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ON THE HIGHER ORDER KOBAYASHI METRICS

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Abstract. In this paper, we prove the product property and the existence of an extremal analytic disc relative to the higher order Kobayashi metric. Also by making use of the upper semicontinuity of the higher order Kobayashi metric, we introduce a pseudodistance and investigate some properties of that pseudodistance related to the usual Kobayashi metric.

1. Introduction

About 1966, S. Kobayashi initiated studying his pseudodistance([3]) and H. L. Royden published the infinitesimal form in [7]. The infinitesimal form that is called as the Kobayashi metric is studied in [1], [6], [7] etc. The higher order Kobayashi metric was introduced by J. Yu in [8] and N. Nikolov also investigated the higher order Kobayashi metric in [5].

Our goal here is the proof of some properties related to the higher order Kobayashi metric as the counterpart for the usual Kobayashi metric. To do this, we introduce some notations which is dealt in this article. By $\mathbb{N}$ and $\mathbb{C}$ we denote the set of natural numbers and the set of complex numbers, respectively. We use the usual inner product $\langle \cdot, \cdot \rangle$ and the

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norm $||\cdot||$ on $\mathbb{C}^n$ which is defined by

$$\langle z, w\rangle := \sum_{j=1}^{n} z_j\bar{w}_j, \quad ||z||^2 := \langle z, z\rangle = \sum_{j=1}^{n} |z_j|^2$$

for all $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$.

Further, by F^c_Ω and K_Ω we denote the Carathéodory metric and the usual Kobayashi metric for some domain Ω.

2. The higher order Kobayashi metric

The higher order Kobayashi metric is introduced in [8] by J. Yu as the generalization of the Kobayashi-Royden metric or simply Kobayashi metric.

Let $D \subset \mathbb{C}^n$ be a domain and denote by $\mathcal{O}(\Delta, D)$ the space of all holomorphic mappings from the unit disk $\Delta \subset \mathbb{C}$ into D. For $t \in D$, we mean by $\mathcal{O}_t(\Delta, D)$ the set $\{\varphi \in \mathcal{O}(\Delta, D) \mid \varphi(0) = t\}$.

For each $m \in \mathbb{N}$ and $(z, X) \in D \times \mathbb{C}^n$, the *m-th order Kobayashi metric* is defined by

(2.1)

$$K^m_D(z, X) := \inf\{|r|^{-1} \mid \exists\psi \in \mathcal{O}_z(\Delta, D) \text{ s.t. } \nu(\psi) \geq m, \psi^{(m)}(0) = m!rX\}$$

where $\nu(\psi)$ stands for the order of vanishing of $\psi - \psi(0)$ at 0. Clearly $K^1_D(z, X)$ is the usual Kobayashi metric. The *singular Kobayashi metric* is defined as follows :

$$K^\infty_D(z, X) := \inf_{m\in\mathbb{N}} K^m_D(z, X). \tag{2.2}$$

Proposition 2.1. Let D be a domain in $\mathbb{C}^n$ and $m \in \mathbb{N}$. Then K^m_D is upper semicontinuous on $D \times \mathbb{C}^n$.

Proof. Fix $(z_0, X_0) \in D \times (\mathbb{C}^n \setminus \{0\})$ and let $K^m_D(z_0, X_0) < A$. Then by the definition of the higher order Kobayashi metric, there exist a

$\phi \in \mathcal{O}_{z_0}(\Delta, D)$ and $r > 0$ such that

$$\nu(\phi) \geq m,\ \phi^{(m)}(0) = m! r X_0 \text{ and } \frac{1}{r} < A.$$

Fix an $\eta \in (0,1)$ arbitrarily small for which $\phi((1-\eta)^{\frac{1}{m}}\overline{\Delta})$ is a relatively compact subset of D. Putting $\delta = \mathrm{dist}(\partial D, \phi((1-\eta)^{\frac{1}{m}}\overline{\Delta}))$, we have $\delta > 0$. Let $(z, X) \in D \times \mathbb{C}^n$ with $||z - z_0|| < \frac{\delta}{4}$ and $r||X - X_0|| < \frac{\delta}{4}$. Define a holomorphic map $\psi : \Delta \longrightarrow D$ by $\psi(\zeta) := \phi((1-\eta)^{\frac{1}{m}}\zeta) + (z - z_0) + r(1-\eta)\zeta^m(X - X_0)$. Then

$$\psi(\Delta) \subset D,\ \psi(0) = z,\ \nu(\psi) \geq m \text{ and } \psi^{(m)}(0) = r(1-\eta)m!X.$$

Hence we have

$$K_D^m(z, X) \leq \frac{1}{r(1-\eta)}.$$

Letting $\eta \to 0^+$, we obtain

$$K_D^m(z, X) \leq \frac{1}{r} < A.$$

Thus K_D^m is upper semicontinuous on $D \times \mathbb{C}^n$. □

Since the infimum of any collection of upper semicontinuous functions is upper semicontinuous, we have the following

Corollary 2.2. Let $D \subset \mathbb{C}^n$ be a domain. Then the singular Kobayashi metric K_D^∞ is also upper semicontinuous on $D \times \mathbb{C}^n$.

Proposition 2.3. Let $D \subset \mathbb{C}^n$ be a domain. Then for each $m \geq 1$, we have

(1) K_D^m has the length decreasing property. In particular, K_D^m is biholomorphically invariant.

(2) $K_D^m \equiv K_\Delta$, the usual Kobayashi metric for the unit disc Δ.

(3) $F_D^c(z, X) \leq K_D^m(z, X) \leq K_D(z, X)$.

(4) $K_D^m(z, \mu X) = |\mu| K_D^m(z, X)$ for all $\mu \in \mathbb{C}$.

Proof. For the detail proofs of (1) $\sim$ (3), one refer to [8].

(4) It is clear in case of $\mu = 0$. Hence assume that $\mu \neq 0$.

Let $\psi \in \mathcal{O}_z(\Delta, D)$ and $r > 0$ with $\nu(\psi) \geq m$ and $\psi^{(m)}(0) = m!r\mu X$. Then since

$$K_D^m(z, X) \leq \frac{1}{r|\mu|},$$

we obtain $|\mu| K_D^m(z, X) \leq \frac{1}{r}$. But since ψ was arbitrary, we have the following

$$|\mu| K_D^m(z, X) \leq K_D^m(z, \mu X). \tag{2.3}$$

Conversely, let $\phi \in \mathcal{O}_z(\Delta, D)$ and $r > 0$ with $\nu(\phi) \geq m$ and $\phi^{(m)}(0) = m!rX$. Then since $\phi^{(m)}(0) = m!rX = m!\frac{r}{\mu}\mu X$, we obtain

$$K_D^m(z, \mu X) \leq \frac{|\mu|}{r} = |\mu| \frac{1}{r}.$$

But since ϕ was arbitrary, we have

$$K_D^m(z, \mu X) \leq |\mu| K_D^m(z, X). \tag{2.4}$$

By (2.3) and (2.4), we reach at the required result. □

Theorem 2.4. Let $D \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^l$ be domains. Then the following formula holds :

$$K_{D\times G}^m((z, w), (X, Y)) = \max\{K_D^m(z, X), K_G^m(w, Y)\}.$$

Proof. By the contraction property (1) of Proposition 2.3 with respect to holomorphic mappings, the inequality " $\geq$ " is easily obtained.

To prove the reverse inequality, suppose that

$$K_{D\times G}^m((z, w), (X, Y)) > A > \max\{K_D^m(z, X), K_G^m(w, Y)\}.$$

Then, by definition, we find $\phi \in \mathcal{O}_z(\Delta, D), \psi \in \mathcal{O}_w(\Delta, G)$ and $r > 0, s > 0$ that satisfy the properties

$$\nu(\phi) \geq m, \quad \phi^{(m)}(0) = m!rX, \quad \tfrac{1}{r} < A$$
$$\nu(\psi) \geq m, \quad \psi^{(m)}(0) = m!sY, \quad \tfrac{1}{s} < A.$$

Without loss of generality, we may assume $0 < r \leq s$. If we define a holomorphic map $f : \Delta \longrightarrow D \times G$ by

$$f(\zeta) = \left(\phi(\zeta), \psi\left((\frac{r}{s})^{\frac{1}{m}}\zeta\right)\right),$$

then $f(0) = (z, w), \nu(f) \geq m$ and $f^{(m)}(0) = m!r(X, Y)$. Hence

$$K^m_{D\times G}((z, w), (X, Y)) \leq \frac{1}{r} < A,$$

contrary to our assumption. □

By applying Lempert's Theorem([2], [4]) with Proposition 2.3, we obtain

Proposition 2.5.([1], [2]) Let $D \subset \mathbb{C}^n$ be an open unit ball with center 0. Then we have

$$K^m_D(z, X) = \left[\frac{||X||^2}{1 - ||z||^2} + \frac{|\langle z, X\rangle|^2}{(1 - ||z||^2)^2}\right]^{\frac{1}{2}}.$$

for all $(z, X) \in D \times \mathbb{C}^n$. In particular, we have

$$K^\infty_D(z, X) = \left[\frac{||X||^2}{1 - ||z||^2} + \frac{|\langle z, X\rangle|^2}{(1 - ||z||^2)^2}\right]^{\frac{1}{2}}.$$

for all $(z, X) \in D \times \mathbb{C}^n$.

From Theorem 2.4 and Proposition 2.5 we have the following

Corollary 2.6. Let $D := \Delta^n \subset \mathbb{C}^n$ be a unit polydisc with center 0. Then we have

$$K^\infty_D(z, X) = \max\left\{\frac{|X_1|}{1 - |z_1|^2}, \cdots, \frac{|X_n|}{1 - |z_n|^2}\right\}.$$

for all $(z, X) \in D \times \mathbb{C}^n$.

Theorem 2.7. Let $D \subset \mathbb{C}^n$ be a taut domain (i.e., $\mathcal{O}(\Delta, D)$ is a normal family). Then for any $(z, X) \in D \times \mathbb{C}^n$, there exists an extremal analytic disc $\phi \in \mathcal{O}_z(\Delta, D)$ for K_D^m, in other words,

$$\nu(\phi) \geq m \quad \text{and} \quad K_D^m(z, X)\phi^{(m)}(0) = m!X.$$

Proof. By definition of the higher order Kobayashi metric, we can choose a sequence $\{\phi_n\} \subset \mathcal{O}_z(\Delta, D)$ and a sequence $\{r_n\}$ of positive real numbers such that

$$\nu(\phi_n) \geq m,\ \phi_n^{(m)}(0) = m!r_n X \text{ and } \frac{1}{r_n} \searrow K_D^m(z, X) \text{ as } n \to \infty.$$

Since $\mathcal{O}(\Delta, D)$ is a normal family, there is a subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$ and $\phi \in \mathcal{O}_z(\Delta, D)$ such that $\{\phi_{n_k}\}$ converges compactly to ϕ on D. Then we have

$$\nu(\phi) \geq m,\ \phi^{(m)}(0) = \lim_{k\to\infty} \phi_{n_k}^{(m)}(0) = \lim_{k\to\infty} m!r_{n_k}X = \frac{m!X}{K_D^m(z, X)}. \quad \square$$

3. A distance induced by Kobayashi metric

We know from Proposition 2.1 that the higher order Kobayashi metric can be used to define the length of a piecewise C^1-curve and then the minimal length of all such curves connecting two fixed points will yield a new pseudodistance.

Define the K_D^m*-length* of a piecewise C^1-curve $\alpha : [0, 1] \longrightarrow D$ by

$$L_m(\alpha) := \int_0^1 K_D^m(\alpha(t), \alpha'(t))dt. \tag{3.5}$$

Then $L_m(\alpha) \in [0, \infty)$ and so we may define a map $d_D^m : D \times D \longrightarrow \mathbb{R}$, which is called the integrated form of K_D^m, by

$$d_D^m(z, w) := \inf_\alpha L_m(\alpha) \tag{3.6}$$

where the infinmum is taken over all piecewise C^1-curves joining z and w.

Theorem 3.1. Let $D \subset \mathbb{C}^n$ be a domain. Then d_D^m is a pseudodistance on D.

Proof. Checking the properties for pseudodistance except the triangle inequality is clear. To show the triangle inequality, let $u, v, w \in D$. Then for any $\epsilon > 0$, there exist two piecewise C^1-curves $\alpha, \beta : [0,1] \longrightarrow D$ such that $\alpha(0) = u, \alpha(1) = v = \beta(0), \beta(1) = w$ and

$$L_m(\alpha) < d_D^m(u,v) + \frac{\epsilon}{2}, \; L_m(\beta) < d_D^m(v,w) + \frac{\epsilon}{2}.$$

Define a curve $\gamma : [0,1] \longrightarrow D$ by

$$\gamma(t) := \begin{cases} \alpha(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Then γ is a piecewise C^1-curve connecting u and w. Moreover, applying (3.5) to γ, we have

$$\begin{aligned} L_m(\gamma) &= \int_0^1 K_D^m(\gamma(t), \gamma'(t))dt \\ &= \int_0^1 K_D^m(\alpha(t), \alpha'(t))dt + \int_0^1 K_D^m(\beta(t), \beta'(t))dt \\ &< d_D^m(u,v) + d_D^m(v,w) + \epsilon. \end{aligned}$$

Applying (3.6) and then using the fact that $\epsilon > 0$ was arbitrary, we get the required triangle inequality

$$d_D^m(u,w) \le d_D^m(u,v) + d_D^m(v,w). \quad \square$$

Proposition 2.3 and the definition of d_D^m induce the following

Proposition 3.2. Let $\Omega \subset \mathbb{C}^l$ and $D \subset \mathbb{C}^n$ be two domains. If $f : \Omega \longrightarrow D$ is a holomorphic map, then $d_\Omega^m(z,w) \ge d_D^m(f(z), f(w))$ for any $z, w \in \Omega$. That is, d_D^m has the distance decreasing property under holomorphic mappings.

It follows from Proposition 2.3 that the pseudodistance d_D^c induced by the Carathéodory metric F_D^c is not larger than d_D^m. Hence we have the following

Proposition 3.3. Let $D \subset \mathbb{C}^n$ be a domain. If D is the Carathéodory hyperbolic(i.e., d^c_D is a distance), then d^m_D is a distance.

Corollary 3.4. Let $D \subset \mathbb{C}^n$ be a bounded domain. Then d^m_D is a distance.

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