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INTUITIONISTIC FUZZY TOPOLOGICAL GROUPS

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Abstract. In this paper, we introduce the concepts of intuitionistic fuzzy subspaces, intuitionistic fuzzy topological groups and intuitionistic fuzzy quotient groups. And we investigate some of their properties.

0. Introduction

In 1965, Zadeh [14] introduced the concept of fuzzy sets as the generalization of ordinary sets. After that time, several researchers [6,7,12,13] have applied the notion of fuzzy sets to algebras and topological group theory.

In 1986, Atanassov [1] introduced the concept of intuitionisitc fuzzy sets as the generalization of fuzzy sets. Recently, Çoker and his colleagues [4,5,8], and Lee and Lee [11] have applied the notion of intuitionistic fuzzy sets to topologies. Also, several researchers [2,3,9,10] introduced the concepts of intuitionistic fuzzy subgroups, subrings and ideals using intuitionistic fuzzy sets.

In this paper, we introduce the concepts of intuitionistic fuzzy subspaces, intuitionistic fuzzy topological groups and intuitionistic fuzzy quotient groups. And we investigate some of their properties.

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1. Preliminaries

We will list some concepts and results needed in the later sections.

For set X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Definition 1.1[1]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set*(in short, *IFS*) in X if $\mu_A : X \to I$ and $\nu_A : X \to I$ are mappings, and $\mu_A + \nu_A \leq 1$.

In this case, μ_A and ν_A denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership(namely $\nu_A(x)$) of each $x \in X$ to A, respectively.

It is clear that every fuzzy set A in X is an IFS in X having the form $A = (\mu_A, \mu_{A^c})$ (See Example 2.4 in [4]).

We will denote the set of all the IFSs in X as $IFS(X)$.

Definitions 1.2[1]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\,]A = (\mu_A, 1 - \mu_A)$, $<> A = (1 - \nu_A, \nu_A)$.

Definition 1.3[4]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(a) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.

(b) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4[4]. $0_\sim = (0,1)$ and $1_\sim = (1,0)$.

Result 1.A[4, Corollary 2.8]. Let A, B, C, D be IFSs in X. Then

(1) $A \subset B$ and $C \subset D \Rightarrow A \cup C \subset B \cup D$ and $A \cap C \subset B \cap D$.

(2) $A \subset B$ and $A \subset C \Rightarrow A \subset B \cap C$.

(3) $A \subset B$ and $B \subset C \Rightarrow A \cup B \subset C$.

(4) $A \subset B$ and $B \subset C \Rightarrow A \subset C$.

(5) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

(6) $A \subset B \Rightarrow B^c \subset A^c$.

(7) $(A^c)^c = A$.

(8) $1_\sim^c = 0$, $0_\sim^c = 1_\sim$.

Definition 1.5[4]. Let X and Y be nonempty sets and let $f : X \to Y$ a mapping. Let $A = (\mu_A, \nu_A)$ be an IFS in X and $B = (\mu_B, \nu_B)$ be IFS on Y. Then

(a) the *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by:

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)),$$

where $f^{-1}(\mu_B) = \mu_B \circ f$.

(b) the *image* of A under f, denoted by $f(A)$, is the IFS in Y defined by:

$$f(A) = (f(\mu_A), f(\nu_A)),$$

where for each $y \in Y$

$$f(\mu_A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu_A(x) & iff^{-1}(y) \neq \emptyset, \\ 0 & iff^{-1}(y) = \emptyset \end{cases},$$

and

$$f(\nu_A)(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \nu_A(x) & iff^{-1}(y) \neq \emptyset, \\ 1 & iff^{-1}(y) = \emptyset. \end{cases},$$

Result 1.B[4, Corollary 2.10]. Let $A, A_i (i \in J)$ be IFSs in X, let $B, B_j (j \in K)$ IFSs in Y and let $f : X \to Y$ a mapping. Then

(1) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

(2) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$.

(3) $A \subset f^{-1}(f(A))$.

If f is injective, then $A = f^{-1}(f(A))$.

(4) $f(f^{-1}(B) \subset B$.

If f is surjective, then $f(f^{-1}(B)) = B$.

(5) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$.

(6) $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$.

(7) $f(\bigcup A_i) = \bigcup f(A_i)$.

(8) $f(\bigcap A_i) \subset \bigcap f(A_i)$.

If f is injective, then $f(\bigcap A_i) = \bigcap f(A_i)$.

(9) $f(1_\sim) = 1_\sim$, if f is surjective and $f(0_\sim) = 0_\sim$.

(10) $f^{-1}(1_\sim) = 1_\sim$ and $f^{-1}(0_\sim) = 0_\sim$.

(11) $[f(A)]^c \subset f(A^c)$, if f is surjective.

(12) $f^{-1}(B^c) = [f^{-1}(B)]^c$.

Definition 1.6[4]. Let X be a set and let $\lambda, \mu \in I$ with $0 \leq \lambda + \mu \leq 1$. Then the IFS $C_{(\lambda,\mu)}$ in X is defined by : for each $x \in X$, $C_{(\lambda,\mu)}(x) = (\lambda, \mu)$, i.e., $\mu_{C_{(\lambda,\mu)}}(x) = \lambda$ and $\nu_{C_{(\lambda,\mu)}}(x) = \mu$.

2. Intuitionistic fuzzy topological spaces and subspaces

Definition 2.1[4]. Let X be a nonempty set and let $T \subset IFS(X)$. Then T is called an *intuitionistic fuzzy topology*(in short, *IFT*) on X in the sense of Lowen, if it satisfies the following axioms :

(T_1) For each $\alpha, \beta \in I$ with $\alpha + \beta \leq 1$, $C_{(\alpha,\beta)} \in T$.

(T_2) For any $G_1, G_2 \in T$, $G_1 \cap G_2 \in T$.

(T_3) For any $\{G_\alpha\}_{\alpha\in\Gamma} \subset T$, $\bigcup_{\alpha\in\Gamma} G_\alpha \in T$.

In this case, the pair (X, T) is called an *intuitionistic fuzzy topological space*(in short, *IFTS*) in the sense of Lowen, and each member of T is called an *intuitionistic fuzzy open set*(in short, *IFOS*) in X. $A \in IFS(X)$ is called an *intuitionistic fuzzy closed set*(in short, *IFCS*) in X if $A^c \in T$.

We will denote the set of all the IFTs on a set X as IFT(X), and the set of all IFOSs and the set of all IFCSs in an IFTS X as $IFO(X)$ and $IFC(X)$, respectively.

Definition 2.2. Let (X, T) be an IFTS and let $A \in IFS(X)$. Then the collection $T_A = \{U \cap A \in IFS(X) : U \in T\}$ is called the *induced intuitionistic fuzzy topology*(in short, *IIFT*) on A. The pair (A, T_A) is called an *intuitionistic fuzzy subspace*(in short, *IFSP*) of (X, T).

Note that T_A does not in general satisfy the axiom $[T_1]$. However the following holds.

Proposition 2.3. T_A satisfies the axioms (T_2) and (T_3):

Proof. Let $G_1, G_2 \in T_A$. Then there exist $U_1, U_2 \in T$ such that $G_1 = U_1 \cap A$ and $G_2 = U_2 \cap A$, i.e., $G_1 = (\mu_{U_1} \wedge \mu_A, \nu_{U_1} \vee \nu_A)$ and $G_2 = (\mu_{U_2} \wedge \mu_A, \nu_{U_2} \vee \nu_A)$. Thus $G_1 \cap G_2 = (\mu_{G_1} \wedge \mu_{G_2}, \nu_{G_1} \vee \nu_{G_2})$

$$= ((\mu_{U_1} \wedge \mu_A) \wedge (\mu_{U_2} \wedge \mu_A), (\nu_{U_1} \vee \nu_A) \vee (\nu_{U_2} \vee \nu_A))$$
$$= ((\mu_{U_1} \wedge \mu_{U_2}) \wedge \mu_A, (\nu_{U_1} \vee \nu_{U_2}) \vee \nu_A)$$
$$= (U_1 \cap U_2) \cap A.$$

Since $U_1, U_2 \in T$, $U_1 \cap U_2 \in T$. So $G_1 \cap G_2 \in T_A$.

Now let $\{G_\alpha\}_{\alpha\in\Gamma} \subset T_A$. Then $G_\alpha \in T_A$ for each $\alpha \in \Gamma$, i.e., there exists a $U_\alpha \in T$ such that $G_\alpha = U_\alpha \cap A$, i.e., $G_\alpha = (\mu_{U_\alpha} \wedge \mu_A, (\nu_{U_\alpha} \vee \nu_A))$ for each $\alpha \in \Gamma$. Thus:

$\bigcup_{\alpha\in\Gamma} G_\alpha = (\bigvee_{\alpha\in\Gamma}(\mu_{U_\alpha} \wedge \mu_A), \bigwedge_{\alpha\in\Gamma}(\nu_{U_\alpha} \vee \nu_A))$
$= ((\bigvee_{\alpha\in\Gamma} \mu_{U_\alpha}) \wedge \mu_A, (\bigwedge_{\alpha\in\Gamma} \nu_{U_\alpha}) \vee \nu_A)$
$= ((\bigcup_{\alpha\in\Gamma} U_\alpha) \cap A.$

Since $\{U_\alpha\}_{\alpha\in\Gamma} \subset T$, $\bigcup_{\alpha\in\Gamma} U_\alpha \in T$. So $\bigcup_{\alpha\in\Gamma} G_\alpha \in T_A$. Hence T_A satisfies the axioms (T_2) and (T_3).

Definition 2.4[4, 10]. Let (X, T_X) and (Y, T_Y) be IFTSs and let $f : X \to Y$ a mapping.

(1) f is said to be *intuitionistic fuzzy continuous*(in short, IF-continuous) if for each $V \in T_Y$, $f^{-1}(V) \in T_X$.

(2) f is said to be *intuitionistic fuzzy open*(in short, IF-open) if for each $U \in T_X$, $f(U) \in T_Y$.

Definition 2.5. Let (A, T_A) and (B, T_B) be IFSPs of IFTSs (X, T_X) and (Y, T_Y), respectively and let $f : X \to Y$ a mapping.

(1) f is called a *mapping of* (A, T_A) *into* (B, T_B), denoted by $f : (A, T_A) \to (B, T_B)$, if $f(A) \subset B$.

(2) A mapping $f : (A, T_A) \to (B, T_B)$is said to be *relatively intuitionistic fuzzy continuous*(in short, *relatively IF-continuous*) if $f^{-1}(V) \cap A \in T_A$, for each $V \in T_B$.

(3) A mapping $f : (A, T_A) \to (B, T_B)$ is said to be *relatively intuitionistic fuzzy open*(in short, *relatively IF-open*) if for each $U \in T_A$, $f(U) \in T_B$.

Proposition 2.6. Let (A, T_A) and (B, T_B) be IFSPs of IFTSs (X, T_X) and (Y, T_Y), respectively and let $f : (X, T_X) \to (Y, T_Y)$ an IF-continuous mapping such that $f(A) \subset B$. Then $f : (A, T_A) \to (B, T_B)$ is relatively IF- continuous.

Proof. Let $V' \in T_B$. Then there exists a $V \in T_Y$ such that $V' = V \cap B$. Since $f : (X, T_X) \to (Y, T_Y)$ is IF- continuous and $V \in T_Y$,

$f^{-1}(V) \in T_X$. On the other hand, $f^{-1}(V') \cap A = f^{-1}(V) \cap f^{-1}(B) \cap A$. Since $f(A) \subset B$ by (2) and (3) of Result 1.B, $A \subset f^{-1}(B)$. Thus $f^{-1}(V') \cap A = f^{-1}(V) \cap A$. So $f^{-1}(V') \cap A \in T_A$. Hence f is relatively IF- continuous.

The following is the immediate result of Definition 2.4:

Proposition 2.7. Let $f : (X, T_X) \to (Y, T_Y)$ and $g : (Y, T_Y) \to (Z, T_Z)$ be IF- continuous[resp. IF- open]. Then $g \circ f : (X, T_X) \to (Z, T_Z)$ is IF- continuous[resp. IF- open].

Proposition 2.7′. Let (A, T_A), (B, T_B), (C, T_C) be IFSPs of IFTSs (X, T_X), (Y, T_Y), (Z, T_Z), respectively. Let $f : (A, T_A) \to (B, T_B)$ and $g : (B, T_B) \to (C, T_C)$ be relatively IF-continuous[resp. relatively IF- open], respectively. Then $g \circ f : (A, T_A) \to (C, T_C)$ is relatively IF- continuous[resp. relatively IF- open].

Proof. Suppose $f : (A, T_A) \to (B, T_B)$ and $g : (B, T_B) \to (C, T_C)$ are relatively IF- continuous, respectively. Let $W \in T_C$. Since g is relatively IF- continuous, $g^{-1}(W) \cap B \in T_B$. Since f is relatively IF- continuous, $f^{-1}[g^{-1}(W) \cap B] \cap A \in T_A$. On the other hand,

$$f^{-1}[g^{-1}(W) \cap B] \cap A = f^{-1}[g^{-1}(W)] \cap f^{-1}(B) \cap A$$
$$= (g \circ f)^{-1}(W) \cap f^{-1}(B) \cap A$$
$$= (g \circ f)^{-1}(W) \cap A. \quad \text{(by (2) and (3) of Result 1.B)}$$

Thus $(g \circ f)^{-1}(W) \cap A \in T_A$. Hence $g \circ f$ is relatively IF- continuous.

Suppose $f : (A, T_A) \to (B, T_B)$ and $g : (B, T_B) \to (C, T_C)$ are relatively IF- open, respectively. Let $U \in T_A$. Since f is relatively IF- open, $f(U) \in T_B$. Since g is relatively IF- open, $g(f(U) \in T_C$. But $g(f(U)) = (g \circ f)(U)$. Thus $(g \circ f)(U) \in T_C$. Hence $g \circ f$ is relatively IF- open.

Definition 2.8. Let T be an IFT on a set X and let $\mathcal{B} \subset T$. Then $\mathcal{B}$ is called a *base* for T if for each $U \in T$ either $U = C_{0_\sim}$ or there exists a $\mathcal{B}' \subset B$ such that $U = \bigcup \mathcal{B}'$.

Definition 2.8′. Let (X, T) be an IFTS, let $A \in IFS(X)$ and let $\mathcal{B} \subset T_A$. Then $\mathcal{B}$ is called a *base* for T_A if for each $U \in T_A$ either $U = C_{0_\sim}$ or there exists a $\mathcal{B}' \subset B$ such that $U = \bigcup \mathcal{B}'$.

It is clear that if $\mathcal{B}$ is a base for an IFT T on a set X and $A \in IFS(X)$, then $\mathcal{B}_A = \{U \cap A : U \in \mathcal{B}\}$ is a base for T_A.

The followings are the immediate results of Definition 2.5, Definition 2.8 and Definition 2.8′:

Proposition 2.9. Let $f : (X, T_X) \to (Y, T_Y)$ be a mapping and let $\mathcal{B}$ a base for T_Y. Then f is IF- continuous if and only if for each $B \in \mathcal{B}$, $f^{-1}(B) \in T_X$.

Proposition 2.9′. Let $(A, T_A), (B, T_B)$ be IFSPs of IFTSs $(X, T_X), (Y, T_Y)$, respectively. Let $\mathcal{B}'$ a base for T_B. Then $f : (A, T_A) \to (B, T_B)$ is relatively IF- continuous if and only if for each $B' \in \mathcal{B}'$, $f^{-1}(B') \cap A \in T_A$.

Definition 2.10. Let $T_1, T_2 \in IFT(X)$. Then T_1 is said to be *finer than* T_2(or T_2 is said to be *coarser than* T_2) if the identity mapping $id_X : (X, T_1) \to (X, T_2)$ is IF- continuous, i.e., $T_2 \subset T_1$.

Definition 2.11. Let $f : X \to Y$ be a mapping and let T_Y an IFT on Y. Then the family $T_{f^{-1}} = \{f^{-1}(U) \in IFS(X) : U \in T_Y\}$ is called the *inverse image of* T_Y *under* f.

It is clear that $T_{f^{-1}}$ is the coarest IFT on X for which $f : (X, T_{f^{-1}}) \to (Y, T_Y)$ is IF- continuous.

Definition 2.12. Let $f : X \to Y$ be a mapping and let T_X an IFT on X. Then the family $T_f = \{U \in IFS(Y) : f^{-1}(U) \in T_X\}$ is called the *image of T_X under f.*

It is clear that T_f is the finest IFT on Y for which $f : (X, T_X) \to (Y, T_f)$ is IF- continuous.

Definition 2.13. Let $\{(X_\alpha, T_\alpha)\}_{\alpha\in\Gamma}$ be a family of IFTSs, let $X = \prod_{\alpha\in\Gamma} X_\alpha$, let (X, T_X) an IFTS and let T the coarest IFT on X for which $\pi_\alpha : (X, T) \to (X_\alpha, T_\alpha)$ is IF- continuous for each $\alpha \in \Gamma$, where π_α is the usual projection. Then T is called the *intuitionistic fuzzy product topology*(in short, *IFPT*) on X and denoted by $\prod_{\alpha\in\Gamma} T_\alpha$, and (X, T) a *intuitionistic fuzzy product space*(in short, *IFPS*).

The following is the immediate result of Definition 2.8 and Definition 2.13:

Proposition 2.14. Let $\{(X_\alpha, T_\alpha)\}_{\alpha\in\Gamma}$ be a family of IFTSs and let (X, T) the IFPS. Then T has as a base the set of finite intersection of IFSs in X of the form $\pi_\alpha^{-1}(U_\alpha)$, where $U_\alpha \in T_\alpha$ for each $\alpha \in \Gamma$.

Let $\{X_j\}, j = 1, \cdots, n$, be a finite family of sets and for each $j = 1, \cdots, n$, let $A_j \in IFS(X_j)$. We define the *product* $A = \prod_{j=1}^n A_j$ of $\{A_j\}$, $j = 1, \cdots, n$, as the IFS in $X = \prod_{j=1}^n X_j$ that has membership mapping and nonmembership mapping, respectively given by : for each $(x_1, \cdots, x_n) \in X$,

$$\mu_A(x) = \mu_{A_1}(x) \wedge \cdots \wedge \mu_{A_n}(x_n) \text{ and } \nu_A(x) = \nu_{A_1} \vee \cdots \vee \nu_{A_n}(x_n).$$

It follows from the above Definition 2.13 and Proposition 2.14 that if X_j has IFT $T_j, j = 1, \cdots, n$, then the IFPT T on X has as a base the set of IFPSs of the form $\prod_{j=1}^n U_j$, where $U_j \in T_j$ for each $j = 1, \cdots, n$.

Proposition 2.15. Let $\{X_j\}, j = 1, \cdots, n$, be a finite family of sets and let $A = \prod_{j=1}^n A_j$ the IFPS in $X = \prod_{j=1}^n X_j$, where $A_j \in IFS(X_j)$ for each $j = 1, \cdots, n$. Then $\pi_j(A) \subset A_j$ for each $j = 1, \cdots, n$.

Proof. Let $x_j \in X_j$. Then $\mu_{\pi_j(A)}(x_j) = \pi_j(\mu_A)(x_j)$

$$= \bigvee_{(z_1,\cdots,z_n)\in\pi_j^{-1}(x_j)} \mu_A(z_1, \cdots, z_n)$$

$$= \bigvee_{(z_1,\cdots,z_n)\in\pi_j^{-1}(x_j)}[\mu_{A_1}(z_1) \wedge \cdots \wedge \mu_{A_n}(z_n)]$$

$$= (\bigvee_{z_1\in X_1} \mu_{A_1}(z_1)) \wedge \cdots \wedge \mu_{A_j}(x_j) \wedge \cdots \wedge (\bigvee_{z_n\in X_n} \mu_{A_n}(z_n))$$

$$\leq \mu_{A_j}(x_j).$$

and

$$\nu_{\pi_j(A)}(x_j) = \pi_j(\nu_A)(x_j)$$

$$= \bigwedge_{(z_1,\cdots,z_n)\in\pi_j^{-1}(x_j)} \nu_A(z_1, \cdots, z_n)$$

$$= \bigwedge_{(z_1,\cdots,z_n)\in\pi_j^{-1}(x_j)}[\nu_{A_1}(z_1) \vee \cdots \vee \nu_{A_n}(z_n)]$$

$$= (\bigwedge_{z_1\in X_1} \nu_{A_1}(z_1)) \vee \cdots \vee \nu_{A_j}(x_j) \vee \cdots \vee (\bigwedge_{z_n\in X_n} \nu_{A_n}(z_n))$$

$$\geq \mu_{A_j}(x_j).$$

Hence $\pi_j(A) \subset A_j$ for each $j = 1, \cdots, n$.

Proposition 2.14′. Let $\{(X_j, T_j)\}, j = 1, \cdots, n$, be a finite family of IFTSs, let (X, T) the IFPS and let $A = \prod_{j=1}^n A_j$, where $A_j \in IFS(X_j)$ for each $j = 1, \cdots, n$. Then the IIFT T_A on A has as a base the set of IFPSs of the form $\prod_{j=1}^n U_j$, where $U_j \in (T_j)_{A_j}$ for each $j = 1, \cdots, n$.

Proof. By the preceding remark, T has a base

$$\mathcal{B} = \{\prod_{j=1}^n U'_j : U'_j \in T_j \text{ for each } j = 1, \cdots, n\}.$$

Thus, by Definition 2.8′, a base for T_A is given by

$$\mathcal{B}_A = \{(\prod_{j=1}^n U'_j) \cap A : U'_j \in T_j \text{ for each } j = 1, \cdots, n\}.$$

On the other hand, $(\prod_{j=1}^n U_j') \cap A = \prod_{j=1}^n (U_j' \cap A_j)$ and $U_j' \cap A_j \in (T_j)_{A_j}$ for each $j = 1, \cdots, n$. Let $U_j = U_j' \cap A_j$ for each $j = 1, \cdots, n$. Then

$\mathcal{B}_A = \{\prod_{j=1}^n U_j : U_j \in (T_j)_{A_j}$ for each $j = 1, \cdots, n\}$.

By an abuse of notation, we will denote the IFSP (A, T_A) by $\prod_{j=1}^n (A_j, (T_j)_{A_j})$.

Proposition 2.16. Let $\{(X_\alpha, T_\alpha)\}_{\alpha \in \Gamma}$ be a family of IFTSs, let (X, T) the IFPS, let (Y, T_Y) an IFTS and let $f : Y \to X$ a mapping. Then $f : (Y, T_Y) \to (X, T)$ is IF- continuous if and only if $\pi_\alpha \circ f : (Y, T_Y) \to (X_\alpha, T_\alpha)$ is IF- continuous for each $\alpha \in \Gamma$.

Proof.($\Rightarrow$): Suppose $f : (Y, T_Y) \to (X, T)$ is IF- continuous. For each $\alpha \in \Gamma$, let $U_\alpha \in T_\alpha$. By the definition of T, $\pi_\alpha^{-1}(U_\alpha) \in T$. By the hypothesis, $f^{-1}(\pi_\alpha^{-1}(U_\alpha)) \in T_Y$. But $f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = (\pi_\alpha \circ f)^{-1}(U_\alpha)$. Thus $(\pi_\alpha \circ f)^{-1}(U_\alpha) \in T_Y$. Hence $\pi_\alpha \circ f : (Y, T_Y) \to (X_\alpha, T_\alpha)$ is IF- continuous.

($\Leftarrow$): Suppose the necessary condition holds and let $U \in T$. By Proposition 2.14, there exists a finitie subset Γ' of Γ such that $U = \cap_{\alpha \in \Gamma'} \pi_\alpha^{-1}(U_\alpha)$ and $U_\alpha \in T_\alpha$ for each $\alpha \in \Gamma'$. Since $\pi_\alpha \circ f : (Y, T_Y) \to (X_\alpha, T_\alpha)$ is IF- continuous for each $\alpha \in \Gamma'$, $(\pi_\alpha \circ f)^{-1}(U_\alpha) \in T_Y$ for each $\alpha \in \Gamma'$. Thus $f^{-1}(\pi_\alpha^{-1}(U_\alpha)) \in T_Y$ for each $\alpha \in \Gamma'$. So $\bigcap_{\alpha \in \Gamma'} f^{-1}(\pi_\alpha^{-1}(U_\alpha)) \in T_Y$. On the other hand, $\bigcap_{\alpha \in \Gamma'} f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = f^{-1}(\bigcap_{\alpha \in \Gamma'} \pi_\alpha^{-1}(U_\alpha)) = f^{-1}(U)$. So $f^{-1}(U) \in T_Y$. Hence $f : (Y, T_Y) \to (X, T_{\pi_\alpha^{-1}})$ is IF- continuous.

The following is the immediate result of Proposition 2.16:

Corollary 2.16. Let $\{(X_\alpha, T_{X_\alpha})\}_{\alpha \in \Gamma}$, $\{(Y_\alpha, T_{Y_\alpha})\}_{\alpha \in \Gamma}$ be two families of IFTSs and let (X, T_X), (Y, T_Y) the respective IFPTSs, where

$X = \prod_{\alpha\in\Gamma} X_\alpha$ and $Y = \prod_{\alpha\in\Gamma} Y_\alpha$. For each $\alpha \in \Gamma$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a mapping. Then the product mapping $f = \prod_{\alpha\in\Gamma} f_\alpha : (X, T_X) \to (Y, T_Y)$ is IF- continuous if and only if $f_\alpha : (X, T_{X_\alpha}) \to (Y, T_{Y_\alpha})$ is IF- continuous for each $\alpha \in \Gamma$, where $f(x) = (f_\alpha(\pi_\alpha(x)))$ for each $x \in \prod_{\alpha\in\Gamma} X_\alpha$.

Proposition 2.16′. Let $\{(X_j, T_j)\}_j$, $j = 1, \cdots, n$, be a finite family of IFTSs, and let (X, T) the IFPS. For each $j = 1, \cdots, n$, let $A_j \in IFS(X_j)$ and let $A = \prod_{j=1}^n A_j \in IFS(X)$. Let (Y, T_Y) be an IFTS and let $B \in IFS(Y)$. Then $f : (B, T_{Y,B}) \to (A, T_A)$ is relatively IF- continuous if and only if $\pi_j \circ f : (B, T_{Y,B}) \to (A_j, (T_j)_{A_j})$ is relatively IF- continuous for each $j = 1, \cdots, n$.

Proof. ($\Rightarrow$): Suppose $f : (B, T_{Y,B}) \to (A, T_A)$ is relatively IF-continuous. Clearly $\pi_j : (X, T) \to (X_j, T_j)$ is IF-continuous for each $j = 1, \cdots, n$ and, by Proposition 2.15, $\pi_j(A) \subset A_j$ for each $j = 1, \cdots, n$. Then, by Proposition 2.6, $\pi_j : (A, T_A) \to (A_j, (T_j)_{A_j})$ is relatively IF-continuous for each $j = 1, \cdots, n$. Hence $\pi_j \circ f : (B, T_{Y,B}) \to (A_j, (T_j)_{A_j})$ is relatively IF-continuous for each $j = 1, \cdots, n$.

($\Leftarrow$): Suppose the necessary condition holds. Let $U = U_1 \times \cdots \times U_n$, where $U_j \in (T_j)_{A_j}$ for each $j = 1, \cdots, n$. By Proposition 2.14', the set of such U forms a base for T_A. On the other hand, $f^{-1}(U) \cap B = f^{-1}[\pi_1^{-1}(U_1) \cap \cdots \cap \pi_n^{-1}(U_n)] \cap B = \bigcap_{j=1}^n ((\pi_1 \circ f)^{-1}[U_j] \cap B)$. Since $\pi_j \circ f : (B, T_{Y,B}) \to (A_j, (T_j)_{A_j})$ is relatively IF-continuous for each $j = 1, \cdots, n$, $f^{-1}(U) \cap B \in T_{Y,B}$. Hence, by Proposition 2.9', $f : (B, T_{Y,B}) \to (A, T_A)$ is relatively IF-continuous.

Corollary 2.16′. Let $\{(X_j, T_{X_j})\}$, $\{(Y_j, T_{Y_j})\}$, $j = 1, \cdots, n$, be two finite families of IFTSs, let (X, T_X), (Y, T_Y) the respective IFPSs. For each $j = 1, \cdots, n$, let $A_j \in IFS(X_j)$, $B_j \in IFS(Y_j)$ and let $A = \prod_{j=1}^n A_j$, $B = \prod_{j=1}^n B_j$ be the IFPSs in X, Y, respectively. If $f_j : (A_j, (T_{X_j})_{A_j}) \to (B_j, (T_{Y_j})_{B_j})$ is relatively IF-continuous for each $j =$

$1, \cdots, n$, then the product mapping $f = \prod_{j=1}^{n} f_j : (A, T_{X,A}) \to (B, T_{Y,B})$ is relatively IF-continuous.

Proof. It is analogous to the proof of from Corollary 2.16.

Proposition 2.17. Let $\{(X_j, T_{X_j})\}$, $\{(Y_j, T_{Y_j})\}$, $j = 1, \cdots, n$, be two finite families of IFTSs and let (X, T_X), (Y, T_Y) the respective IFPSs. If $f_j : (X_j, T_{X_j}) \to (Y_j, T_{Y_j})$ is IF-open for each $j = 1, \cdots, n$, then the product mapping $f = \prod_{j=1}^{n} f_j : (X, T_X) \to (Y, T_Y)$ is IF-open.

Proof. Let $\mathcal{B} = \{\prod_{j=1}^{n} U_j \in IFS(X) : U_j \in T_{X_j}$ for each $j = 1, \cdots, n\}$ and let $C_{0_\sim} \neq U \in T_X$. Since $\mathcal{B}$ is a base for T_X, there is a $\mathcal{B}' \subset B$ such that $U = \bigcup \mathcal{B}'$. Since each member of $\mathcal{B}'$ is of the form $\prod_{j=1}^{n} U_j$, we can consider $\mathcal{B}'$ as $\{\prod_{j=1}^{n} U_{j,\alpha}\}_{\alpha \in \Gamma}$ indexed by Γ. Then $U = \bigcup_{\alpha \in \Gamma} \prod_{j=1}^{n} U_{j,\alpha}$. Let $y \in Y$ such that $f^{-1}(y) \neq \emptyset$. Then

$$
\begin{aligned}
&\mu_{f(U)}(y) = f(\mu_U)(y) \\
&= \bigvee_{z \in f^{-1}(y)} \mu_U(z) \\
&= \bigvee_{z \in f^{-1}(y)} \mu_{\bigcup_{\alpha \in \Gamma} \prod_{j=1}^{n} U_{j,\alpha}}(z) \\
&= \bigvee_{z \in f^{-1}(y)} \bigvee_{\alpha \in \Gamma} \mu_{\prod_{j=1}^{n} U_{j,\alpha}}(z) \\
&= \bigvee_{\alpha \in \Gamma} \bigvee_{z_1 \in f_1^{-1}(y_1)} \cdots \bigvee_{z_n \in f_n^{-1}(y_n)} [\mu_{U_{1,\alpha}}(z_1) \wedge \cdots \wedge \mu_{U_{n,\alpha}}(z_n)] \\
&= \bigvee_{\alpha \in \Gamma} [(\bigvee_{z_1 \in f_1^{-1}(y_1)} \mu_{U_{1,\alpha}}(z_1) \wedge \cdots \wedge (\bigvee_{z_n \in f_n^{-1}(y_n)} \mu_{U_{n,\alpha}}(z_n))] \\
&= \bigvee_{\alpha \in \Gamma} [\mu_{f_1(U_{1,\alpha})}(y_1) \wedge \cdots \wedge \mu_{f_n(U_{n,\alpha})}(y_n)] \\
&= \bigvee_{\alpha \in \Gamma} \mu_{\prod_{j=1}^{n} f_j(U_{j,\alpha})}(y) \\
&= \mu_{\bigcup_{\alpha \in \Gamma} \prod_{j=1}^{n} f_j(U_{j,\alpha})}(y)
\end{aligned}
$$

and

$$
\begin{aligned}
&\nu_{f(U)}(y) = f(\nu_U)(y) \\
&= \bigwedge_{z \in f^{-1}(y)} \nu_U(z) \\
&= \bigwedge_{z \in f^{-1}(y)} \nu_{\bigcup_{\alpha \in \Gamma} \prod_{j=1}^{n} U_{j,\alpha}}(z) \\
&= \bigwedge_{z \in f^{-1}(y)} \bigwedge_{\alpha \in \Gamma} \nu_{\prod_{j=1}^{n} U_{j,\alpha}}(z) \\
&= \bigwedge_{\alpha \in \Gamma} \bigwedge_{z_1 \in f_1^{-1}(y_1)} \cdots \bigwedge_{z_n \in f_n^{-1}(y_n)} [\nu_{U_{1,\alpha}}(z_1) \vee \cdots \vee \nu_{U_{n,\alpha}}(z_n)] \\
&= \bigwedge_{\alpha \in \Gamma} [(\bigwedge_{z_1 \in f_1^{-1}(y_1)} \nu_{U_{1,\alpha}}(z_1)) \vee \cdots \vee (\bigwedge_{z_n \in f_n^{-1}(y_n)} \nu_{U_{n,\alpha}}(z_n))]
\end{aligned}
$$

$= \bigwedge_{\alpha\in\Gamma}[\nu_{f_1(U_{1,\alpha})}(y_1) \vee \cdots \vee \nu_{f_n(U_{n,\alpha})}(y_n)]$
$= \bigwedge_{\alpha\in\Gamma} \nu_{\prod_{j=1}^n f_j(U_{j,\alpha})}(y)$
$= \nu_{\bigcup_{\alpha\in\Gamma}\prod_{j=1}^n f_j(U_{j,\alpha})}(y).$

Thus $f(U) = \bigcup_{\alpha\in\Gamma}\prod_{j=1}^n f_j(U_{j,\alpha})$. Since f_j is IF-open for each $j = 1,\cdots,n$, $f_j(U_{j,\alpha}) \in IFO(X_j)$ for each $j = 1,\cdots,n$. Then $\prod_{j=1}^n f_j(U_{j,\alpha}) \in IFO(Y)$. So $\bigcup_{\alpha\in\Gamma}\prod_{j=1}^n f_j(U_{j,\alpha}) \in IFO(Y)$. Hence f is IF-open.

Proposition 2.17′. Let $\{(X_j, T_{X_j})\}$, $\{(Y_j, T_{Y_j})\}$, $j = 1,\cdots,n$, be two finite families of IFTSs and let (X, T_X), (Y, T_Y) the respective IFPSs. For each $j = 1,\cdots,n$, let $A_j \in IFS(X_j)$, $B_j \in IFS(Y_j)$ and let $A = \prod_{j=1}^n A_j$, $B = \prod_{j=1}^n B_j$ be the IFPSs in X, Y respectively. If $f_j : A_j \to B_j$ is relatively IF-open for each $j = 1,\cdots,n$, then the product mapping $f = \prod_{j=1}^n f_j : (A, T_A) \to (B, T_B)$ is relatively IF-open.

Proof. Let $\mathcal{B} = \{\prod_{j=1}^n U_j \in IFS(A) : U_j \in (T_{X_j})_{A_j}$ for each $j = 1,\cdots,n\}$. Then by Proposition 2.14', $\mathcal{B}$ is a base for T_A. Let $U \in T_A$. Then there is $\mathcal{B}' \subset B$ such that $\bigcup\mathcal{B}' = U$. We can consider $\mathcal{B}'$ as $\{\prod_{j=1}^n U_{j,\alpha}\}_{\alpha\in\Gamma}$ indexed by Γ. Then $U = \bigcup_{\alpha\in\Gamma}\prod_{j=1}^n U_{j,\alpha}$. As in the proof of Proposition 2.17, it follows that

$$f(U) = \bigcup_{\alpha\in\Gamma}\prod_{j=1}^n f_j(U_{j,\alpha}).$$

Since f_j is relatively IF-open foe each $j = 1,\cdots,n$, $f(U) \in T_B$. Hence f is relatively IF-open.

Lemma 2.18. Let (X_1, T_1), (X_2, T_2) be IFTSs. Then the constant mapping $c : (X_2, T_2) \to (X_1, T_1)$ given by $c(x_2) = a_0 \in X_1$ for each $x_2 \in X_2$, is IF-continuous.

Proof. Let $U \in T_1$ and let $x_2 \in X_2$. Then

$$\mu_{c^{-1}(U)}(x_2) = c^{-1}(\mu_U)(x_2) = \mu_U c(x_2) = \mu_U(a_0).$$

Similarly, we have $\nu_{c^{-1}(U)}(x_2) = \nu_U(a_0)$. Let $\mu_U(a_0) = \alpha$ and let $\nu_U(a_0) = \beta$. Consider $C_{(\alpha,\beta)}$. Since $U \in IFS(X_1)$, $\alpha + \beta \leq 1$. Then

$C_{(\alpha,\beta)} \in IFO(X_2)$. On the other hand, $\mu_{c^{-1}(U)}(x_2) = \alpha = \mu_{C_{(\alpha,\beta)}}(x_2)$ and $\nu_{c^{-1}(U)}(x_2) = \beta = \nu_{C_{(\alpha,\beta)}}(x_2)$. Thus $c^{-1}(U) = C_{(\alpha,\beta)}$. So $c^{-1}(U) \in IFO(X_2)$. Hence c is IF-continuous.

Proposition 2.19. Let (X_1, T_1), (X_2, T_2) be IFTSs and let (X, T) the IFPS. Then for each $a_1 \in X_1$, the mapping $i : (X_2, T_2) \to (X, T)$ defined by $i(x_2) = (a_1, x_2)$ for each $x_2 \in X_2$, is IF-continuous.

Proof. By Lemma 2.18, the constant mapping $i_1 : (X_2, T_2) \to (X_1, T_1)$ given by $i_1(x_2) = a_1$ for each $x_2 \in X_2$, is IF-continuous. On the other hand, the identity mapping $i_2 : (X_2, T_2) \to (X_2, T_2)$ is IF-continuous. Hence, by Proposition 2.16, i is IF-continuous.

Proposition 2.19′. Let (X_1, T_1), (X_2, T_2) be IFTSs and let (X, T) the IFPS. Let A_1, A_2 be IFSs in X_1, X_2, respectively and let A the IFPS in X. Let $a_1 \in X_1$ such that $\mu_{A_1}(a_1) \geq \mu_{A_2}(x_2)$ and $\nu_{A_1}(a_1) \leq \nu_{A_2}(x_2)$ for each $x_2 \in X_2$. Then the mapping $i : (A_2, (T_2)_{A_2}) \to (A, T_A)$ given by $i(x_2) = (a_1, x_2)$ for each $x_2 \in X_2$, is relatively IF-continuous.

Proof. Let $(x_1, x_2) \in X$. Then

$$\begin{aligned}
\mu_{i(A_2)}(x_1, x_2) &\doteq i(\mu_{A_2})(x_1, x_2) \\
&= \begin{cases} \bigvee_{x_2' \in i^{-1}(x_1,x_2)} \mu_{A_2}(x_2') & if i^{-1}(x_1, x_2) \neq \emptyset, \\ 0 & if otherwise \end{cases} \\
&= \begin{cases} \mu_{A_2}(x_2) & if x_1 = a_1, \\ 0 & if otherwise \end{cases},
\end{aligned}$$

and

$$\begin{aligned}
\nu_{i(A_2)}(x_1, x_2) &= i(\nu_{A_2})(x_1, x_2) \\
&= \begin{cases} \bigwedge_{x_2' \in i^{-1}(x_1,x_2)} \nu_{A_2}(x_2') & if i^{-1}(x_1, x_2) \neq \emptyset, \\ 1 & if otherwise \end{cases}
\end{aligned}$$

$$= \begin{cases} \nu_{A_2}(x_2) & if x_1 = a_1, \\ 1 & if otherwise \end{cases},$$

On the other hand, $\mu_A(x_1, x_2) = \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2)$ and $\nu_A(x_1, x_2) = \nu_{A_1}(x_1) \vee \nu_{A_2}(x_2)$. By the hypothesis, $\mu_A(x_1, x_2) \geq \mu_{A_2}(x_2)$ and $\nu_A(x_1, x_2) \leq \nu_{A_2}(x_2)$. Thus $\mu_A(x_1, x_2) \geq \mu_{i(A)}(x_1, x_2)$ and $\nu_A(x_1, x_2) \leq \nu_{i(A)}(x_1, x_2)$. So $i(A) \subset A$. The proof of the relative IF-continuity of i is analogous to the proof of the IF-continuity of i in Proposition 2.19. This completes the proof.

3. Intuitionistic fuzzy groups

Throughout this section and next sections, we will denote X as a group.

Definition 3.1[10]. Let $G \in IFS(X)$. Then G is called an *intuitionistic fuzzy group*(in short, *IFG*) in X if it satisfies the following conditions :

(i) $\mu_G(xy) \geq \mu_G(x) \wedge \mu_G(y)$ and $\nu_G(xy) \leq \nu_G(x) \vee \nu_G(y)$ for each $x, y \in X$,

(ii) $\mu_G(x^{-1}) \geq \mu_G(x)$ and $\nu_G(x^{-1}) \leq \nu_G(x)$ for each $x \in X$.

Result 3.A[10, Proposition 2.6]. If $G \in IFG(X)$, then $\mu_G(x^{-1}) = \mu_G(x)$, $\nu_G(x^{-1}) = \nu_G(x)$ and $\mu_G(e) \geq \mu_G(x)$, $\nu_G(e) \leq \nu_G(x)$ for each $x \in X$, where e is the unity of X.

Result 3.B[10, Proposition 2.9]. $G \in IFG(X)$ if and only if $\mu_G(xy^{-1}) \geq \mu_G(x) \wedge \mu_G(y)$ and $\nu_G(xy^{-1}) \leq \nu_G(x) \vee \nu_G(y)$ for each $x, y \in X$.

Definition 3.2[9]. Let A be an IFS in a set X. Then A is sand to have the *sup property* if for any $T \subset X$, there exists a $t_0 \in T$ such that

$A(t_0) = \bigcup_{t \in T} A(t)$, i.e., $\mu_A(t_0) = \bigwedge_{t \in T} \mu_A(t)$ and $\nu_A(t_0) = \bigvee_{t \in T} \nu_A(t)$.

Result 3.C[10, Proposition 2.13]. Let $f : X \to Y$ be a group homomorphism.

(1) If $G' \in IFG(Y)$, then $f^{-1}(G') \in IFG(X)$.

(2) If $G \in IFG(X)$ and G has the sup property, then $f(G) \in IFG(Y)$.

Definition 3.3. Let $f : X \to Y$ be a group homomorphism and let $G \in IFG(X)$. G is said to be *IF-invariant* if for any $x_1, x_2 \in X$,

$f(x_1) = f(x_2) \Rightarrow \mu_G(x_1) = \mu_G(x_2)$ and $\nu_G(x_1) = \nu_G(x_2)$.

It is clear that if G is IF-invariant, then $f(G) \in IFG(Y)$.

For each $G \in IFG(X)$, let $X_G = \{x \in X : \mu_G(x) = \mu_G(e)$ and $\nu_G(x) = \nu_G(e)\}$. Then it is clear that X_G is a subgroup of X. For each $a \in X$, let $\rho_a : X \to X$ and $\lambda_a : X \to X$ be the right and left translations of X into itself, defined by $\rho_a(x) = xa$ and $\lambda_a(x) = ax$, respectively for each $x \in X$. Then clearly $\rho_a^{-1} = \rho_{a^{-1}}$ and $\lambda_a^{-1} = \lambda_{a^{-1}}$.

Proposition 3.4. If $G \in IFG(X)$, then $\rho_a(G) = \lambda_a(G) = G$, for each $a \in X_G$.

Proof. Let $a \in X_G$ and let $x \in X$. Then

$$\begin{aligned}
\mu_{\rho_a(G)}(x) &= \rho_a(\mu_G)(x) \\
&= \textstyle\bigvee_{y \in \rho_a^{-1}(x)} \mu_G(y) = \bigvee_{x=\rho_a(y)=ya} \mu_G(y) = \bigvee_{y=xa^{-1}} \mu_G(y) \\
&\geq \mu_G(xa^{-1}) = \mu_G(x) \wedge \mu_G(a^{-1}) = \mu_G(x) = \mu_G(xa^{-1}a) \\
&\geq \mu_G(xa^{-1}) \wedge \mu_G(a) = \mu_G(xa^{-1}) \wedge \mu_G(e) = \mu_G(xa^{-1}) \\
&= \textstyle\bigvee_{y=xa^{-1}} \mu_G(y) = \bigvee_{y \in \rho_a^{-1}(x)} \mu_G(y) = \mu_{\rho_a(G)}(x).
\end{aligned}$$

Thus $\mu_{\rho_a(G)} = \mu_G$.

On the other hand,

$\nu_{\rho_a(G)}(x) = \rho_{a-}(\nu_G)(x)$
$= \bigwedge_{y\in\rho_a^{-1}(x)} \nu_G(y) = \bigwedge_{x=\rho_a(y)=ya} \nu_G(y) = \bigwedge_{y=xa^{-1}} \nu_G(y) = \nu_G(xa^{-1})$
$\leq \nu_G(x) \vee \nu_G(a^{-1}) = \nu_G(x) \vee \nu_G(e) = \nu_G(x) = \nu_G(xa^{-1}a)$
$\leq \nu_G(xa^{-1}) \vee \nu_G(a) = \nu_G(xa^{-1}) \vee \nu_G(e) = \nu_G(xa^{-1})$
$= \bigwedge_{y=xa^{-1}} \nu_G(y) = \bigwedge_{y\in\rho_a^{-1}(x)} \nu_G(y) = \nu_{\rho_a(G)}(x).$

Thus $\nu_{\rho_a(G)} = \nu_G$. So $\rho_a(G) = G$. Similarly, we have $\lambda_a(G) = G$. Hence $\rho_a(G) = \lambda_a(G) = G$.

Proposition 3.5. Let $f : X \to Y$ be a group homomorphism and let $G \in IFG(X)$. If G is IF-invariant, then $f(U \cap G) = f(U) \cap f(G)$ for each $U \in IFG(X)$.

Proof. Let $U \in IFS(X)$ and let $V = U \cap G$. Let $y \in Y$ such that $f^{-1}(y) \neq \emptyset$. Then :

$\mu_{f(V)}(y) = f(\mu_V)(y)$
$= \bigvee_{x\in f^{-1}(y)} \mu_V(x) = \bigvee_{x\in f^{-1}(y)} \mu_{U\cap G}(x) = \bigvee_{x\in f^{-1}(y)}[\mu_U(x)\wedge\mu_G(x)]$
$= (\bigvee_{x\in f^{-1}(y)} \mu_U(x)) \wedge (\bigvee_{x\in f^{-1}(y)} \mu_G(x))$ (G is IF-invariant)
$= \mu_{f(U)}(y) \wedge \mu_{f(G)}(y)$
$= \mu_{f(U)\cap f(G)}(y).$

Thus $\mu_{f(V)} = \mu_{f(U)\cap f(G)}$. Similarly, we have $\nu_{f(V)} = \nu_{f(U)\cap f(G)}$. Hence $f(U \cap G) = f(U) \cap f(G)$ for each $U \in IFG(X)$.

4. Intuitionistic fuzzy topological groups

Proposition 4.1. Let $G \in IFG(X)$ and let $\alpha : X \times X \to X$ and $\beta : X \to X$ be the mappings defined relatively as follows: for any $x, y \in X$,

$\alpha(x, y) = xy$ and $\beta(x) = x^{-1}$.

Then $\alpha(G \times G) \subset G$ and $\beta(G) \subset G$.

Proof. Let $z \in X$. Then

$$\mu_{\alpha(G\times G)}(z) = \alpha(\mu_{G\times G})(z)$$
$$= \bigvee_{(x,y)\in\alpha^{-1}(z)} \mu_{G\times G}(x,y)$$
$$= \bigvee_{z=xy}[\mu_G(x) \wedge \mu_G(y)]$$
$$\leq \mu_G(xy)$$
$$= \mu_G(z)$$

and

$$\nu_{\alpha(G\times G)}(z) = \alpha(\nu_{G\times G})(z)$$
$$= \bigwedge_{(x,y)\in\alpha^{-1}(z)} \nu_{G\times G}(x,y)$$
$$= \bigwedge_{z=xy}[\nu_G(x) \vee \nu_G(y)]$$
$$\geq \nu_G(xy)$$
$$= \nu_G(z).$$

Hence $\alpha(G \times G) \subset G$. Similarly, we have $\beta(G) \subset G$.

Let $G \in IFG(X)$ and let T be a given IFT on X. Then, by Definition 2.2, (G, T_G) is an IFSP of (X, T) and $(G, T_G) \times (G, T_G)$ an IFSP of $(X, T) \times (X, T)$.

Definition 4.2. Let T be an IFT on X. Let $G \in IFG(X)$ and let (G, T_G) be the IFSP of (X, T). Then G is called an *intuitionisti fuzzy topological group* (in short, *IFTG*) in X if it satisfies the following two conditions :

(i) The mapping $\alpha : (G, T_G) \times (G, T_G) \to (G, T_G)$ is relatively IF-continuous.

(ii) The mapping $\beta : (G, T_G) \to (G, T_G)$ is relatively IF-continuous.

An IFG structure and IIFT are said to be *compatible* if they satisfy (i) and (ii).

Proposition 4.3. Let T be an IFT on X, let $G \in IFG(X)$ and let $\gamma : X \times X \to X$ be the mapping defined by $\gamma(x, y) = xy^{-1}$ for any $x, y \in X$. Then G is an IFTG in X if and only if the mapping $\gamma : (G, T_G) \times (G, T_G) \to (G, T_G)$ is relatively IF-continuous.

Proof. ($\Rightarrow$): Suppose G is an IFTG in X. Then, by Definition 4.2, $\alpha : G \times G \to G$, $\beta : G \to G$ are relatively IF-continuous. Moreover, $\gamma = \alpha \circ (1_X \times \beta)$ it is clear that $\gamma(G \times G) \subset G$. By Corollary 2.16', $1_X \times \beta$ is relatively IF-continuous. Then $\alpha \circ (1_X \times \beta)$ is relatively IF-continuous. Hence γ is relatively IF-continuous.

($\Leftarrow$): Suppose the necessary condition holds. By Result 3.A, $\mu_G(e) \geq \mu_G(x)$ and $\nu_G(e) \leq \nu_G(x)$ for each $x \in X$. By Proposition 2.19', the canonical injection $i : G \to G \times G$ given by $i(y) = (e, y)$ for each $y \in X$, is relatively IF-continuous. But $\beta = \gamma \circ i$. So β is relatively IF-continuous. Thus $1_X \times \beta$ is relatively IF-continuous and $\alpha = \gamma \circ (1_X \times \beta)$. So α is relatively IF-continuous. Hence G is an IFTG in X.

Definition 4.4. Let (A, T_A) and (B, T_B) be IFSPs of IFTSs (X, T_X) and (Y, T_Y), respectively and let $f : X \to Y$ a bijection.

(1) f is called an *intuitionistic fuzzy homeomorphism*(in short, *IF-homeomorphism*) if it is IF- continuous and IF- open.

(2) $f : (A, T_A) \to (B, T_B)$ is called a *relatively intuitionistic fuzzy homeomorphism*(in short, *relatively IF-homeomorphism*) if it is relatively IF- continuous and relatively IF- open and $f(A) = B$.

Let G be an IFTG in a group X carrying IFT T. Then, in general, the translations $\rho_a, \lambda_a, a \in X$ are not relatively IF-continuous mappings of (G, T_G) into itself. However, we have the following special case.

Proposition 4.5. Let X be a group with IFT T and let G be an IFTG in X. Then for each $a \in X_G$, the translations ρ_a, λ_a are relatively IF-homeomorphisms of (G, T_G) into itself.

Proof. Proposition 3.4, $\rho_a(G) = \lambda_a(G) = G$ for each $a \in X_G$. Let $a \in X_G$ and let $i : G \to G \times G$ be the canonical injection given by $i(y) = (a, y)$ for each $y \in X$. Then $\lambda_a = \alpha \circ i$. Since $a \in G_e$, $\mu_G(a) = \mu_G(e)$ and $\nu_G(a) = \nu_G(e)$. Thus $\mu_G(a) \geq \mu_G(y)$ and $\nu_G(a) \leq \nu_G(y)$ for each $y \in X$. It follows from Proposition 2.19' that $i : (G, T_G) \to (G, T_G) \times (G, T_G)$ is relatively IF-continuous. By the hypothesis, α is relatively IF-continuous. So λ_a is relatively IF-continuous. Moreover, $\lambda_a^{-1} = \lambda_{a^{-1}}$. The relatively IF- continuity of ρ_a and ρ_a^{-1} is shown similarly.

5. Homomorphisms

Let $f : X \to Y$ be a group homomorphism, let Y have an IFT T_Y and let G be an IFTG in Y. Then the mapping f given rise to on IFT T on X, the *inverse image* under f of T_Y, and, by Result 3.C(1), it also given rise to an IFG in X, the inverse image $f^{-1}(G)$ of G.

Proposition 5.1. Let $f : X \to Y$ be a group homomorphism, let $T_Y \in IFT(Y)$, let T the inverse image of T_Y under f and let $(G, (T_Y)_G)$ an IFTG in Y. Then the inverse image $f^{-1}(G)$ of G is an IFTG in X.

Proof. Consider the mapping $\gamma_X : X \times X \to X$ defined by $\gamma_X(x_1, x_2) = x_1 x_2^{-1}$ for any $x_1, x_2 \in X$. We well show that the mapping $\gamma_X : (f^{-1}(G), T_{f^{-1}(G)}) \times (f^{-1}(G), T_{f^{-1}(G)}) \to (f^{-1}(G), T_{f^{-1}(G)})$ is relatively IF-continuous. Since T is the inverse image of T_Y under f, $f : (X, T) \to (Y, T_Y)$ is IF-continuous. Moreover $f(f^{-1}(G)) \subset G$. By Proposition 2.6, $f : (f^{-1}(G), T_{f^{-1}(G)}) \to (G, (T_Y)_G)$ is relatively IF-continuous. Let $U \in T_{f^{-1}(G)}$. Then there exists a $V \in (T_Y)_G$ such that $f^{-1}(V) = U$. Let $(x_1, x_2) \in X \times X$. Then :

$$\mu_{\gamma_X^{-1}(U)}(x_1, x_2) = \gamma_X^{-1}(\mu_U)(x_1, x_2) = \mu_U(\gamma_X(x_1, x_2)) = \mu_U(x_1 x_2^{-1})$$

$= \mu_{f^{-1}(V)}(x_1x_2^{-1}) = f^{-1}(\mu_V)(x_1x_2^{-1}) = \mu_V(f(x_1x_2^{-1}))$
$= \mu_V(f(x_1)f(x_2^{-1})) = \mu_V(f(x_1)(f(x_2))^{-1})$.

Thus $\mu_{\gamma_X^{-1}(U)}(x_1, x_2) = \mu_V(f(x_1)(f(x_2))^{-1})$. Similarly, we have $\nu_{\gamma_X^{-1}(U)}(x_1, x_2) = \nu_V(f(x_1)(f(x_2))^{-1})$. By the hypothesis, the mapping $\gamma_Y : (G, T_G) \times (G, T_G) \to (G, T_G)$ given by $\gamma_Y(y_1, y_2) = y_1y_2^{-1}$ for any $y_1, y_2 \in Y$, is relatively IF-continuous. By Corollary 2.16', the product mapping

$$f \times f : (f^{-1}(G), T_{f^{-1}(G)}) \times (f^{-1}(G), T_{f^{-1}(G)}) \to (G, T_G)$$

is relatively IF-continuous. Now let $(x_1, x_2) \in X \times X$. Then:

$$\mu_V(f(x_1)(f(x_2))^{-1}) = \mu_{\gamma_Y^{-1}(V)}(f(x_1), f(x_2)) = \mu_{(f\times f)^{-1}(\gamma_Y^{-1}(V))}(x_1, x_2)$$

and

$$\nu_V(f(x_1)(f(x_2))^{-1}) = \nu_{\gamma_Y^{-1}(V)}(f(x_1), f(x_2)) = \nu_{(f\times f)^{-1}(\gamma_Y^{-1}(V))}(x_1, x_2).$$

Thus $\gamma_X^{-1}(U) \cap (f^{-1}(G) \times f^{-1}(G))$
$= (f \times f)^{-1}(\gamma_Y^{-1}(V)) \cap (f^{-1}(G) \times f^{-1}(G))$
$= [\gamma_Y \circ (f \times f)]^{-1}(V) \cap (f^{-1}(G) \times f^{-1}(G))$.

So $\gamma_X^{-1}(U) \cap (f^{-1}(G) \times f^{-1}(G)) \in T_{f^{-1}(G)} \times T_{f^{-1}(G)}$, i.e., $\gamma_X : (f^{-1}(G), T_{f^{-1}(G)}) \times (f^{-1}(G), T_{f^{-1}(G)}) \to (f^{-1}(G), T_{f^{-1}(G)})$ is relatively IF-continuous. Moreover, by Result 3.C(1), $f^{-1}(G)$ is an IFG in X. Hence, by Proposition 4.3, $f^{-1}(G)$ is an IFTG in X.

Proposition 5.2. Let $f : X \to Y$ be a group homomorphism, let $T_X \in IFT(X)$, let T the image of T_X under f and let G an IFTG in X. If G is IF-invariant, then the image $f(G)$ of G is an IFTG in Y.

Proof. Suppose G is f-invariant. By the remark in Definition 3.3, $f(G)$ is an IFG in Y. Let $U \in T_X$. Then, by Result 1.B(4), $U \subset f^{-1}(f(U))$. Thus there exists a family $\{U_\alpha\}_{\alpha\in\Gamma} \subset T_X$ such that $f^{-1}(f(U)) = \bigcup_{\alpha\in\Gamma} U_\alpha$. So $f^{-1}(f(U)) \in T_X$. Since T is the image of T_X under f, $f(U) \in T$. So f is IF-open. Now let $U \in (T_X)_G$. Then there exists a $U' \in T_X$ such that $U = U' \cap G$. Since G is f-invariant, by Proposition 3.5, $f(U) = f(U') \cap f(G)$. Since f is IF-open, $f(U') \in T$. Then $f(U) \in T_{f(G)}$.

Thus $f : (G, (T_X)_G) \to (f(G), T_{f(G)})$ is relatively IF-open. By Proposition 2.17', the product mapping $f \times f : (G, (T_X)_G) \times (G, (T_X)_G) \to (f(G), T_{f(G)})$ is relatively IF-open.

Let $V \in T_{f(G)}$ and let $(x_1, x_2) \in X \times X$. Then

$$\begin{aligned}
&\mu_{[\gamma_Y \circ (f\times f)]^{-1}(V)}(x_1, x_2) = [\gamma_Y \circ (f \times f)]^{-1}(\mu_V)(x_1, x_2)\\
&= \mu_V[\gamma_Y \circ (f \times f)](x_1, x_2)\\
&= \mu_V \gamma_Y(f(x_1), f(x_2))\\
&= \mu_V(f(x_1)(f(x_2))^{-1})\\
&= \mu_V(f(x_1)f(x_2^{-1})) \quad \text{(Since } f \text{ is a homomorphism)}\\
&= \mu_V(f(x_1x_2^{-1})) \quad \text{(Since } f \text{ is a homomorphism)}\\
&= \mu_V f(\gamma_X(x_1, x_2))\\
&= \mu_V(f \circ \gamma_X)(x_1, x_2)\\
&= (f \circ \gamma_X)^{-1}(\mu_V)(x_1, x_2)\\
&= \mu_{(f\circ\gamma_X)^{-1}(V)}(x_1, x_2)
\end{aligned}$$

, where $\gamma_X : X \times X \to X$ is the mapping given by $\gamma_X(x_1, x_2) = x_1x_2^{-1}$ for each $(x_1, x_2) \in X \times X$. Thus $\mu_{[\gamma_Y\circ(f\times f)]^{-1}(V)} = \mu_{(f\circ\gamma_X)^{-1}(V)}$, i.e., $\mu_{(f\times f)^{-1}[\gamma_Y^{-1}(V)]} = \mu_{\gamma_X^{-1}(f^{-1}(V))}$. Similarly, we have $\nu_{(f\times f)^{-1}[\gamma_Y^{-1}(V)]} = \nu_{\gamma_X^{-1}(f^{-1}(V))}$. So $(f \times f)^{-1}[\gamma_Y^{-1}(V)] = \gamma_X^{-1}(f^{-1}(V))$. Since G is IFTG in X, $\gamma_X : (G, (T_X)_G) \times (G, (T_X)_G) \to (G, (T_X)_G)$ is relatively IF-continuous. Since T is the image of T_X under f, $f : (G, (T_X)_G) \to (f(G), T_{f(G)})$ is relatively IF-continuous. Then $f \times f : (G, (T_X)_G) \times (G, (T_X)_G) \to (f(G), T_{f(G)}) \times (f(G), T_{f(G)})$ is relatively IF-continuous. Thus $(f\times f)\circ\gamma_Y : (G, (T_X)_G) \times (G, (T_X)_G) \to (f(G), T_{f(G)})$ is relatively IF-continuous. Since A is IF-invariant,

$$(f \times f)^{-1}[\gamma_Y^{-1}(V) \cap (f(G) \times f(G))] = (f \times f)^{-1}[\gamma_Y^{-1}(V)] \cap (G \times G).$$

So $(f\times f)^{-1}[\gamma_Y^{-1}(V)\cap(f(G)\times f(G))] \in (T_X)_G\times(T_X)_G$. Since $f\times f$ is relatively IF-open, $(f\times f)(f\times f)^{-1}[\gamma_Y^{-1}(V)\cap(f(G)\times f(G))] \in T_{f(G)}\times T_{f(G)}$. But $(f\times f)(f\times f)^{-1}[\gamma_Y^{-1}(V)\cap(f(G)\times f(G))] = \gamma_Y^{-1}(V)\cap(f(G)\times f(G))$. So $\gamma_Y^{-1}(V)\cap(f(G)\times f(G)) \in T_{f(G)}\times T_{f(G)}$. Hence $f(G)$ is an IFTG in Y.

Let X be a group carrying a fuzzy topology T, let G an IFTG in X, let N a normal subgroup of X and let φ the canonical homomorphism of X onto the quotient group X/N.

Proposition 5.3. If G is constant on N, then G is φ-invariant. Hence, by Proposition 5.2, $\varphi(G)$ is an IFTG in X/N. In this case, $\varphi(G)$ is called an *intuitionistic fuzzy quotient group*(in short, *IFQG*) and denoted by G/N.

Proof. Suppose $\varphi(x_1) = \varphi(x_2)$ for any $x_1, x_2 \in X$. Then $x_1N = x_2N$. Thus there exist $n_1, n_2 \in N$ such that $x_1n_1 = x_2n_2$. Since G is constant on N, $\mu_G(x) = \mu_G(e)$ and $\nu_G(x) = \nu_G(e)$, for each $x \in N$. Then :

$$\begin{aligned}\mu_G(x_1) &= \mu_G(x_2n_2n_1^{-1}) \\ &\geq \mu_G(x_2) \wedge \mu_G(n_2n_1^{-1}) \\ &= \mu_G(x_2) \wedge \mu_G(e) \quad (n_2n_1^{-1} \in N) \\ &= \mu_G(x_2)\end{aligned}$$

and

$$\begin{aligned}\nu_G(x_1) &= \nu_G(x_2n_2n_1^{-1}) \\ &\leq \nu_G(x_2) \vee \nu_G(n_2n_1^{-1}) \\ &= \nu_G(x_2) \vee \nu_G(e) \quad (n_2n_1^{-1} \in N) \\ &= \nu_G(x_2).\end{aligned}$$

Similarly, we have $\mu_G(x_2) \geq \mu_G(x_1)$ and $\nu_G(x_2) \leq \nu_G(x_1)$. Thus $\nu_G(x_1) = \nu_G(x_2)$. Hence G is φ-invariant.

The following is the immediate result of Proposition 5.2 and Proposition 5.3:

Proposition 5.4. Let X be a group having a fuzzy topology T, let G an IFTG in X, and let N a normal subgroup of X. Let T_φ be the image of T under the canonical homomorphism φ. If G is constant on N, then the

IFQG G/N is an IFTG in X/N. In this case, T_φ is called an *intuitionistic fuzzy quotient topology*(in short, *IFQT*) on X/N and G/N is called an *intuitionistic fuzzy quotient topological group*(in short, *IFQTG*) on X/N.

Proposition 5.5. Let $f : X \to Y$ be a group epimorphism. Let $T_X \in IFT(X)$, $T_Y \in IFT(Y)$ and let f be both IF-continuous and IF-open. Let G be an IFTG in X such that G is constant on the kernel $f^{-1}(e)$ of f. Let the quotient group $X/_{f^{-1}(e)}$ have the IFQT T. Then

(1) The IFGs $G/_{f^{-1}(e)}$ and $f(G)$ are IFTGs in $X/_{f^{-1}(e)}$ and Y, respectively.

(2) The canonical isomorphism $k : X/_{f^{-1}(e)} \to Y$ given by $k(af^{-1}(e)) = f(a)$ for each $a \in X$, is a relative IF-homeomorphism of $G/_{f^{-1}(e)}$ onto $f(G)$.

it Proof.(1) From Proposition 5.4, it is clear that $G/_{f^{-1}(e)}$ is an IFTG in $X/_{f^{-1}(e)}$. Let $T_{f^{-1}}$ be the inverse of T_X under f and let $V \in T_{f^{-1}}$. Then $f^{-1}(V) \in T_X$. Since f is surjective and IF-open, $V = f(f^{-1}(V)) \in T_Y$. Thus $T_{f^{-1}} \subset T_Y$. Now let $V \in T_Y$. Since f is IF-continuous, $f^{-1}(V) \in T_X$. Then $V \in T_{f^{-1}}$, i.e., $T_Y \subset T_{f^{-1}}$. So $T_{f^{-1}} = T_Y$. Hence, by Proposition 5.2, $f(G)$ is an IFTG in Y.

(2) Let $\varphi : X \to X/f^{-1}(e)$ be the canonical epimorphism. Then clearly $k : X/f^{-1}(e) \to Y$ is bijective and $f = k \circ \varphi$. Let $V' \in (T_Y)_{f(G)}$. Since $f : (G, (T_X)_G) \to (f(G), (T_Y)_{f(G)})$ is relatively IF-continuous, $f^{-1}(V') = \varphi^{-1}(k^{-1}(V')) \in (T_X)_G$. Since G is an IFTG in X such that G is constant on $f^{-1}(e)$, by Proposition 6.1, $G/f^{-1}(e)$ is an IFQG in $X/f^{-1}(e)$. Since $\varphi : (G, (T_X)_G) \to (G/f^{-1}(e), T_{G/f^{-1}(e)})$ is relatively IF-open, $\varphi(f^{-1}(V')) \in T_{G/f^{-1}(e)}$. But $\varphi(f^{-1}(V')) = k^{-1}(V')$. Thus $k^{-1}(V') \in T_{G/f^{-1}(e)}$. So $k : (G/f^{-1}(e), T_{G/f^{-1}(e)}) \to (f(G), (T_Y)_{f(G)})$ is relatively IF-continuous. Now let $U \in T_{G/f^{-1}(e)}$. Since T is the image of T_X under φ, $\varphi^{-1}(U) \in (T_X)_G$. On the other hand, $\varphi^{-1}(U) = f^{-1}(k(U))$. Thus $f^{-1}(k(U)) \in (T_X)_G$. Since $f : (G, (T_X)_G) \to (f(G), (T_Y)_{f(G)})$ is

relatively IF-open $k(U) \in (T_Y)_{f(G)}$. So k is relatively IF-open. This completes the proof.

Let $\{X_j\}$, $j = 1, 2, \cdots, n$, be a finite family of groups and let X the product group. For each $j = 1, 2, \cdots, n$, let $T_j \in IFT(X_j)$ and let G_j an IFTG in X_j. We define a complex mapping $G = (\mu_G, \nu_G) : X \times X \to I \times I$ as follows : for each $x = (x_1, \cdots, x_n) \in X$,

$$\mu_G(x) = \mu_{G_1}(x_1) \wedge \cdots \wedge \mu_{G_n}(x_n)$$

and

$$\nu_G(x) = \nu_{G_1}(x_1) \vee \cdots \vee \nu_{G_n}(x_n).$$

Proposition 5.6. Let G is an IFG in X. In this case, G is called the *intuitionistic fuzzy product group*(in short, *IFPG*) of the IFGs G_j, $j = 1, 2, \cdots, n$ and denoted by $G = \prod_{j=1}^n G_j$.

Proof. It is clear that $G \in IFS(X)$ from the definition of G. Let $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in X$. Then

$$\begin{aligned}
&\mu_G(xy^{-1}) = \mu_G(x_1y_1^{-1}, \cdots, x_ny_n^{-1}) \\
&= \mu_{G_1}(x_1y_1^{-1}) \wedge \cdots \wedge \mu_{G_n}(x_ny_n^{-1}) \\
&\geq [\mu_{G_1}(x_1) \wedge \mu_{G_1}(y_1)] \wedge \cdots \wedge [\mu_{G_n}(x_n) \wedge \mu_{G_n}(y_n)] \\
&= [\mu_{G_1}(x_1) \wedge \cdots \wedge \mu_{G_n}(x_n)] \wedge [\mu_{G_1}(y_1) \wedge \cdots \wedge \mu_{G_n}(y_n)] \\
&= \mu_G(x) \wedge \mu_G(y)
\end{aligned}$$

and

$$\begin{aligned}
&\nu_G(xy^{-1}) = \nu_G(x_1y_1^{-1}, \cdots, x_ny_n^{-1}) \\
&= \nu_{G_1}(x_1y_1^{-1}) \vee \cdots \vee \nu_{G_n}(x_ny_n^{-1}) \\
&\leq [\nu_{G_1}(x_1) \vee \nu_{G_1}(y_1)] \vee \cdots \vee [\nu_{G_n}(x_n) \vee \nu_{G_n}(y_n)] \\
&= [\nu_{G_1}(x_1) \vee \cdots \vee \nu_{G_n}(x_n)] \vee [\nu_{G_1}(y_1) \vee \cdots \vee \nu_{G_n}(y_n)] \\
&= \nu_G(x) \vee \nu_G(y).
\end{aligned}$$

Hence G is an IFG in X.

Proposition 5.7. Let $\{X_j\}$, $j = 1, 2, \cdots, n$, be a finite family of groups, and for each $j = 1, 2, \cdots, n$, let $T_j \in IFT(X_j)$ and let G_j an IFTG in X_j. Let the product group $X = \prod_{j=1}^{n} X_j$ have the IFPT T. Then the IFPG $G = \prod_{j=1}^{n} G_j$ is an IFTG in X. In this case, G is called an *intuitionistic fuzzy product topological group*(in short, $IFPT(G)$).

We will denote the set of all IFPTG's in X as $IFPTG(X)$.

Proof. We define a mapping $\gamma_1 : X \times X \to (X_1 \times X_1) \times \cdots \times (X_n \times X_n)$ as follows : for any $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots y_n) \in X$,

$\gamma_1(x, y) = ((x_1, y_1), \cdots (x_n, y_n))$.

For each $j = 1, \cdots, n$, let $\pi_j : (X_1 \times X_1) \times \cdots \times (X_n \times X_n) \to X_j \times X_j$ be the mapping defined by as follows :

$\pi_j((x_1, y_1), \cdots, (x_n, y_n)) = (x_j, y_j)$

for each $((x_1, y_1), \cdots, (x_n, y_n)) \in (X_1 \times X_1) \times \cdots \times (X_n \times X_n)$.

Then we can easily see that $\pi_j \circ \gamma_1 : X \times X \to X_j \times X_j$ is IF-continuous for each $j = 1, \cdots n$. Thus, by Proposition 1.6, γ_1 is IF-continuous. Moreover,

$\gamma_1(G \times G) \subset (G_1 \times G_1) \times \cdots \times (G_n \times G_n)$.

So, by Proposition 2.6, $\gamma_1 : (G \times G) \to (G_1 \times G_1) \times \cdots \times (G_n \times G_n)$ is relatively IF-continuous.

Now we define a mapping $\gamma_2 : (X_1 \times X_1) \times \cdots \times (X_n \times X_n) \to X$ as follows : for each $((x_1, y_1), \cdots, (x_n, y_n)) \in (X_1 \times X_1) \times \cdots \times (X_n \times X_n)$.

$\gamma_2((x_1, y_1), \cdots (x_n, y_n)) = (x_1 y_1^{-1}, \cdots, x_n y_n^{-1})$.

For each $j = 1, \cdots, n$, let $f_j : X_j \times X_j \to X_j$ be the mapping defined by $f_j((x_j, y_j)) = x_j y_j^{-1}$ for each $(x_j, y_j) \in X_j \times X_j$. Then, by Proposition 4.3, $f_j : G_j \times G_j \to G_j$ is relatively IF-continuous, for each $j = 1, \cdots, n$. Moreover, $\gamma_2 = \prod_{j=1}^{n} f_j$. So, by Corollary 2.16', $\gamma_2 : (G_1 \times G_1) \times \cdots \times (G_n \times G_n) \to G$ is relatively IF-continuous.

Let $\gamma : X \times X \to X$ be the mapping defined as follows :

$$\gamma(x, y) = xy^{-1}$$

for each $(x,y) \in X \times X$. Then clearly $\gamma = \gamma_2 \circ \gamma_1$. Thus γ is IF-continuous. So $\gamma : G \times G \to G$ is relatively IF-continuous. Hence G is an IFTG in X.

Proposition 5.8. Let $\{X_j\}$, $j = 1, \cdots, n$, be a finite family of groups, and for each $j = 1, \cdots n$, let $T_j \in IFT(X_j)$, N_j a normal subgroup of X_j and $G_j \in IFTG(X_j)$ such that G_j is constant on N_j. Let the quotient groups X/N, where $N = \prod_{j=1}^n N_j$ and X_j/N_j, $j = 1, \cdots, n$, have the respective IFQT's and let the product groups $X = \prod_{j=1}^n X_j$ and $\prod_{j=1}^n (X_j/N_j)$ the respective IFPT's. Let $G = \prod_{j=1}^n G_j \in IFPTG(X)$. Then the canonical isomorphism c of X/N onto $\prod_{j=1}^n (X_j/N_j)$ is a relatively IF-homeomorphism of the IFQTG G/N onto the IFPTG $\prod_{j=1}^n (X_j/N_j)$.

Proof. Let $\varphi : X \to X/N$ be the canonical epimorphism defined by $\varphi(x) = [x]$ for each $x \in X$ and for each $j = 1, \cdots, n$, let $\varphi_j : X_j \to X_j/N_j$ the canonical epimorphism defined by $\varphi_j(x_j) = [x_j]$ for each $x_j \in X_j$. Let $\prod_{j=1}^n \varphi_j : X \to \prod_{j=1}^n (X_j/N_j)$ be the surjective product mapping. Then clearly $c \circ \varphi = \prod_{j=1}^n \varphi_j$. Let $[x] \in X/N$. Then

$$\begin{aligned} &\mu_{G/N}([x]) = \mu_G(x) \\ &= \mu_{\prod_{j=1}^n G_j}(x_1, \cdots, x_n) \\ &= \mu_{G_1}(x_1) \wedge \cdots \wedge \mu_{G_n}(x_n) \\ &= \mu_{G_1/N_1}([x_1]) \wedge \cdots \wedge \mu_{G_n/N_n}([x_n]) \\ &= \mu_{\prod_{j=1}^n (G_j/N_j)}(c([x])) \end{aligned}$$

and

$$\begin{aligned} &\nu_{G/N}([x]) = \nu_G(x) \\ &= \nu_{\prod_{j=1}^n G_j}(x_1, \cdots, x_n) \\ &= \nu_{G_1}(x_1) \vee \cdots \vee \nu_{G_n}(x_n) \\ &= \nu_{G_1/N_1}([x_1]) \vee \cdots \vee \nu_{G_n/N_n}([x_n]) \\ &= \nu_{\prod_{j=1}^n (G_j/N_j)}(c([x])). \end{aligned}$$

By Proposition 5.4 and Proposition 5.7, G/N and $\prod_{j=1}^{n}(G_j/N_j)$ are IFTG's. Let V' be open in the induced IFT on $\prod_{j=1}^{n}(G_j/N_j)$. By Proposition 2.6 and Proposition 2.16, $\prod_{j=1}^{n}\varphi_j : G \to \prod_{j=1}^{n} G_j/N_j$ is relatively IF-continuous. Then $\varphi^{-1}(c^{-1}(V')) = (\prod_{j=1}^{n}\varphi_j)^{-1}(V')$ is open in the induced IFT on G. Since $\varphi : G \to G/N$ is relatively IF-open, $c^{-1}(V')$ is open in the induced IFT on G/N. So $c : G/N \to \prod_{j=1}^{n}(G_j/N_j)$ is relatively IF-continuous.

Now let U be open in the induced IFT on G/N. Then $\varphi^{-1}(U)$ is open in the induced IFT on $\prod_{j=1}^{n}(G_j/N_j)$. By Proposition 2.17', since $\prod_{j=1}^{n}\varphi_j$ is relatively IF-open, $(\prod_{j=1}^{n}\varphi_j)(\varphi^{-1}(U)) = c(U)$ is open in the induced IFT on $\prod_{j=1}^{n}(G_j/N_j)$. So c is relatively IF-open. Hence $c : G/N \to \prod_{j=1}^{n}(G_j/N_j)$ is a relatively IF-homeomorphism.

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