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A GEOMETRIC PROOF OF THE ROBINSON-SCHENSTED-KNUTH CORRESPONDENCE

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Abstract. In this paper, we give a proof of the Robinson-Schensted-Knuth correspondence by using the geometric. construction. We represent a generalized permutation in the first quadrant of the Cartesian plane and find a corresponding pair of semi-standard tableaux of same shape. This work extends the classical geometric construction of Viennot [10] for Robinson-Schensted correspondence.

1. Introduction

Robinson-Schensted correspondence [9] is a combinatorial construction relating permutations and pairs of standard tableaux having the same shape. It has been widely studied in the last few years. Knuth [6] generalized the Robinson-Schensted correspondence by giving a bijection between generalized permutations and pairs of semi-standard tableaux of the same shape. The bijection given by Kunth is called the Robinson-Schensted-Knuth correspondence.

The Robinson-Schensted-Knuth correspondence provided a combinatorial proof of a fundamental identity involving Schur function and resolved the counting problems of semi-standard tableaux [1],[3]. After

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Knuth's generalization to semi-standard tableaux, it was followed by various analogs of Robinson-Schensted correspondence for shifted tableaux [7], skew tableaux [4], oscillating tableaux [2],[4],[5]. Viennot gave a different proof for Robinson-Schensted correspondence by using a geometric construction [8],[10].

In this paper, we extend the Viennot's geometric construction to generalized permutations and we obtain a bijection between generalized permutations and pairs of semi-standard tableaux of same shape.

2. Definitions and notations

S_n denotes by the set of permutations of $[n]=\{1, 2, ..., n\}$. A generalized permutation on $[n]$ is a two line array of positive integers of $[n]$, $\pi = \begin{pmatrix} u_1 & u_2 & \dots & u_k \\ v_1 & v_2 & \dots & v_k \end{pmatrix}$ where $u_1 \leq u_2 \leq ... \leq u_k$, and $v_i \geq v_j$ if $u_i = u_j$ for $i < j$. $\hat{\pi}$ denotes the top row of π and $\check{\pi}$ its bottom row. We denote by GP the set of generalized permutations.

We use the notation $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ for both partition and the corresponding Ferrers diagram being the largest part λ_1 in the bottom row and the smallest part in the top row. A standard tableau T of shape λ is a labeling of the cells of λ with $1, ..., n = |\lambda|$ so that the rows and columns are strictly increasing. A semi-standard tableau S of shape α is a labelling of the cells of α with positive integers so that the rows are strictly increasing and the columns are weakly increasing. The label of the cell in the i^{th} row and j^{th} column of a tableau S is denoted by $S_{i,j}$.

We assume that the reader is familiar with the the Robinson-Schensted correspondence and Robinson-Schensted-Knuth correspondence (see [3],[6],[9] for a survey).

The Robinson-Schensted correspondence is denoted

$$\tau \overset{R-S}{\longleftrightarrow} (P(\tau), Q(\tau))$$

where τ is a permutation and $P(\tau)$ and $Q(\tau)$ are standard tableaux of same shape.

The Robinson-Schensted-Knuth correspondence is denoted

$$\pi \overset{R-S-K}{\longleftrightarrow} (P(\pi), Q(\pi))$$

where π is a generalized permutation and $P(\pi)$ and $Q(\pi)$ are semi-standard tableaux of same shape.

3.Geometric description of a generalized permutation

In this section, we give a proof of the Robinson-Schensted-Knuth correspondence by using an analogue of Viennot's geometric construction and our proof is different from the proof given by Knuth.

We standardize a generalized permutation in order to represent it in the first quadrant of the Cartesian plane. In the following we introduce a way to standardize a generalized permutation to permutation and we show its geometric description. Then we give a bijection between generalized permutations and pairs of semi-standard tableaux of same shape.

Let $\mathbb{N}$ be the set of positive integers. We consider the new alphabet $\ddot{\mathbb{N}} = \mathbb{N} \cup \{i^{(h)} : i, h \in \mathbb{N}\}$ such that

$$i < i^{(1)} < i^{(2)} < ... < i + 1 < (i + 1)^{(1)} < (i + 1)^{(2)} < ...$$

Definition3.1. (i) A standard tableau A on $\ddot{\mathbb{N}}$ of shape λ is a labeling of the cells of λ with alphabet of $\ddot{\mathbb{N}}$ so that the rows and columns are strictly increasing.

(ii) Two line array $\begin{pmatrix} u_1 & u_2 & \cdots & u_k \\ v_1 & v_2 & \cdots & v_k \end{pmatrix}$ is a permutation on $\ddot{\mathbb{N}}$ if $u_1 < u_2 < \cdots < u_k$ for $u_i, v_i \in \ddot{\mathbb{N}}$ $i = 1, ..., k$ and all of the u_i's and v_i's are distinct each other.

We define a map $\phi(\pi) = \sigma$ from the set of generalized permutations on $\mathbb{N}$ to the set of permutations on $\ddot{\mathbb{N}}$ as follows:

for a generalized permutation $\pi = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}$,

(i) if $u_j = u_{j+1} = \ldots = u_{j+m}$ in $\hat{\pi}$, then $\hat{\sigma}$ is obtained from $\hat{\pi}$ by changing u_{j+1} into $u_j^{(1)}$, ..., u_{j+m} into $u_j^{(m)}$

(ii) if $v_i = v_k = v_l = \ldots = v_h$, $i < k < l < \ldots < h$, for m v_i's, then $\check{\sigma}$ is obtained from $\check{\pi}$ by changing v_i into $v_i^{(m-1)}$, v_k into $v_i^{(m-2)}$, ..., v_h into v_i.

The map ϕ is bijective. The contents of σ are distinct each other and $\hat{\sigma}$ is an increasing sequence. So σ is a permutation on $\ddot{\mathbb{N}}$.

For example,

$$\pi = \begin{pmatrix} 1 & 2 & 2 & 3 & 3 & 5 \\ 4 & 1 & 1 & 4 & 2 & 5 \end{pmatrix} \xrightarrow{\phi} \sigma = \begin{pmatrix} 1 & 2 & 2^{(1)} & 3 & 3^{(1)} & 5 \\ 4^{(1)} & 1^{(1)} & 1 & 4 & 2 & 5 \end{pmatrix}$$

Now, we represent $\binom{u_i^{(k)}}{v_i^{(j)}} \in \sigma$ by a vertex with coordinates $(u_i^{(k)}, v_i^{(j)})$ in the first quadrant of the Cartesian plane.

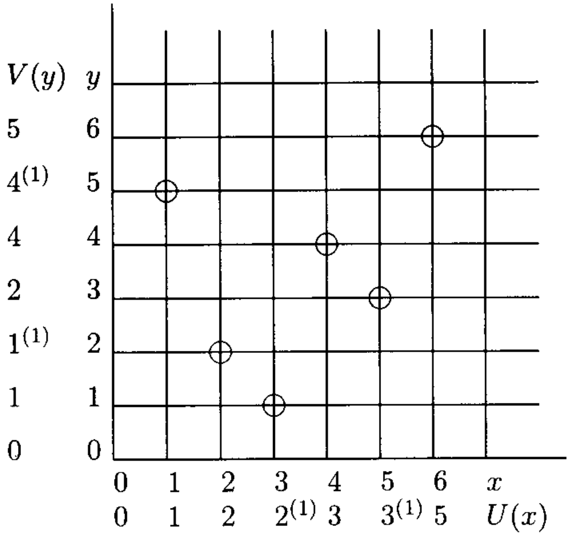
- Define a map U : abscissas x $(x = 0, 1, 2, \ldots, n) \to \{0\} \cup \hat{\sigma}$ with

$$U(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^{th} \text{ lowest element of } \hat{\sigma} & \text{else} \end{cases}$$

- Define a map V : abscissas y $(y = 0, 1, 2, \ldots, n) \to \{0\} \cup \check{\sigma}$ with

$$V(y) = \begin{cases} 0 & \text{if } y = 0 \\ y^{th} \text{ lowest element of } \check{\sigma} & \text{else} \end{cases}$$

Using our example σ, we obtain the following figure. In the figure, we see only one vertex on each vertical line and on each horizontal line.



Representation of σ

Figure 1.

The shadow $S(\gamma)$ of a permutation γ is the union of the set of points (x, y) such that $x \geq u$, $y \geq v$ for (u, v) of the representation of γ.

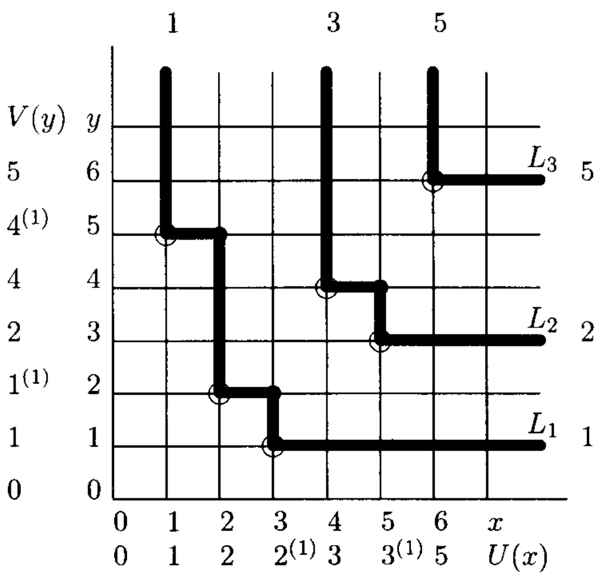
Shadow lines of γ are defined recursively. The first shadow line L_1 of γ is the boundary of $S(\gamma)$. To construct the shadow line L_{i+1} of γ remove the points of the representation of γ lying on L_i and construct the shadow line of the remaining points. This procedure ends when there is no remaining point on the plane.

The SW-corners (southwest corners) of a shadow line are the points of the representation of γ located on this line. The NE-corners (northeast corners) of a shadow line are the points (x, y) of the shadow line such that $(x + 1, y)$ and $(x, y + 1)$ don't belong to this shadow line. The shadow lines make up the shadow diagram of σ.

Definition3.2. Given a permutation, let $L_1, L_2, ...L_k$ be its shadow lines. If (x_1, y_1) is the leftmost SW-corner of the shadow line L_i then

x_1 is called **the x-coordinate of L_i** and is denoted by x_{L_i}. If (x_2, y_2) is the rightmost SW-corner of the shadow line L_i then y_2 is called **the y-coordinate of L_i** and is denoted by y_{L_i}.

In our example, there are three shadow lines, and the x-coordinates $(x_{L_1}, x_{L_2}, x_{L_3}) = (1, 3, 5)$ are shown above and the y-coordinates $(y_{L_1}, y_{L_2}, y_{L_3}) = \{1, 2, 5\}$ are to the left of the following shadow diagram, respectively. The white circles mark SW-corners and the black circles mark NE-corners (cf. Figure 2).



Description of shadow lines of σ

Figure 2.

On the other hand, using our example π and σ, The Robinson-Schensted correspondence and the Knuth correspondence are given by the following forms.

$$\sigma \overset{R-S}{\longrightarrow} P(\sigma) = \begin{array}{|l|l|l|} \hline 4^{(1)} & & \\ \hline 1^{(1)} & 4 & \\ \hline 1 & 2 & 5 \\ \hline \end{array} \qquad Q(\sigma) = \begin{array}{|l|l|l|} \hline 2^{(1)} & & \\ \hline 2 & 3^{(1)} & \\ \hline 1 & 3 & 5 \\ \hline \end{array}$$

$$\pi \overset{R-S-K}{\longrightarrow} P(\pi) = \begin{array}{|l|l|l|} \hline 4 & & \\ \hline 1 & 4 & \\ \hline 1 & 2 & 5 \\ \hline \end{array} \qquad Q(\pi) = \begin{array}{|l|l|l|} \hline 2 & & \\ \hline 2 & 3 & \\ \hline 1 & 3 & 5 \\ \hline \end{array}$$

We see in the correspondences above that the x-coordinates of the shadow lines consist of the first row of the standard tableaux $P(\sigma)$ and the y-coordinates of the shadow lines consist of the first rows of the standard tableaux $Q(\sigma)$. Moreover each row of $P(\pi)$ (resp. $Q(\pi)$) is equal to each row of $P(\sigma)$ (resp. $Q(\sigma)$) if the exponents of the labels in each row of $P(\sigma)$ are removed (resp. $Q(\sigma)$).

Lemma 3.1. Let ϕ be a generalized permutation on $\mathbb{N}$ and $\sigma = \phi(\pi)$ be a permutation on $\ddot{\mathbb{N}}$ with $\sigma = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}$. Let the shadow diagram of σ be constructed as before. Suppose the vertical line $x = k$ intersects i of the shadow lines. Let Y_j be the y-coordinate of the the lowest point of the intersection with L_j, $1 \leq j \leq n$. Then the first row of the $P_k = P(\begin{pmatrix} u_1 & u_2 & \cdots & u_k \\ v_1 & v_2 & \cdots & v_k \end{pmatrix})$ with $0 \leq k \leq n$ is

$$R_1 = Y_1 Y_2 ... Y_i$$

Proof. The lemma is trivial for $k = 0$. Assume the result holds for the line $x = k$ and consider $x = k + 1$.

(i) If $v_{k+1} > Y_i$ then the white circle (u_{k+1}, v_{k+1}) starts a new shadow line. So none of the values $Y_1Y_2...Y_i$ change and we obtain a new intersection,

$$Y_{i+1} = v_{k+1}$$

(ii) On the other hand,

$$Y_1 < ... < Y_{j-1} < v_{k+1} < Y_j < ... < Y_i$$

then (u_{k+1}, v_{k+1}) is added to the line L_j. Thus the lowest coordinates on becomes $Y_j' = v_{k+1}$ and all other Y-values stay the same. so the first row of P_{k+1} is

$$Y_1...Y_{j-1} < Y_j' < ... < Y_i$$

$\diamondsuit$

Theorem 3.1. Let $\sigma = \phi(\pi)$ for a generalized permutation π . If $\sigma \overset{R-S}{\longleftrightarrow} (P(\sigma), Q(\sigma))$ and the shadow diagram of σ has j shadow lines, then the first row of $P(\sigma)$ (resp. $Q(\sigma)$) is obtained from the y-coordinates (resp. x-coordinates) of the shadow lines. We have, for all j, $P_{1,j} = y_{L_j}$, $Q_{1,j} = x_{L_j}$.

Proof. The statement for $P(\sigma)$ is just the case $k = n$ of Lemma 3.1. As for $Q(\sigma)$, the entry u_k is added to the first row of Q_{k-1} in cell $(1, i)$ when v_k is greater than every element of the first row of P_{k-1}. But the previous lemma's proof shows that this happens precisely when the line $x = k$ intersects shadow line L_i in a vertical ray. In other words, $x_{L_i} = u_k = Q_{1,i}$. $\diamondsuit$

Consider the NE-corners of shadow lines. These are marked with black circles in the shadow diagram. If such a corner has coordinates (u_k, s), then we have $v_k < s$. According to the Lemma 1, s must be displaced from the first row of P_{k-1} by insertion of v_k. So the black circles correspond to the elements inserted into the second row during the construction of P. Thus we can get the rest of the two tableaux by iterating the shadow diagram construction. The figure 3 shows the second and third rows by thickened and dotted lines, respectively.

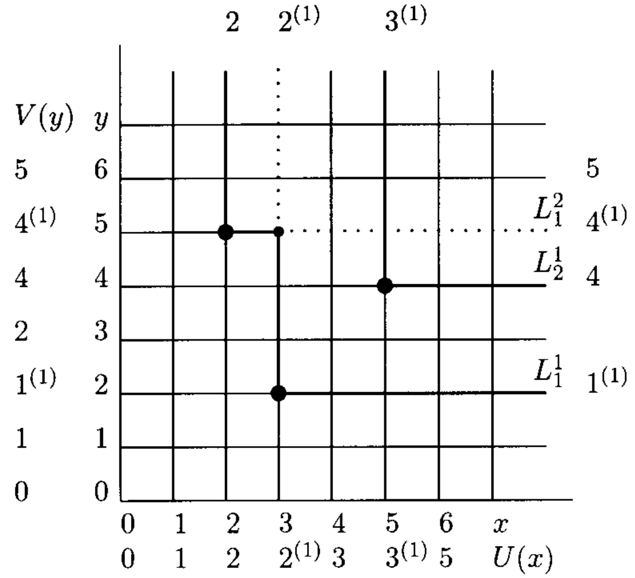


Figure 3

Definition 3.3. The i^{th} *skeleton* of a permutation τ on $\ddot{\mathbb{N}}$, $\tau^{(i)}$, is defined inductively by $\tau^{(1)} = \tau$ and

$$\tau^{(i)} = \begin{pmatrix} s_1 & s_2 & \dots & s_m \\ t_1 & t_2 & \dots & t_m \end{pmatrix}$$

where $(s_1, t_1), \dots, (s_m, t_m)$ are the NE-corners of the shadow diagram of $\tau^{(i-1)}$ listed in lexicographic order. The shadow lines for $\tau^{(i)}$ are denoted $L_j^{(i)}$

Theorem 3.2. Suppose $\sigma \overset{R-S}{\longleftrightarrow} (P(\sigma), Q(\sigma))$. Then $\sigma^{(i)}$ is a partial permutation of σ such that

$$\sigma^{(i)} \overset{R-S}{\longleftrightarrow} (P(\sigma)^i, Q(\sigma)^i)$$

where $P(\sigma)^i$ (resp. $Q(\sigma)^i$)) consists of the rows i and above of $P(\sigma)$ (resp. $Q(\sigma)$). Furthermore, $P(\sigma)_{i,j} = y_{L_j^{(i)}}$ and $Q(\sigma)_{i,j} = x_{L_j^{(i)}}$ for all i, j.

Proof. Induct on i. Let $\sigma^{(i)} \overset{R-S}{\longleftrightarrow} (P(\sigma)^i, Q(\sigma)^i)$. If $i = 1$, then y-coordinates (resp. x-coordinates) of shadow diagram of σ consist of the first row of $P(\sigma)$ (resp. $Q(\sigma)$) by Lemma 3.1 and Theorem 3.1. So the result holds for $i = 1$. Suppose the result holds for $i-1$ and we consider $\sigma^{(i)}$.

$\sigma^{(i)}$ is composed of the NE-corners of shadow diagram of $\sigma^{(i-1)}$. If such a corner has coordinates (u_k, s), and (u_k, v_k) SW-corner of shadow diagram of $\sigma^{(i-1)}$, then $v_k < s$. According to the Lemma 3.1, s must be displaced from the first row of P_{k-1} by insertion of v_k. So s is the element inserted in the second row during the construction of $P(\sigma)^{i-1}$. Thus $P(\sigma)^i$ consists of the rows i and above of $P(\sigma)$. Finally, $P(\sigma)_{i,j} = y_{L_j^{(i)}}$ and $Q(\sigma)_{i,j} = x_{L_j^{(i)}}$ for all i, j. $\diamondsuit$

Example 3.1. Using our example σ, the skeltons of σ are given:

$$\sigma^{(1)} = \sigma = \begin{pmatrix} 1 & 2 & 2^{(1)} & 3 & 3^{(1)} & 5 \\ 4^{(1)} & 1^{(1)} & 1 & 4 & 2 & 5 \end{pmatrix},$$

$$\sigma^{(2)} = \begin{pmatrix} 2 & 2^{(1)} & 3^{(1)} \\ 4^{(1)} & 1^{(1)} & 4 \end{pmatrix},$$

$$\sigma^{(3)} = \begin{pmatrix} 2^{(1)} \\ 4^{(1)} \end{pmatrix}.$$

We have the following correspondences:

$$\sigma^1 \xrightarrow{R-S} P(\sigma^{(1)}) = \begin{array}{lll} 4^{(1)} & & \\ 1^{(1)} & 4 & \\ 1 & 2 & 5 \end{array} \qquad Q(\sigma) = \begin{array}{lll} 2^{(1)} & & \\ 2 & 3^{(1)} & \\ 1 & 3 & 5 \end{array}$$

$$\sigma^{(2)} \xrightarrow{K} P(\sigma^{(2)}) = \begin{array}{ll} 4^{(1)} & \\ 1^{(1)} & 4 \end{array} \qquad Q(\sigma^{(2)}) = \begin{array}{ll} 2^{(1)} & \\ 2 & 3^{(1)} \end{array}$$

$$\sigma^{(3)} \xrightarrow{K} P(\sigma^{(3)}) = \begin{array}{l} 4^{(1)} \end{array} \qquad Q(\sigma^{(3)}) = \begin{array}{l} 2^{(1)} \end{array}$$

Theorem 3.3. For a generalized permutation, we have

$$P(\pi^{-1}) = Q(\pi) \text{ and } Q(\pi^{-1}) = P(\pi)$$

Proof. Taking the inverse of a generalized permutation π on $\mathbb{N}$ gives the inverse of the permutation $\sigma = \phi(\pi)$ on $\ddot{\mathbb{N}}$. Taking the inverse of a permutation σ corresponds to reflecting the shadow diagram in the line $y = x$. We have $P(\sigma^{-1}) = Q(\sigma)$ and $Q(\sigma^{-1}) = P(\sigma)$. The result holds by removing the exponents of the contents of $P(\sigma^{-1})$ and $Q(\sigma^{-1})$. $\diamondsuit$

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