Honam Mathematical J. **26**(2004), pp. 3–15

# THE IDEMPOTENT FUZZY MATRICES

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**Abstract.** In the fuzzy theory, a matrix $A$ is idempotent if $A^2 = A$. The idempotent fuzzy matrices are important in various applications and have many interesting properties. Using the upper diagonal completion process, we have the zero patterns of idempotent fuzzy matrix, that is, the idempotent Boolean matrices. In addition, we give the construction of all idempotent fuzzy matrices for each dimension $n$.

## 1. Introduction

The theory of fuzzy matrix is very useful in the discussion of fuzzy relations. We can represent basic propositions of the theory of fuzzy relations in terms of matrix operations. Futhermore we can deal with the fuzzy relations in the matrix form. In the study of the theory of a fuzzy matrix, a canonical form of some fuzzy matrices has received increasing attention. For example, Kim and Roush [1, 2] studied the canonical form of an idempotent matrix in 1980 and 1981. Hashimoto [3] studied the canonical form of a transitive matrix in 1983. Their main result is:

*For an idempotent (transitive, strongly transitive) fuzzy matrix $R$ there exists a permutation matrix $P$ such that $T = [t_{ij}] = P \times R \times P^T$ satisfies $t_{ij} \leq t_{ji}$, for $i > j$.*

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Received October 31, 2003. Revised November 21, 2003.

2000 Mathematics Subject Classification: Primary 15A15.

This paper was supported by Woosuk university.

The above $T$ was called the canonical form of an idempotent (transitive, strongly transitive) fuzzy matrix by these authors. Clearly, the study of this kind of canonical forms is important to develop the theory of a fuzzy matrix.

We define some operations for fuzzy matrices whose elements are in the unit interval $[0, 1]$. First, for $x, y \in [0, 1]$, we define $x \vee y$ and $x \wedge y$ as follows:

$$x \vee y = max(x, y)$$

$$x \wedge y = min(x, y).$$

Next we define the following matrix operations for $n \times n$ fuzzy matrices $R = [r_{ij}]$ and $S = [s_{ij}]$:

$$\cdot\, R \vee S = [r_{ij} \vee s_{ij}];$$

$$\cdot\, R \wedge S = [r_{ij} \wedge s_{ij}];$$

$$\cdot\, R \times S = [(r_{i1} \wedge s_{1j}) \vee (r_{i2} \wedge s_{2j}) \vee \cdots (r_{in} \wedge s_{nj})];$$

$$\cdot\, R^T = [r_{ji}] \quad \text{(the transpose of } R\text{)};$$

$$\cdot\, R \leq S \quad \text{if and only if} \quad r_{ij} \leq s_{ij} \quad \text{for all} \quad i, j.$$

We define some special kinds of fuzzy matrices. A fuzzy matrix $R$ is said to be:

$$\cdot\, \text{transitive if} \quad R^2 \leq R;$$

$$\cdot\, \text{idempotent if} \quad R^2 = R$$

Accordingly, every idempotent fuzzy matrix is transitive.

## 2. The Properties of Idempotent Fuzzy Matrices

First we are examine some basic properties of idempotent fuzzy matrices. They are useful in the following discussion. We know that all $1 \times 1$ fuzzy matrices are idempotent. Hence, in this paper, we deal only with square fuzzy matrix that dimension $n$, $n \geq 2$. Let $F_I$ be the set of all idempotent fuzzy matrices.

**Lemma 2.1.** The set of all idempotent fuzzy matrices, $F_I$, is closed under the following operations:

(i) permutation similarity

(ii) transposition.

**Lemma 2.2** (In [3]). For an idempotent fuzzy marix $R$ there exists a permutation matrix $P$ such that $T = [t_{ij}] = P \times R \times P^T$ satisfies $t_{ij} \leq t_{ji}$, for $i > j$.

We know the fact that the condition $t_{ij} \leq t_{ji}$ in the above Lemma is changeable to $t_{ij} \geq t_{ji}$ by Lemma 2.1-(ii). Thus we will use the later one from now on.

**Definition 2.3.** Let $A$ be a fuzzy matrix. For some nonzero entry $a$ of $A$, $Z(a) = [z_{ij}]$ is defined by

$$z_{ij} = \begin{cases} 1 & if \quad a \leq a_{ij} \\ 0 & if \quad otherwise. \end{cases}$$

$Z(a) = [z_{ij}]$ is called a *zero pattern* of $A$.

**Example 2.4.** If a fuzzy matrix

$$A = \begin{bmatrix} 0.2 & 0.7 & 1 \\ 0.7 & 0 & 0.2 \\ 0.3 & 1 & 0 \end{bmatrix},$$

then all its zero patterns are

$$Z(1) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Z(0.7) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$Z(0.3) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \ and \ Z(0.2) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

**Lemma 2.5** (In [1]). A fuzzy matrix is idempotent if and only if all its zero patterns are idempotent.

By the above Lemma 2.5, we examine the properties of $(0,1)$-fuzzy matrices and obtain a theorem and canonical form of the $(0,1)$-fuzzy matrices. Thus we will be able to characterize the structure of the set of all idempotent fuzzy matrices, $F_I$. Let $B_I$ be the set of all idempotent $(0,1)$-fuzzy matrices. In fact, $B_I$ is the set of idempotent Boolean matrices and is same as the set of all zero patterns that is idemopotent fuzzy matrces. In this paper, the index set $\{1,2,3,\cdots n-1,n\}$ will be denoted by $N$.

**Remark 2.6.** For an $n \times n$ fuzzy matrix $A$, $A$ is in $B_I$ and $a_{ij} = 1$ if and only if there exists a $k \in N$ such that $a_{ik} = 1 = a_{kj}$. This statement follows from the fact that since $A = [a_{ij}]$ is idempotent, we must have that $a_{ij} = \sum_{k=1}^{k=n} a_{ik}a_{kj}$ for all $i,j \in N$. Another way of writing this is to observe that $A \in B_I$ if and only if for all $i$ $R_i = \sum_j R_j$ where $j = \{k | e_k \leq R_i\}$ and $C_i = \sum_j C_j$ where $j = \{k | e^k \leq C_i\}$, $R_i(C_j)$ is an row(column) vector, and $e_k(e^k)$ is in the standard basis.

## 3. The Idempotent $(0,1)$-fuzzy Matrices

A matrix $A$ of order $n \geq 2$ is said to be *reducible* if there exists a permutation matrix $P$ such that

$$PAP^T = \begin{pmatrix} B & C \\ O & D \end{pmatrix}$$

where $B$ and $D$ are square matrices of order at least one. Otherwise $A$ is called *irreducible*.

**Lemma 3.1.** Let $A = [a_{ij}]$ be an $n \times n$ $(2 \leq n)$ idempotent fuzzy matrix. If $a_{ij} = 0$ for some $i$ and $j$ in $N$, then each product $a_{ik}a_{kj} = 0$ for all $k$ in $N$.

*Proof.* It is an immediate consequence of the fuzzy matrix product. $\square$

**Lemma 3.2** (In [4]). If $A = [a_{ij}]$ is an $n \times n$ irreducible idempotent Boolean matrix, then $A$ is entrywise nonzero.

From Lemma 3.2, we know that an irreducible $n \times n$ idempotent Boolean matrix is entrywise nonzero and use only 1 or 0 entries in $B_I$. Consequently, an irreducible idempotent $(0,1)$-fuzzy matrix may be taken to be an entrywise 1 matrix, that is, $J_n = [1_{ij}]$. We state this formally as follows:

**Theorem 3.3.** If $A = [a_{ij}]$ is an $n \times n$ irreducible idempotent $(0,1)$-fuzzy matrix, that is, $A$ is an irreducible idempotent zero patterns, then it is entrywise 1 (up to equivalences, as stated in Lemma 2.1).

**Corollary 3.4.** If $A = [a_{ij}]$ is an $n \times n$ idempotent $(0,1)$-fuzzy matrix, then $A$ does not contain $k$-zeros where $0 < k < (n-1)$.

*Proof.* Suppose that $A$ is an $n \times n$ irreducible idempotent $(0,1)$-fuzzy matrix. Then $A$ has no zeros by Theorem 3.3. We assume that $A$ is reducible. Then $A$ have at least $(n-1)$-zeros by the definition of reducible. □

Let an $n \times n$ matrix $A$ be given. Then either $A$ is irreducible or there exists a permutation matrix $P$ such that

$$PAP^T = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_k \end{bmatrix}$$

in which $A_i$ is either irreducible or is $1 \times 1$ zero matrix, $i = 1, \cdots, k$. This is called the *Frobenious normal form* of $A$. If $A$ is a reducible idempotent fuzzy matrix in Frobenious normal form, then it is clear that each irreducible diagonal block of $A$ is idempotent fuzzy matrix. In the remainder of this paper, we use the results of above section and assume that each nonzero irreducible diagonal block $A_{ii}$ of $A$ is entrywise 1, that is, $A_{ii} = J_i$ which are same size. Futher, in terms of block multiplication where $P = A^2$,

$$P_{ij} = A_{ii}A_{ij} + A_{i,i+1}A_{i+1,j} + \cdots + A_{ij}A_{jj} = A_{ij}.$$

The key question here is: *What are the possible zero patterns of the off-diagonal blocks $A_{ij}$ so that the above equation is satisfied?* The remainder of this paper is dedicated to answering this question and all nonzero matrix(or block) means entrywise 1 matrix, that is, $J = [1_{ij}]$.

**Lemma 3.5** (In [4]). If $A$ is an $n \times n$ reducible Boolean matrix such that $A_{ii}$ and $A_{jj}$ are entrywise nonzero diagonal blocks, then $A$ is Boolean idempotent only if the entries of $A_{ij}$ are obtained as follows:

(i) $A_{ij}$ contains only 1's or

(ii) $A_{ij}$ is 0-block.

**Lemma 3.6** (In [4]). Let $A$ be an $n \times n$ reducible Boolean idempotent matrix where $A_{ii}$ is an $n_i \times n_i$ entrywise nonzero diagonal block and $A_{jj}$ is an 0-block. If $A_{ij}$ contains a 0-entry, then $A_{ij}$ contains an 0-column.

**Lemma 3.7.** Let $A$ be an $n \times n$ reducible Boolean idempotent matrix where $A_{ii}$ is an 0-block and $A_{jj}$ is an $n_j \times n_j$ entrywise nonzero diagonal block. If $A_{ij}$ contains a 0-entry, then $A_{ij}$ contains an 0-row.

*Proof.* To simplify notation, let $A_{jj} = H = [h_{ij}]$ and $A_{ij} = B = [b_{ij}]$. Assume $b_{ir} = 0$ for some $r$ in $N_j$. By Boolean idempotence, we know that the product entry $(BH)_{ir} = 0$ and can state results for $A_{ij}$ analogous to those given in Lemma 3.6. □

Since a Boolean matrix is idempotent only if each off-diagonal block $P_{ij}$ is defined, we obtain the following:

**Lemma 3.8** (In [4]). Let $A$ be an $n \times n$ reducible Boolean idempotent matrix. If $A_{ii}$ is entrywise nonzero and $A_{jj}$ is 0-block, then $A$ is Boolean idempotent only if $A_{ij} = 0$ or $A_{ij}$ contains only 1's.

**Lemma 3.9.** Let $A$ be an $n \times n$ reducible Boolean idempotent matrix. If $A_{ii}$ is 0-block and $A_{jj}$ is entrywise nonzero, then $A$ is Boolean idempotent only if $A_{ij} = 0$ or $A_{ij}$ contains only 1's.

*Proof.* By Boolean idempotence, we can state results for $A_{ij}$ analogous to those given in Lemma 3.7 and Lemma 3.8. □

If $A$ is an $m \times m$ block reducible matrix, then the off-diagonal blocks $A_{i,i+k}$ lie on the $k$-th superdiagonal for all $k = 1, 2, \cdots, m-1$. Due to the triangular structure of $A$, each $P_{i,i+k}$ in the product matrix $P = A^2$ is independent of all terms above the $k$-th superdiagonal. This independence allows us to complete the zero pattern of $A$ so that $A = A^2$, as described in the following:

**Algorithm 3.10** ([4]). (The upper diagonal completion process) Let $A = [A_{ij}]$ be an $m \times m$ reducible, partial block idempotent Boolean matrix in modified Frobenius normal form. We can Determine the entry of each off-diagonal block as follows:

(i) Start with 1-st superdiagonal. Determine the entries of each off-diagonal block $A_{i,i+1}$ using Lemma 3.5 if $A_{ii}$ and $A_{i+1,i+1}$ are entrywise nonzero , Lemma 3.8 for each diagonal block of $A_{i+1,i+1}$ if $A_{ii}$ is an entrywise nonzero and $A_{i+1,i+1}$ is a 0-block, or Lemma 3.9 for each diagonal block of $A_{ii}$ so that $A_{i-1,i}A_{i,i+1}$ is unambiguously defined if $A_{ii}$ is a 0-block and $A_{i+1,i+1}$ is an entrywise nonzero Boolean matrix. Move up to the next diagonal (if there is one).

(ii) For each unspecified entry $A_{i,i+k}$ on the $k$-th superdiagonal, $k = 2, 3, \cdots, m-1$, if $P_{i,i+k} = A_{ii}A_{i,i+k} + A_{i,i+k}A_{i+k,i+k}$ use step (i) with $i+k$ replacing it, otherwise let $A_{i,i+k} = A_{i,i+1}A_{i+1,i+k}$. When all blocks are specified on this diagonal, move up to the next diagonal, if there is one, increase $k$ by 1 for all $k = 2, 3, \cdots, m-2$, and repeat (ii).

**Lemma 3.11** ([4]). Let $A$ be an $n \times n$ reducible Boolean idempotent matrix in modified Frobenious normal form, each of whose diagonal blocks is entrywise nonzero or 0-block. Then $A$ is idempotent only if each off-diagonal block $A_{ij}$ is obtained using Algorithm 3.10.

**Lemma 3.12** ([4]). A reducible Boolean matrix $A$, in modified Frobenius normal form, each of whose nonzero diagonal block is entrywise 1, is idempotent if and only if the entry of each off-diagonal block is obtained using the upper diagonal completion process.

**Example 3.13.** Let $A$ be an idempotent $(0,1)$-fuzzy matrix. Then

$$A = \begin{bmatrix} 1 & 1 & 1 & * & * & * & * & * & * \\ 1 & 1 & 1 & * & * & * & * & * & * \\ 1 & 1 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & ? & ? \\ 0 & 0 & 0 & 0 & 0 & * & * & ? & ? \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ is 0 or 1 and ? is determined by the specified number $*$ that is on the lower superdiagonal than it by using the upper diagonal completion process.

Since the set of Boolean idempotent matrices is same as the zero patterns of the idempotent fuzzy matrix, we have the following Theorem;

**Theorem 3.14.** A reducible zero patterns of the fuzzy matrix $A$, in modified Frobenius normal form, each of whose nonzero diagonal block is entrywise 1, is idempotent if and only if the entry of each off-diagonal block is obtained using the upper diagonal completion process.

## 4. The Construction of The Idempotent Fuzzy Matrices

In this section, using the conditions and properties in the above sections we construct all zero patterns and idempotent fuzzy matrix from $n = 2$ to $n = 3$.

[$Case,\ n = 2$] There are all zero patterns of idempotent fuzzy matrices except the zero matrix as follows;

$$A(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A(3) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$A(4) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A(5) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A(6) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since there exist three maximal chains such that $A(1) < A(3) < A(5) < A(6)$, $A(2) < A(4) < A(5) < A(6)$, and $A(1) < A(4) < A(5) < A(6)$, we construct all $2 \times 2$ idempotent fuzzy matrix such that

$$\begin{bmatrix} a & b \\ d & c \end{bmatrix}, \begin{bmatrix} b & c \\ d & a \end{bmatrix}, \begin{bmatrix} a & c \\ d & b \end{bmatrix}$$

where fuzzy numbers are $0 < d \leq c \leq b \leq a \leq 1$.

[$Case,\ n = 3$] There exist zero patterns of idempotent fuzzy matrices except the zero matrix as follows;

$$A(1a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A(1b) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A(1c) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A(2a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A(2b) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A(2c) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A(2d) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A(2e) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A(2f) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A(2g) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A(2h) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A(2i) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A(3a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A(3b) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A(3c) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A(3d) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A(3e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A(3f) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A(3g) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A(3h) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A(3i) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A(3j) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A(3k) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A(3l) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A(4a) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A(4b) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A(4c) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A(4d) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A(4e) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A(4f) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A(4g) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A(4h) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A(4i) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A(4j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A(4k) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A(4l) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$A(5a) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A(5b) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, A(5c) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A(5d) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A(5e) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A(5f) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A(5g) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A(6a) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A(6b) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$A(6c) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A(7a) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A(7b) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$A(9a) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Since there exist 222's maximal chains, we construct all $3 \times 3$ idempotent fuzzy matrix such that if we choose a chain

(*) $\quad A(1*) < A(2*) < A(3*) < A(4*) < A(5*) < A(6*) < A(7*) < A(9*)$

then the fuzzy matrix

$$\begin{aligned} A &= aA(1*) + b(A(2*) - A(1*)) + c(A(3*) - A(2*)) \\ &\quad + d(A(4*) - A(3*)) + e(A(5*) - A(4*)) + f(A(6*) - A(5*)) \\ &\quad + g(A(7*) - A(6*)) + h(A(9*) - A(7*)) \end{aligned}$$

is idempotent where

$$0 < h \leq g \leq f \leq e \leq d \leq c \leq b \leq a \leq 1.$$

**Example 4.1.** If we choose a maximal chain of zero patterns,

$$A(1a) < A(2b) < A(3g) < A(4e) < A(5d) < A(6c) < A(7a) < A(9a),$$

then the fuzzy matrix $A$ is an idempotent fuzzy matrix that has the form

$$A = \begin{bmatrix} a & e & d \\ g & f & c \\ h & h & b \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0.9 & 0.4 & 0.5 \\ 0.2 & 0.3 & 0.6 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}$$

is an idempotent fuzzy matrix where

$$a = 0.9,\ b = 0.7,\ c = 0.6,\ d = 0.5,\ e = 0.4,\ f = 0.3,\ g = 0.2,\ h = 0.1.$$

In addition, when we choose a sub-chain of any maximal chain $(*)$ by the above method the idempotent fuzzy matrix is constructed. For example, if we choose a sub-chain $A(1*) < A(3*) < A(5*) < A(7*) < A(9*)$ of the above maximal chain $(*)$ then the fuzzy matrix

$$\begin{aligned} A = aA(1*) &+ c(A(3*) - A(1*)) + e(A(5*) - A(3*)) \\ &+ g(A(7*) - A(5*)) + h(A(9*) - A(7*)) \end{aligned}$$

is idempotent where

$$0 < h \leq g \leq e \leq c \leq a \leq 1.$$

**Remark 4.2.** In fact, we have all $n \times n$ zero patterns of idempotent fuzzy matrix by the upper diagonal completion process in Lemma 3.8. Therefore we construct all idempotent fuzzy matrices by the above construction and Theorem 3.12.

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