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ON THE CYCLICTY OF ADJOINTS OF WEIGHTED SHIFTS

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Abstract. We provide some sufficient conditions for the adjoint of a unilateral weighted shift operator on a Hilbert space to be cyclic.

Introduction

Let H be a separable Hilbert space and $B(H)$ the algebra of all bounded operators on H. A unilateral weighted shift operator on H is an operator S given by $Se_n = w_n e_{n+1}$, where $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for H and $\{w_n\}_{n=0}^{\infty}$ is a sequence of complex numbers. It was shown in [1] that S is bounded if and only if $\{w_n\}_n$ is bounded operator and in this case

$$\|S^n\| = \sup_k |w_k w_{k+1} \dots w_{k+n-1}|, \qquad n = 1, 2, 3, \dots.$$

The unilateral weighted shift operators are also studied in [1, 3, 4].

Proposition ([2]). If $\{\lambda_n\}_n$ are complex numbers of modulus 1, then S is unitarily equivalent to the weighted shift operator with weight sequence $\{\bar{\lambda}_{n+1} w_n \lambda_n\}$.

Therefore S is unitary equivalent to the weighted shift operator with weight sequence $\{|w_n|\}_n$ and so we may assume that the weights $\{w_n\}_n$ be non-negative real numbers.

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Definition. Let $A \in B(H)$ and $f \in H$. We say that f is a cyclic vector of A if $H = span\{A^n f : n = 0, 1, 2, \ldots\}$ and in this case A is called cyclic. Here $span\{\cdot\}$ is the closed linear span of the set $\{\cdot\}$.

Note that the adjoint of a unilateral weighted shift operator is a backward shift operator. Indeed if T denotes the adjoint of the unilateral weighted shift with weight sequence $\{w_n\}_n$, then

$$(*) \qquad Te_n = \begin{cases} w_{n-1}e_{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases},$$

and

$$\|T^n\| = \sup_{k>n}(w_{k-1} \ldots w_{k-n}).$$

Main Results

Throughout of this paper T denotes the operator defined by $(*)$ in the Introduction. In the main theorems of the present paper we give some sufficient conditions for the backward weighted shift operator T to be cyclic.

The following Lemma is needed for the proof of our main theorem.

Lemma 1. Let $\mathcal{M}$ be a closed subspace of H. Also let for every vector $f = \sum_{n=0}^{\infty} a_n e_n$ in $\mathcal{M}$,

$$|a_n| \leq \gamma_n \|f\|, n \in \mathbb{N} \cup \{0\}$$

where $\{\gamma_n\}_n$ is independent of f and $\sum_{n=0}^{\infty} \gamma_n^2 < \infty$. Then $\mathcal{M}$ is finite dimensional.

Proof. See [4]. □

Theorem 2. Let $f = \sum_{n=0}^{\infty} a_n e_n$ be a vector in H with infinitely many $a_n \neq 0$, and let the sequence $\{w_n\}_n$ satisfy the following conditions:

(1) $\forall n \in \mathbb{N} \qquad w_n \geq 1$.

(2) $\delta = \sup\{w_k \ldots w_{k+n-1} : \ k \geq N, \ n \geq N\} < \infty$ for all sufficiently large positive integer N.

(3) $\sum_{k \geq N} (k+1)(w_{k-N} \ldots w_{k-1})^{-2} < \infty$ for some large positive integer N.

Then f is a cyclic vector of T.

Proof. Put $\mathcal{M} = span\{T^n f : n \geq 0\}$. If $\mathcal{M} \neq H$, then there is a nonzero $g = \sum_{n \geq 0} b_n e_n$ in $\mathcal{M}^{\perp}$ such that $< T^n f, g >= 0$ for all $n = 0, 1, 2, \ldots$. Let $m_0 = \min\{k : b_k \neq 0\}$. Since

$$\begin{aligned} T^n f &= \sum_{k \geq n} a_k (w_{k-1} \ldots w_{k-n}) e_{k-n} \\ &= \sum_{k \geq 0} a_{k+n} (w_{k+n-1} \ldots w_k) e_k, \end{aligned}$$

we get

$$< T^n f, g >= \sum_{k \geq m_0} a_{k+n} (w_{k+n-1} \ldots w_k) \bar{b}_k = 0.$$

So

$$a_{m_0+n} (w_{m_0+n-1} \ldots w_{m_0}) \bar{b}_{m_0} = - \sum_{k \geq m_0+1} a_{k+n} (w_{k+n-1} \ldots w_k) \bar{b}_k.$$

Therefore

$$|a_{m_0+n}| \leq \frac{1}{|b_{m_0}|} \sum_{k \geq m_0+1} |a_{k+n}||b_k| \frac{w_{k+n-1} \ldots w_k}{w_{m_0+n-1} \ldots w_{m_0}}.$$

Now choosing N as in condition (3), we can choose g such that $b_k = 0$ for $m_0 < k < m_0 + N$. In that case

$$|a_{m_0+n}| \leq \frac{\|g\|}{|b_{m_0}|} \sum_{k \geq m_0+N} \frac{w_{k+n-1} \ldots w_k}{w_{m_0+n-1} \ldots w_{m_0}} |a_{k+n}|.$$

But by conditions (1) and (2) we have

$$\begin{aligned}\frac{w_{k+n-1}\cdots w_k}{w_{m_0+n-1}\cdots w_{m_0}} &\leq (w_{k+n-1}\cdots w_k)\frac{w_{k+n-N}\cdots w_{k+n-1}}{w_{k+n-N}\cdots w_{k+n-1}} \\ &\leq \delta^2(w_{k+n-N}\cdots w_{k+n-1})^{-1}.\end{aligned}$$

Now by the Hölder inequality we get $|a_{m_0+n}| < \beta_{m_0+n}\|f\|$, where

$$\begin{aligned}\beta_{m_0+n} &= \frac{\|g\|}{|b_{m_0}|}\delta^2\left(\sum_{k\geq m_0+N}(w_{k+n-N}\cdots w_{k+n-1})^{-2}\right)^{-\frac{1}{2}} \\ &= \beta_{m_0+n}\|f\|.\end{aligned}$$

Thus if

$$\gamma_i = \begin{cases} 1 & i < m_0+n \\ \beta_i & i \geq m_0+n \end{cases},$$

then for all i we have $|a_i| \leq \gamma_i\|f\|$. Note that $\{\gamma_i\}_i$ is independent of f because $< g, h >= 0$ for all h in $\mathcal{M}$. Now if we show that $\sum_{i\geq 0}\gamma_i^2 < \infty$, then by the Lemma 1, $\mathcal{M}$ is finite dimensional which contradicts our assumption that $a_n \neq 0$ for infinitely many n, therefore $\mathcal{M} = H$ and so f is a cyclic vector.

To show that $\sum_{i\geq 0}\gamma_i^2 < \infty$ put

$$\alpha_i = \left(\sum_{k\geq m_0+N}(w_{k+i-N}\cdots w_{k+i-1})^{-2}\right)^{\frac{1}{2}}, \qquad i \geq m_0.$$

It is sufficient to show that $\sum_{i\geq m_0} \alpha_i^2 < \infty$:

$$\begin{aligned}
\sum_{i\geq m_0} \alpha_i^2 &= \sum_{i\geq m_0} \sum_{k\geq m_0+N} (w_{k+i-N}\cdots w_{k+i-1})^{-2} \\
&\leq \sum_{i\geq m_0} \sum_{k\geq N} (w_{k+i-N}\cdots w_{k+i-1})^{-2} \\
&= [(w_{m_0}\cdots w_{m_0+N-1})^{-2} + (w_{m_0+1}\cdots w_{m_0+N})^{-2} \\
&\quad +(w_{m_0+2}\cdots w_{m_0+N+1})^{-2} + \cdots] \\
&\quad +[(w_{m_0+1}\cdots w_{m_0+N})^{-2} + (w_{m_0+2}\cdots w_{m_0+N+1})^{-2} \\
&\quad +(w_{m_0+3}\cdots w_{m_0+N+2})^{-2} + \cdots] \\
&\quad +[(w_{m_0+2}\cdots w_{m_0+N+1})^{-2} + (w_{m_0+3}\cdots w_{m_0+N+2})^{-2} \\
&\quad +(w_{m_0+4}\cdots w_{m_0+N+3})^{2} + \cdots] + \cdots \\
&= \sum_{k=0}^{\infty} (k+1)(w_{m_0+k}\cdots w_{m_0+k+N-1})^{-2} \\
&= \sum_{k\geq N+m_0} (k+1-(N+m_0))(w_{k-1}\cdots w_{k-N})^{-2} \\
&\leq \sum_{k\geq N} (k+1)(w_{k-1}\cdots w_{k-N})^{-2}
\end{aligned}$$

that is finite by condition (3). At this time the proof is complete. □

Theorem 3. Let $f = \sum_{n=0}^{\infty} a_n e_n$ be a vector in H with $a_n \neq 0$ and $\lim_{i\to\infty} \sum_{k=i+1}^{\infty} |\frac{a_{k+n}}{a_{i+n}}|^2 = 0$ for all $n \in \mathbb{N}\cup\{0\}$. If the sequence $\{w_n\}_n$ satisfies:

(1) $\forall n \in \mathbb{N} \qquad w_n > 0$,

(2) $\delta = \sup\{\frac{w_{i+n-1}\cdots w_i}{w_{m+n-1}\cdots w_n} : i > m \geq 0,\ n \in \mathbb{N}\} < \infty$, then f is a cyclic vector of T.

Proof. Put $\mathcal{M} = span\{T^n f : n = 0, 1, 2, \ldots\}$. We have

$$T^n f = a_n(w_{n-1}\cdots w_0)e_0 + \sum_{i\geq n+1} a_i(w_{i-1}\cdots w_{i-n})e_{i-n}.$$

Therefore

$$\left\|\frac{T^n f}{a_n(w_{n-1}\dots w_0)} - e_0\right\|^2 = \sum_{i\geq n+1} \frac{|a_i|^2}{|a_n|^2}\left(\frac{w_{i-1}\dots w_{i-n}}{w_{n-1}\dots w_0}\right)^2 \leq \delta^2 \sum_{i\geq n+1}\frac{|a_i|^2}{|a_n|^2}.$$

Now by the assumption we get

$$\lim_n \frac{T^n f}{a_n(w_{n-1}\dots w_0)} = e_0$$

and so $e_0 \in \mathcal{M}$. Now by induction we prove that $e_m \in \mathcal{M}$ for all $m \in \mathbb{N}$. For this let $e_0, \dots, e_{m-1} \in \mathcal{M}$ and note that

$$\begin{aligned} T^n f &= \sum_{i=0}^{\infty} a_{i+n}(w_{i+n-1}\dots w_i)e_i = \sum_{i=0}^{m-1} a_{i+n}(w_{i+n-1}\dots w_i)e_i \\ &\quad + a_{m+n}(w_{m+n-1}\dots w_n)e_m + \sum_{i=m+1}^{\infty} a_{i+n}(w_{i+n-1}\dots w_i)e_i. \end{aligned}$$

So we have

$$\left\|\left(\frac{T^n f}{a_{m+n}(w_{m+n-1}\dots w_m)} - \sum_{i=0}^{m-1}\frac{a_{i+n}}{a_{m+n}}\frac{w_{i+n-1}\dots w_i}{w_{m+n-1}\dots w_m}e_i\right) - e_m\right\|^2$$

$$\begin{aligned} &= \sum_{i\geq m+1} \left|\frac{a_{i+n}}{a_{m+n}}\right|^2 \left(\frac{w_{i+n-1}\dots w_i}{w_{m+n-1}\dots w_m}\right)^2 \\ &\leq \delta^2 \sum_{i\geq m+1}\left|\frac{a_{i+n}}{a_{m+n}}\right|^2 = \delta^2 \sum_{i\geq n+1}\left|\frac{a_{m+i}}{a_{m+n}}\right|^2. \end{aligned}$$

By letting $n \longrightarrow \infty$, we conclude that $e_m \in \mathcal{M}$. Thus indeed $e_n \in \mathcal{M}$ for all $n \in \mathbb{N}\cup\{0\}$. Hence $\mathcal{M} = H$ and so f is a cyclic vector of T. □

Corollary 4. Let $f = \sum_{n=0}^{\infty} a_n e_n \in H$, $a_n \neq 0$ and $\lim_{i\to\infty} \sum_{k=i+1}^{\infty} |\frac{a_{k+n}}{a_{i+n}}|^2 = 0$ for all $n \in \mathbb{N}\cup\{0\}$. If $w_n > 0$ and $w_{n+1} \leq w_n$ for all n, then f is a cyclic vector of T.

Proof. It is easy to see that condition (2) of Theorem 3 is consistent and indeed $\delta \leq 1$. This completes the proof. □

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