Honam Mathematical J. **26**(2004), No. 4, pp. 391–399

AN ITERATIVE METHOD FOR NONLINEAR MIXED IMPLICIT VARIATIONAL INEQUALITIES

JAE UG JEONG

Abstract. In this paper, we develop an iterative algorithm for solving a class of nonlinear mixed implicit variational inequalities in Hilbert spaces. The resolvent operator technique is used to establish the equivalence between variational inequalities and fixed point problems. This equivalence is used to study the existence of a solution of nonlinear mixed implicit variational inequalities and to suggest an iterative algorithm for solving variational inequalities. In our results, we do not assume that the mapping is strongly monotone.

1. Introduction

In recent years, variational inequalities have been extended and generalized in different directions using novel and innovative techniques both for its own sake and for its applications. A useful and important generalization of variational inequality is a mixed variational inequality containing the nonlinear term. Due to the presence of the nonlinear term, the projection method cannot be used to study the existence of a solution of the mixed variational inequalities. These facts motivated us to develop another technique. This technique is related to the resolvent of the maximal monotone operator. Hassouni and Moudafi[2] modified

Received July 27,2004; Accepted September 10,2004.

2000 Mathematics Subject Classification: 49J40, 90C33.

Key words and phrases: Variational inequality, resolvent operator, iterative algorithm.

This paper was supported by Dong Eui University Research Grants in 2004.

and extended this technique for a class of general mixed variational inequalities.

In this paper, we develop an iterative algorithm for solving a class of nonlinear mixed implicit variational inequalities in Hilbert spaces. The resolvent operator technique is used to establish the equivalence between variational inequalities and fixed point problems. This equivalence is used to study the existence of a solution of nonlinear mixed implicit variational inequalities and to suggest an iterative algorithm for solving variational inequalities. In our results, we do not assume that the mapping is strongly monotone.

2. Preliminaries

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $<\cdot,\cdot>$. Let $g : H \to H$ be a nonlinear operator with $g(H)\bigcap \mathrm{dom}\partial\phi \neq \phi$, where $\partial\phi$ denotes the subdifferential of a proper, convex and lower semicontinuous function $\phi : H \to R\bigcup\{+\infty\}$. We consider the following nonlinear mixed implicit variational inequality problem:

Find $u \in H$ such that

$$(2.1) \qquad < u, v - g(u) > \geq \phi(g(u)) - \phi(v), \quad {}^\forall v \in H.$$

If the function ϕ is the indicator function of the closed convex set K in H, that is,

$$\phi(u) = I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise,} \end{cases}$$

then problem (2.1) is equivalent to finding $u \in H$. $g(u) \in K$ such that

$$(2.2) \qquad < u. v - g(u) > \geq 0, \quad {}^\forall v \in K.$$

which is called the implicit variatinal inequality problem.

Definition 2.1[1]. Let H be a Hilbert space and let $G : H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J_\rho^G : H \to H$ defined by

$$J_\rho^G(u) = (I + \rho G)^{-1}(u), \quad {}^\forall u \in H,$$

is said to be the resolvent operator of G where I is the identity mapping on H.

Lemma 2.1[3]. Let X be a reflexive Banach space endowed with a strictly convex norm and $\phi : X \to R \bigcup \{+\infty\}$ be a proper convex lower semicontinuous function. Then $\partial\phi : X \to 2^{X^*}$ is a maximal monotone mapping.

Lemma 2.2[1]. For a given $u \in H$, $u \in H$ satisfies the inequality

$$< u - z, v - u > + \rho\phi(v) - \rho\phi(u) \geq 0, \quad {}^\forall v \in H,$$

if and only if

$$u = J_\rho^{\partial\phi}(z),$$

where $J_\rho^{\partial\phi} = (I + \rho\partial\phi)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

Furthermore, $J_\rho^{\partial\phi}$ is a nonexpansive operator, that is,

$$\|J_\rho^{\partial\phi}(u) - J_\rho^{\partial\phi}(v)\| \leq \|u - v\|, \quad {}^\forall u, v \in H.$$

Definition 2.2. A mapping $g : H \to H$ is said to be

(i) monotone if

$$< g(u) - g(v), u - v > \geq 0, \quad {}^\forall u, v \in H;$$

(ii) δ-Lipschitz continuous if there exists a constant $\delta \geq 0$ such that

$$\|g(u) - g(v)\| \leq \delta\|u - v\|.$$

Remark 2.1. It is well known that $J_\rho^{\partial\phi} = P_K$ if $\phi = I_K$, where P_K denotes the projection mapping.

3. Main Results

In this section, we shall suggest an iterative algorithm for finding approximate solutions of the problem (2.1). Then we show that the sequence of approximate solutions strongly converges to the exact solution of the problem (2.1).

Lemma 3.1. u^* is a solution of the problem (2.1) if and only if u^* satisfies the relation

$$g(u) = J_\rho^{\partial\phi}(g(u) - \rho u). \tag{3.1}$$

Proof. Let u^* satisfy the relation (3.1), that is

$$g(u^*) = J_\rho^{\partial\phi}(g(u^*) - \rho u^*).$$

The equality holds if and only if

$$-u^* \in \partial\phi(g(u^*)),$$

by the definition of $J_\rho^{\partial\phi}$. The relation holds if and only if

$$\phi(v) - \phi(g(u^*)) \geq < -u^*, v - g(u^*) >, \quad {}^\forall v \in H,$$

by the definition of the subdifferential $\partial\phi$. Hence u^* is the solution of

$$< u^*, v - g(u^*) > \geq \phi(g(u^*)) - \phi(v), \quad {}^\forall v \in H.$$

Remark 3.1. From Theorem 3.1, we see that the nonlinear mixed implicit variational inequality (2.1) is equivalent to the problem (3.1).

Let $\rho > 0$ be a constant, $g : H \to H$ a single-valued operator and $\phi : H \to R\bigcup\{+\infty\}$ a proper, convex and lower semicontinuous functions.

Set

$$H(u,\rho)=\frac{1}{\rho}(g(u)-J_{\rho}^{\partial\phi}(g(u)-\rho)),$$

$$D(u,\rho)=g(u-H(u,\rho))-J_{\rho}^{\partial\phi}(g(u)-\rho u), \tag{3.2}$$

$$\bar{H}(u,\rho)=\frac{1}{\rho}(g(u)-P_K(g(u)-\rho)),$$

and

$$\bar{D}(u,\rho)=g(u-\bar{H}(u,\rho))-P_K(g(u)-\rho u). \tag{3.3}$$

Remark 2.1 and 3.1 imply that $u\in H$ is a solution of the problem (2.1) if and only if $H(u,\rho)=0$ and $u\in H$ is a solution of the probelm (2.2) if and only if $\bar{H}(u,\rho)=0$. This facts enables us to suggest the following algorithms.

Algorithm 3.1. For any given $u_0\in H$, the iterative sequence $\{u_k\}$ is defined by

$$u_{k+1}=u_k-\frac{1}{\rho}D(u_k,\rho),\quad k=0,1,2,\cdots. \tag{3.4}$$

If ϕ is the indicator function of K in H, then Algorithm 3.1 reduces to the following algorithm.

Algorithm 3.2. For any given $u_0\in H$, the iterative sequence $\{u_k\}$ is defined by

$$u_{k+1}=u_k-\frac{1}{\rho}\bar{D}(u_k,\rho),\quad k=0,1,2,\cdots. \tag{3.5}$$

Theorem 3.1. Let the single-valued operator $g:H\to H$ be monotone and Lipschitz continuous with constant $\delta>0$. If $\rho>\max\{2\sqrt{1+\delta^2},(\sqrt{2}+1)\delta\}$, then the nonlinear mixed implicit variational inequality (2.1) has a solution $u\in H$.

Proof. From Lemma 3.1, it follows that the nonlinear mixed implicit variational inequality (2.1) is equivalent to the fixed point problem

$$u = F(u) \equiv u - g(u) + J_{\rho}^{\partial\phi}(g(u) - \rho u). \tag{3.6}$$

In order to prove the existence of a solution of (2.1), it is sufficient to show that the problem (3.6) has a fixed point. Thus, for all $u_1, u_2 \in H$, $u_1 \neq u_2$, we have

$$\begin{aligned} \|F(u_1) - F(u_2)\| &\leq \|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad + \|J_{\rho}^{\partial\phi}(g(u_1) - \rho u_1) - J_{\rho}^{\partial\phi}(g(u_2) - \rho u_2)\| \\ &\leq \|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad + \|g(u_1) - g(u_2) - \rho(u_1 - u_2)\| \\ &\leq 2\|u_1 - u_2 - (g(u_1) - g(u_2))\| + (1 - \rho)\|u_1 - u_2\|. \end{aligned} \tag{3.7}$$

Since $g : H \to H$ is monotone and Lipschitz continuous,

$$\begin{aligned} &\|u_1 - u_2 - (g(u_1) - g(u_2))\|^2 \\ &= \|u_1 - u_2\|^2 - 2 < g(u_1) - g(u_2), u_1 - u_2 > + \|g(u_1) - g(u_2)\|^2 \\ &\leq (1 + \delta^2)\|u_1 - u_2\|^2. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we have

$$\begin{aligned} \|F(u_1) - F(u_2)\| &\leq [2\sqrt{1 + \delta^2} + (1 - \rho)]\|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|, \end{aligned}$$

where $\theta = 2\sqrt{1 + \delta^2} + 1 - \rho$. From $\rho > 2\sqrt{1 + \delta^2}$, it follows that $\theta < 1$. So, the map $F(u)$ defined by (3.6) has a fixed point $u \in H$ satisfying the nonlinear mixed implicit variational inequality (2.1).

Theorem 3.2. Let $g : H \to H$ be a monotone mapping and $\phi : H \to R\bigcup\{+\infty\}$ be a proper, convex and lower semicontinuous functions. Then for any solution $u^* \in H$ of the problem (2.1), the following inequality holds:

$$< u - u^*, D(u,\rho) > \geq < H(u,\rho), D(u,\rho) >, \quad {}^\forall u \in H. \tag{3.9}$$

Proof. Let u^* be a solution of the problem (2.1). It follows that

$$< u^*, J_\rho^{\partial\phi}(g(u) - \rho u) - g(u^*) > \geq \phi(g(u^*)) - \phi(J_\rho^{\partial\phi}(g(u) - \rho u)), {}^\forall u \in H. \tag{3.10}$$

By Lemma 2.2, we have

$$\begin{aligned} &< J_\rho^{\partial\phi}(g(u) - \rho u) - (g(u) - \rho u), g(u^*) - J_\rho^{\partial\phi}(g(u) - \rho u) > \\ &\qquad \geq \rho\phi(J_\rho^{\partial\phi}(g(u) - \rho u)) - \rho\phi(g(u^*)), \quad {}^\forall u \in H, \end{aligned}$$

that is,

$$\begin{aligned} &< H(u,\rho) - u, J_\rho^{\partial\phi}(g(u) - \rho u) - g(u^*) > \\ &\qquad \geq \phi(J_\rho^{\partial\phi}(g(u) - \rho u)) - \phi(g(u^*)), \quad {}^\forall u \in H. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we get

$$< H(u,\rho) - (u - u^*), J_\rho^{\partial\phi}(g(u) - \rho u) - g(u^*) > \geq 0. \tag{3.12}$$

Since g is monotone, we have

$$< u^* - (u - H(u,\rho)), g(u^*) - g(u - H(u,\rho)) > \geq 0, {}^\forall u \in H. \tag{3.13}$$

It folows from (3.12) and (3.13) that

$$< u - u^*, D(u,\rho) > \geq < H(u,\rho), D(u,\rho) >, \quad {}^\forall u \in H.$$

This completes the proof.

Theorem 3.3. Let the single-valued operator $g : H \to H$ be monotone and Lipschitz continuous with constant $\delta > 0$ and $\{u_k\}$ be the iterative sequence generated by Algorithm 2.1. If $\rho > \max\{2\sqrt{1+\delta^2}, (\sqrt{2}+1)\delta\}$, then $\{u_k\}$ converges to a solution $\bar{u} \in H$ of the problem (2.1).

Proof. By Theorem 3.1, the nonlinear mixed implicit variational inequality (2.1) has a solution $u^* \in H$. From (3.4) we obtain

(3.14)
$$\|u_{k+1} - u^*\|^2 = \|u_k - u^*\|^2 + \frac{1}{\rho^2}\|D(u_k,\rho)\|^2 - \frac{2}{\rho} < u_k - u^*, D(u_k,\rho) > .$$

Since g is monotone and Lipschitz continuous, we have

$$\begin{aligned}(3.15)\|D(u_k,\rho)\|^2 &= \|g(u_k - H(u_k,\rho)) - g(u_k) - (u_k - \rho H(u_k,\rho) - u_k)\|^2 \\ &= \|g(u_k - H(u_k,\rho)) - g(u_k)\|^2 + \rho^2\|H(u_k,\rho)\|^2 \\ &\quad -2\rho < g(u_k - H(u_k,\rho)) - g(u_k), (u_k - H(u_k,\rho) - u_k > \\ &\leq (\delta^2 + \rho^2)\|H(u_k,\rho)\|^2.\end{aligned}$$

By Theorem 3.2, we get

$$(3.16) \qquad < u_k - u^*, D(u_k,\rho) > \geq < H(u_k,\rho), D(u_k,\rho) > .$$

Further, we have

$$\begin{aligned}(3.17) \qquad & < H(u_k,\rho), D(u_k,\rho) > \\ = \ & < H(u_k,\rho), g(u_k - H(u_k,\rho)) - g(u_k) + \rho H(u_k,\rho) > \\ = \ & \rho\|H(u_k,\rho)\|^2 - < (u_k - H(u_k,\rho)) - u_k, g(u_k - H(u_k,\rho)) - g(u_k) > \\ \geq \ & \rho\|H(u_k,\rho)\|^2 - \|u_k - H(u_k,\rho) - u_k\|\|g(u_k - H(u_k,\rho)) - g(u_k)\| \\ \geq \ & (\rho - \delta)\|H(u_k,\rho)\|^2.\end{aligned}$$

It follows from (3.14)-(3.17) that

$$\begin{aligned} & \|u_{k+1} - u^*\|^2 \\ \leq \ & \|u_k - u^*\|^2 + \frac{1}{\rho^2}(\delta^2 + \rho^2)\|H(u_k,\rho)\|^2 - \frac{2}{\rho} < H(u_k,\rho), D(u_k,\rho) > \\ \leq \ & \|u_k - u^*\|^2 + \frac{1}{\rho^2}(\delta^2 + \rho^2)\|H(u_k,\rho)\|^2 - \frac{2}{\rho}(\rho - \delta)\|H(u_k,\rho)\|^2 \\ = \ & \|u_k - u^*\|^2 - \frac{1}{\rho^2}(\rho^2 - 2\rho\delta - \delta^2)\|H(u_k,\rho)\|^2.\end{aligned}$$

Snce $\rho > (\sqrt{2} + 1)\delta$ we know that $\{u_k\}$ is bounded. Therefore, there exists a subsequence $\{u_{k_j}\} \subset \{u_k\}$ and a point $\bar{u} \in H$ such that $u_{k_j} \to \bar{u}$

as $j \to \infty$. A simple induction leads to $u_k \to \bar{u}$ as $k \to \infty$. Now by using the continuity of the operator $g, D, H, J_\rho^{\partial\phi}$ and Algorithm 3.1, we have $H(\bar{u}, \rho) = 0$, that is, $g(\bar{u}) = J_\rho^{\partial\phi}(g(\bar{u}) - \rho\bar{u}) \in H$. By Lemma 3.1, it follows that $\bar{u} \in H$ which satisfies the inequality (2.1) and $u_k \to \bar{u}$ strongly in H. This completes the proof.

For $\phi = I_K$, Theorem 3.3 reduces to the following Corollary 3.1.

Corollary 3.1. Let the single-valued operator $g : H \to H$ be monotone and Lipschitz continuous with constant $\delta > 0$ and $\{u_k\}$ be the iterative sequence generated by Algorithm 3.2. If $\rho > \max\{2\sqrt{1+\delta^2}, (\sqrt{2}+1)\delta\}$, then $\{u_k\}$ converges to a solution $\bar{u} \in H$ of the problem (2.2).

References

[1] H. Brezis, Operateur Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert, North-Holland, Amsterdam (1973)

[2] A. Hassouni and A. Moudafi, A perturbed algorithm for variational inclusions, J. Math. Anal. Appl. 185(1994) 706-712

[3] D. Pascali and S. Sburlan, Nonlinear Mappings of Monotone Type, Sijthoff and Noordhoff, Romania (1978)

Jae Ug Jeong
Department of Mathematics.
Dong Eui University,
Pusan 614-714 South Korea
E-mail :jujeong@deu.ac.kr