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ON THE CONVERGENCE FOR WEIGHTED SUMS OF ASYMPTOTICALLY ALMOST NEGATIVELY ASSOCIATED RANDOM VARIABLES

TAE-SUNG KIM AND HYUN-CHULL KIM

Abstract. ABSTRACT. In this paper we derive complete convergences for weighted sums of asymptotically almost negatively associated random variables.

1. Introduction

The concept of complete convergence introduced by Hsu and Robbins(1947) is as follows. A sequence $\{U_n, n \geq 1\}$ of random variables converges completely to a constant C if $\sum_{n=1}^{\infty} P\{|U_n - C| > \epsilon\} < \infty$ *for all* $\epsilon > 0$. Since then, there have been many authors who devote the study to complete convergence for sums and weighted sums of i.i.d. random variables (see, e.g. Chow(1966), Thrum(1987), Gut(1993) and Li. et al.(1995)). Recently, these complete convergences were generalized to negatively associated(NA) sequences by Liang and Su(1999). Recall that a finite family $\{X_1, \cdots, X_n\}$ is said to be negatively associated(NA) if for any disjoint subsets $A, B \subset \{1, \cdots, n\}$ and any real coordinatewise nondecreasing functions $f : R^A \to R$ and $g : R^B \to R$, $Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$.
An infinite family of random variables is *negatively associated*(NA) if

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every finite subfamily is negatively associated(NA). This concept was introduced by Joag-Dev and Proschan(1983). By inspecting the proof of Matula's(1992) maximal inequality for NA sequences, Chandra and Ghosal(1996) found that one can also allow positive correlations provided they are small. Primarily motivated by this they introduced the following dependence condition: A sequence $\{X_n, n \geq 1\}$ of random variables is called *asymptotically almost negatively associated*(AANA) if there is a nonnegative sequence $q(m) \to 0$ such that

$$
\begin{aligned}
&Cov(f(X_m), g(X_{m+1}, \cdots, X_{m+k})) \\
&\leq q(m)(var(f(X_m))var(g(X_{m+1}, \cdots, X_{m+k})))^{1/2} \qquad (1)
\end{aligned}
$$

for all $m, k \geq 1$ and for all coordinatewise increasing continuous functions f and g whenever the right side of (1) is finite. The family of AANA sequences contains NA(in particular, independent) sequences (with $q(m) = 0$, $\forall\, m \geq 1$) and some more sequences of random variables which are not much deviated from being negatively associated.
In this paper we study complete convergence for weighted sums of AANA sequence, which is supposed to have not been studied in the literature.

2. Preliminaries

We start this section with the properties of AANA random variables which can be obtained easily from the definition of AANA random variables.

Lemma 2.1 Let $\{X_n, n \geq 1\}$ be a sequence of AANA. Then $\{f_n(X_n), n \geq 1\}$ is a sequence of AANA random variables and let $f_n(\cdot), n = 1, 2, \cdots$, be nondecreasing functions.

Lemma 2.2(Chandra and Ghosal(1996) Let $X_1, \cdots, X_n$ be mean zero, square integrable random variables such that condition (1) holds for

$1 \le m < k+m \le n$ and for all coordinatewise nondecreasing continuous functions f and g whenever the right-hand side of condition (1) is finite. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and $\sigma_k^2 = EX_k^2,\ 1 \le k \le n$. Then we have

$$E(\max_{1\le k\le n}\sum_{i=1}^{k} X_i)^2 \le (A + (1+A^2)^{1/2})^2 \sum_{k=1}^{n} \sigma_k^2. \tag{2}$$

Proof. See the proof of Theorem 1 in Chandra and Ghosal(1996).

Lemma 2.3 Let $X_1, \cdots, X_n$ be AANA random variables satisfying conditions in Lemma 2.2. Then

$$E(\sum_{k=1}^{n} X_k)^2 \le (A + (1+A^2)^{1/2})^2 \sum_{k=1}^{n} \sigma_k^2. \tag{3}$$

Proof . The proof is based on the ideas of the proof of Theorem 1 in Chandra and Ghosal(1996). To prove (3) set

$$Y_k = X_k/(\sum_{j=1}^{n}\sigma_j^2)^{1/2},\ 1 \le k \le n. \tag{4}$$

Note that $Y_1, Y_2, \cdots$, are AANA by Lemma 2.1 and that $\sum_{k=1}^{n}\tau_k^2 = 1$, where $\tau_k^2 = EY_k^2$. And set

$$T_k = Y_k + Y_{k+1} + \cdots + Y_n,\ 1 \le k \le n. \tag{5}$$

Then we have

$$T_k = Y_k + T_{k+1}$$

and consequently it follows from (1) that

$$ET_k^2 \le \tau_k^2 + ET_{k+1}^2 + 2q(k)\tau_k(ET_{k+1}^2)^{1/2},\ 1 \le k \le n-1.$$

Define a sequence $\{\xi_k, 1 \le k \le n\}$ by

$$\begin{cases} \xi_k^2 = \tau_k^2 + \xi_{k+1}^2 + 2q(k)\tau_k\xi_{k+1},\ 1 \le k \le n-1, \\ \xi_n^2 = \tau_n^2. \end{cases}$$

From the definition of ξ_k we have

$$ET_k^2 \le \xi_k^2,\ 1 \le k \le n. \tag{6}$$

Note that $\{\xi_k\}$ is decreasing. Thus

$$\xi_k^2 \le \tau_k^2 + \xi_{k+1}^2 + 2q(k)\tau_k\xi_1, \ 1 \le k \le n-1.$$

Substituting sequentially and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\xi_1^2 \quad & \le 1 + 2\xi_1 \sum_{k=1}^{n-1} q(k)\tau_k \\
& \le 1 + 2\xi_1 (\sum_{k=1}^{n-1} q^2(k))^{1/2} (\sum_{k=1}^{n-1} \tau_k^2)^{1/2} \\
& \le 1 + 2\xi_1 A.
\end{aligned}$$

Hence

$$(\xi_1 - A)^2 \le 1 + A^2. \tag{7}$$

Combining (6) and (7) we have

$$ET_1^2 \le \xi_1^2 \le (A + (1 + A^2)^{1/2})^2 \tag{8}$$

Thus by (4) and (8) the desired result (3) follows.

3. Main results

Theorem 3.1 Let $\{X_k, k \ge 1\}$ be a sequence of AANA random variables with $EX_k = 0$ and $EX_k^2 \le B < \infty$ *for all* $k \ge 1$. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and let $\{a_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of real numbers satisfying the condition

$$\sum_{k=1}^{n} a_{nk}^2 = O(n^\delta) \ as \ n \to \infty, \ |a_{nk}| = O(1), \ 1 \le k \le n, \ n \ge 1 \tag{9}$$

for some $0 < \delta < 1$. Assume

$$\sum_{m=1}^{\infty} q^2(m) < \infty. \tag{10}$$

Then, $\forall\ \epsilon > 0$ and some δ' such that $\delta < \delta' \leq 1$,

$$\sum_{n=1}^{\infty} n^{-\delta'} P(\mid \sum_{i=1}^{n} a_{ni} X_i \mid > \epsilon n^{1/2}) < \infty. \tag{11}$$

Proof. To prove (11) it suffices to show that

$$\sum_{n=1}^{\infty} n^{-\delta'} P(\mid \sum_{i=1}^{n} a_{ni}^{+} X_i \mid > \epsilon n^{1/2}) < \infty,\ \forall\ \epsilon > 0, \tag{12}$$

$$\sum_{n=1}^{\infty} n^{-\delta'} P(\mid \sum_{i=1}^{n} a_{ni}^{-} X_i \mid > \epsilon n^{1/2}) < \infty,\ \forall\ \epsilon > 0, \tag{13}$$

where $a_{ni}^{+} = a_{ni} \vee 0,\ a_{ni}^{-} = (-a_{ni}) \vee 0$. We will prove (12) only, since the proof of (13) is analogous. From the definition of an AANA sequence(see Lemma 2.1), we know that $\{a_{ni}^{+} X_i, 1 \leq i \leq n, n \geq 1\}$ is still an AANA sequence, and hence by applying (3) of Lemma 2.3 we have

$$\sum_{n=1}^{\infty} n^{-\delta'} P(\mid \sum_{i=1}^{n} a_{ni}^{+} X_i \mid > \epsilon n^{1/2})$$
$$\leq \epsilon^{-2} \sum_{n=1}^{\infty} n^{-1-\delta'} (A + (1 + A^2)^{1/2})^2 \sum_{k=1}^{n} a_{nk}^2 E X_k^2.$$

Note that conditions (9) and $EX_k^2 \leq B < \infty$ imply

$$\sum_{k=1}^{n} a_{nk}^2 E X_k^2 = O(n^{\delta})\ as\ n \to \infty\ for\ some\ 0 < \delta < 1. \tag{14}$$

Hence, by (10) and (14) we have, for some $0 < \delta < \delta' \leq 1$

$$\sum_{n=1}^{\infty} n^{-1-\delta'} (A + (1 + A^2)^{1/2})^2 \sum_{k=1}^{n} a_{nk}^2 E X_k^2$$
$$<< (A + (1 + A^2)^{1/2})^2 \sum_{n=1}^{\infty} n^{-(1+\delta' - \delta)} < \infty,$$

where $a << b$ means $a = O(b)$. The proof is complete.

Remark. Let $\{X_k, k \geq 1\}$ be a sequence AANA random variables with $EX_k = 0$. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$. Assume that (10) holds and that

$$\sum_{i=1}^{n} EX_i^2 = O(n^\delta) \; for \; some \; 0 < \delta < 1.$$

Then, $\forall \, \epsilon > 0$ and δ' such that $\delta < \delta' \leq 1$

$$\sum_{n=1}^{\infty} n^{-\delta'} P(|\sum_{i=1}^{n} X_i | > \epsilon n^{1/2}) < \infty.$$

By the similar method of the proof of Theorem 3.1 from (2) of Lemma 2.2 Theorem 3.2 follows :

Theorem 3.2. Let $\{X_k, k \geq 1\}$ be a sequence of AANA random variables with $EX_k = 0$ and $EX_k^2 \leq B < \infty \; for \; all \; k \geq 1$. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and let $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of real numbers satisfying the condition (9). If (10) holds then $\forall \, \epsilon > 0$ and δ' such that $\delta < \delta' \leq 1$

$$\sum_{n=1}^{\infty} n^{-\delta'} P(| \max_{1 \leq k \leq n} \sum_{i=1}^{k} a_{ni} X_i | > \epsilon n^{1/2}) < \infty. \tag{15}$$

Theorem 3.3 Let $\{X, X_k, k \geq 1\}$ be sequence of identically distributed AANA random variables with $EX = 0$ and $EX^2 < \infty$. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and let $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of real numbers satisfying (9). If (15) holds then for some δ' such that $\delta < \delta' \leq 1$ we have

$$\sum_{n=1}^{\infty} n^{-\delta'} \sum_{j=1}^{n} P(| a_{nj} X | > n^{1/2}) < \infty. \tag{16}$$

Proof. Obviously (15) implies

$$\sum_{n=1}^{\infty} n^{-\delta'} P(\max_{1\le j\le n} | a_{nj}X_j |> n^{1/2}) < \infty , \tag{17}$$

$$P(\max_{1\le j\le n} | a_{nj}X_j |> n^{1/2}) \to 0 \ as \ n \to \infty. \tag{18}$$

Note that

$$P(\max_{1\le j\le n} | a_{nj}X_j |> n^{1/2}) = \sum_{j=1}^{n} P(| a_{nj}X_j |> n^{1/2}, \max_{1\le i\le j-1} | a_{ni}X_i |\le n^{1/2}).$$

Hence, we deduce that

$$\sum\nolimits_{j=1}^{n} P(| a_{nj}X_j |> n^{1/2}) = P(\max\nolimits_{1\le j\le n} | a_{nj}X_j |> n^{1/2})$$
$$+\sum_{j=1}^{n} P(| a_{nj}X_j |> n^{1/2}, \max_{1\le i\le j-1} | a_{ni}X_i |> n^{1/2}). \tag{19}$$

Also, we have

$$\sum_{j=1}^{n} P(| a_{nj}X_j |> n^{1/2}, \max_{1\le i\le j-1} | a_{ni}X_i |> n^{1/2})$$
$$= \sum\nolimits_{j=1}^{n} E\{I(| a_{nj}X_j |> n^{1/2})I(\max_{1\le i\le j-1} | a_{ni}X_i |> n^{1/2})\}$$
$$= \sum\nolimits_{j=1}^{n} \{E[I(| a_{nj}X_j |> n^{1/2})I(\max_{1\le i\le j-1} | a_{ni}X_i |> n^{1/2})]$$
$$-EI(| a_{nj}X_j |> n^{1/2})EI(\max_{1\le i\le j-1} | a_{ni}X_i |> n^{1/2})\}$$
$$+\sum\nolimits_{j=1}^{n} \{EI(| a_{nj}X_j |> n^{1/2})EI(\max_{1\le i\le j-1} | a_{ni}X_i |> n^{1/2})\}$$
$$\le E\sum_{j=1}^{n} [I(| a_{nj}X_j |> n^{1/2}) - P(| a_{nj}X_j |> n^{1/2})]I(\max_{1\le i\le n} | a_{ni}X_i |> n^{1/2})$$
$$+\sum\nolimits_{j=1}^{n} P(| a_{nj}X_j |> n^{1/2})P(\max\nolimits_{1\le i\le n} | a_{ni}X_i |> n^{1/2}) = II + III. \tag{20}$$

Define

$$Y_{nj} = \begin{cases} a_{nj}X_j, & if \ a_{nj} \ge 0 , \\ -a_{nj}X_j, & if \ a_{nj} < 0 . \end{cases}$$

Then from the definition of AANA the random variables $\{Y_{nj}, 1 \leq j \leq n, n \geq 1\}$ are AANA and $\{I(Y_{nj} > n^{1/2})\}$ are still AANA(see Lemma 2.1). By applying the Cauchy-Schwarz inequality and Lemma 2.2, we get

$$
\begin{aligned}
\mid II \mid \quad & = \mid E\sum_{j=1}^{n}[I(\mid a_{nj}X_j \mid> n^{1/2}) - P(\mid a_{nj}X_j \mid> n^{1/2})] \\
& \times I(\max_{1\leq i\leq n} \mid a_{ni}X_i \mid> n^{1/2}) \mid \\
& \leq [E(\sum_{j=1}^{n} I(\mid a_{nj}X_j \mid> n^{1/2}) - EI(\mid a_{nj}X_j \mid> n^{1/2}))^2 \\
& \times E(I(\max_{1\leq i\leq n} \mid a_{ni}X_i \mid> n^{1/2}))^2]^{1/2} \\
& = [Var(\sum_{j=1}^{n} I(\mid a_{nj}X_j \mid> n^{1/2}))P(\max_{1\leq i\leq n} \mid a_{ni}X_i \mid> n^{1/2})]^{1/2} \\
& \leq [2\{Var[\sum_{j=1}^{n} I(Y_{nj} > n^{1/2})] + Var[\sum_{j=1}^{n} I(Y_{nj} < -n^{1/2})]\} \\
& \times P(\max_{1\leq i\leq n} \mid a_{ni}X_i \mid> n^{1/2})]^{1/2} \\
& \leq [8\{\sum_{j=1}^{n} P(Y_{nj} > n^{1/2}) + \sum_{j=1}^{n} P(Y_{nj} < -n^{1/2})\} \\
& \times P(\max_{1\leq i\leq n} \mid a_{ni}X_i \mid> n^{1/2})(A + (1 + A^2)^{1/2})^2]^{1/2} \\
& \leq \frac{1}{2}\sum_{j=1}^{n} P(\mid a_{nj}X \mid> n^{1/2}) + 4\{(A + (1 + A^2)^{1/2})^2 \\
& \times P(\max_{1\leq i\leq n} \mid a_{ni}X_i \mid> n^{1/2})\} \qquad (21)
\end{aligned}
$$

by $\sqrt{ab} \leq \frac{a+b}{2}$. From (19)-(21) we have

$$\frac{1}{2}\sum_{j=1}^{n} P(\mid a_{nj}X \mid > n^{1/2})$$
$$\leq \{1 + 4(A + (1 + A^2)^{1/2})^2\} P(\max_{1\leq i\leq n} \mid a_{ni}X_i \mid > n^{1/2})$$
$$+\sum_{j=1}^{n} P(\mid a_{nj}X \mid > n^{1/2}) P(\max_{1\leq i\leq n} \mid a_{ni}X_i \mid > n^{1/2}) \tag{22}$$

and from (18) we get

$$\sum_{j=1}^{n} P(\mid a_{nj}X \mid > n^{1/2}) << P(\max_{1\leq i\leq n} \mid a_{ni}X_i \mid > n^{1/2}) \tag{23}$$

for sufficiently large n, where $a << b$ means $a = O(b)$. Therefore, from (17) and (23)

$$\sum_{n=1}^{\infty} n^{-\delta'} \sum_{j=1}^{n} P(\mid a_{nj}X \mid > n^{1/2}) < \infty.$$

The proof is complete.

Corollary 3.4 Let $\{X_k, k \geq 1\}$ be a sequence of AANA random variables with $EX_k = 0$ and $EX_k^2 < \infty$ *for all* $k \geq 1$ and let $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of real numbers. Assume that (15) holds and that for $\delta > 0$ small enough,

$$P(\max_{1\leq i\leq n} \mid a_{ni}X_i \mid > \epsilon n^{1/2}) < \delta \ \forall \ \epsilon > 0$$

for sufficiently large n. Then, $\forall \ \epsilon > 0$

$$\sum_{j=1}^{n} P(\mid a_{nj}X_j \mid > \epsilon n^{1/2}) << P(\max_{1\leq i\leq n} \mid a_{ni}X_i \mid > \epsilon n^{1/2})$$

for sufficiently large n.

Theorem 3.5 Let $\{X, X_k, k \geq 1\}$ be a sequence of identically distributed AANA random variables with $EX = 0$. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$

and let $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of real numbers satisfying

$$\sum_{k=1}^{n} a_{nk}^2 = O(1), 1 \leq k \leq n, \ as\ n \to \infty. \tag{24}$$

If (11) and

$$N(n, m+1) =: \#\{k \geq 1 :| a_{nk} |\geq (m+1)^{-1/2}\} \asymp O(m) \ as\ n \to \infty (25)$$

hold then the following statements are equivalent:

$$(i)\ EX^2 < \infty,$$
$$(ii)\ \sum_{n=1}^{\infty} \frac{1}{n} P(| \sum_{i=1}^{n} a_{ni} X_i |> \epsilon n^{1/2}) < \infty \ \forall\ \epsilon > 0.$$

Proof. $(i) \Rightarrow (ii)$ As in the proof of Theorem 3.1 by Lemma 2.3 we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{n} P(| \sum_{i=1}^{n} a_{ni}^{+} X_i |> \epsilon n^{1/2}) \\ &\leq \epsilon^{-2} \sum_{n=1}^{\infty} \frac{1}{n^2} (A + (1 + A^2)^{1/2})^2 EX^2 \sum_{k=1}^{n} a_{nk}^2 \\ &<< \epsilon^{-2} (A + (1 + A^2)^{1/2})^2 \sum_{n=1}^{\infty} n^{-2} < \infty. \end{aligned}$$

Similarly, we also have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(| \sum_{i=1}^{n} a_{ni}^{-} X_i |> \epsilon n^{1/2}) < \infty.$$

Thus (i) implies (ii).

It follows from (24) and (25) that

$$\begin{aligned}
&\sum_{j=1}^{n} P(\mid a_{nj}X \mid > n^{1/2}) \\
&= \sum\nolimits_{j=1}^{\infty} \sum_{(j+1)^{-1} \leq a_{ni}^2 < j^{-1}} P(\mid a_{nj}X \mid \geq n^{1/2}) \\
&= \sum_{j=1}^{\infty} (N(n,j+1) - N(n,j)) \sum_{m=nj}^{\infty} P(m \leq X^2 < m+1) \\
&\asymp \sum_{m=n}^{\infty} (m/n) P(m \leq X^2 < m+1). \qquad (26)
\end{aligned}$$

(26) yields

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n} P(\mid a_{nj}X \mid > n^{1/2}) &\asymp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=n}^{\infty} (m/n) P(m \leq X^2 < m+1) \\
&= \sum_{m=1}^{\infty} mP(m \leq X^2 < m+1) \sum_{n=1}^{m} (1/n^2) \\
&\asymp \sum_{m=1}^{\infty} mP(m \leq X^2 < m+1), \qquad (27)
\end{aligned}$$

where $a \asymp b$ means $a = O(b)$ and $b = O(a)$. Finally, by (ii) and (27) we obtain

$$EX^2 \asymp \sum_{n=1}^{\infty} mP(m \leq X^2 < m+1) < \infty.$$

Thus, the proof is complete.

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Tae Sung Kim
Division of Mathematics and Informational Statistics and Institute
of Basic Natural Science, WonKwang University,
Iksan, 570-749, Korea
E-mail : starkim@wonkwang.ac.kr

Hyun Chull Kim
Division of Computer and Information Science
Daebul University, Yangam. Chonnam 526-720, Korea
E-mail : kimhc@mail.daebul.ac.kr