\right).$$

Proof. Upon substituting $n=m+1$ in Theorem 2.3 and simplifying, then we have

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(m+1)}^{s}}\right)=\frac{1}{(r+2)}\ E(X_{u(m)}^{r-s+2})-\frac{1}{(r+2)}\ E\left(\frac{X_{u(m-1)}^{r+2}}{X_{u(m)}^{s}}\right).$$

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