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POSITIVE INTERPOLATION PROBLEMS IN ALG$\mathcal{L}$

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Abstract. Given operators X and Y acting on a Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $AX = Y$. An interpolating operator for n-operators satisfies the equation $AX_i = Y_i$, for $i = 1, 2, \cdots, n$. In this article, we obtained the following : Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let X and Y be operators acting on $\mathcal{H}$. Then the following statements are equivalent.

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$, A is positive and every E in $\mathcal{L}$ reduces A.

(2) $\sup \left\{ \frac{\|\sum_{i=1}^{n} E_i Y f_i\|}{\|\sum_{i=1}^{n} E_i X f_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H} \right\} < \infty$

and $< \sum_{i=1}^{n} E_i Y f_i, \sum_{i=1}^{n} E_i X f_i > \geq 0$, $n \in \mathbb{N}$, $E_i \in \mathcal{L}$ and $f_i \in H$.

1. Introduction

The equation $Ax = y$ in Hilbert space has been considered by a number of authors. The problem is this: Given Hilbert space vectors x and y, when is there a bounded linear operator A (usually satisfying some other conditions) that maps x to y ? The "other conditions" that have been of interest to us involve restricting A to lie in the algebra associated with a subspace lattice. Lance [15] initiated the discussion by considering a nest $\mathcal{N}$ and asking what conditions on x and y will guarantee the existence of an operator A in Alg$\mathcal{N}$ such that $Ax = y$.

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This result was used to find a new proof of Ringrose's characterization of the Jacobson radical. Hopenwasser [9] extended Lance's result to the case where the nest $\mathcal{N}$ is replaced by an arbitrary commutative subspace lattice $\mathcal{L}$; the conditions in both cases read the same. Munch [16] considered the problem of finding a Hilbert-Schmit operator A in Alg$\mathcal{N}$ that maps x to y, whereupon Hopenwasser [10] again extended to Alg$\mathcal{L}$. In [1], authors studied the problem of finding A so that $Ax = y$ and A is required to lie in certain ideals contained in Alg$\mathcal{L}$ (for a nest $\mathcal{L}$); in particular, they considered the ideal of compact operators, the Jacobson radical, and Larson's ideal $\mathcal{R}^{\infty}$.

Roughly speaking, when an operator maps one thing to another, we think of the operator as the interpolating operator and the equation representing the mapping as the interpolation equation. The equations $Ax = y$ and $AX = Y$ are indistinguishable if spoken aloud, but we mean the change to capital letters to indicate that we intend to look at fixed operators X and Y, and ask under what conditions there will exist an operator A satisfying the equation $AX = Y$. Let x and y be vectors in a Hilbert space. Then $< x, y >$ means the inner product of vectors x and y.

Note that the "vector interpolation" problem is a special case of the "operator interpolation" problem. Indeed, if we denote by $x \otimes u$ the rank-one operator defined by the equation $x \otimes u(w) =< w, u > x$, and if we set $X = x \otimes u$, and $Y = y \otimes u$, then the equations $AX = Y$ and $Ax = y$ represent the same restriction on A.

Let $\mathcal{L}$ be a (commutative) subspace lattice on a Hilbert space $\mathcal{H}$. In this paper, we investigate positive interpolation problems in Alg$\mathcal{L}$: Let X and Y be operators acting on $\mathcal{H}$. When does there exist a positive operator A in Alg$\mathcal{L}$ such that $Y = AX$

The simplest case of the operator interpolation problem relaxes all restrictions on A, requiring it simply to be a bounded operator. In this case, the existence of A is nicely characterized by the well-known factorization theorem of Douglas [7]:

Theorem D. Let Y and X be bounded operators on a Hilbert space $\mathcal{H}$. The following statements are equivalent:

(1) $range(Y^*) \subseteq range(X^*)$;

(2) $Y^*Y \leq \lambda^2 X^*X$ for some $\lambda \geq 0$;

(3) there exists a bounded operator A on $\mathcal{H}$ so that $AX = Y$.

Moreover, if (1), (2), and (3) are valid, then there exists a unique operator A so that

(a) $\|A\|^2 = \inf\{\mu : Y^*Y \leq \mu X^*X\}$;

(b) $ker[Y^*] = ker[A^*]$; and

(c) $range[A^*] \subseteq range[X]^-$.

First, we establish some notation and conventions. A (commutative) subspace lattice $\mathcal{L}$ is a strongly closed lattice of (commutative) projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and 1 lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is a subspace lattice on $\mathcal{H}$, then Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$.

2. Results

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice of orthogonal projections acting on $\mathcal{H}$ containing 0 and I. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $M^\perp$ the orthogonal complement of M. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

Definition. Let $\mathcal{H}$ be a Hilbert space and A be an operator acting on $\mathcal{H}$. A is *positive* if $< Ax, x > \geq 0$ for all x in $\mathcal{H}$.

Theorem 1. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$.
Let X and Y be operators acting on $\mathcal{H}$. Assume that $rangeX$ is dense in $\mathcal{H}$.
If $\sup\left\{\frac{\|\sum_{i=1}^{n} E_iYf_i\|}{\|\sum_{i=1}^{n} E_iXf_i\|} : n \in \mathbb{N}, f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L}\right\} < \infty$ and $< Yf, Xf > \geq 0$, for all $f \in \mathcal{H}$, then there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$, A is positive and every E in $\mathcal{L}$ reduces A.

Proof. By Theorem 1 [13], there is an operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A. By the second condition of hypothesis, $< Af, f > \geq 0$ for all f in $rangeX$. Since the $rangeX$ is dense in $\mathcal{H}$, $< Af, f > \geq 0$ for all f in $\mathcal{H}$. Hence A is positive.

Theorem 2. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let X and Y be operators acting on $\mathcal{H}$. If there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$, A is positive and every E in $\mathcal{L}$ reduces A, then

$\sup\left\{\frac{\|\sum_{i=1}^{n} E_iYf_i\|}{\|\sum_{i=1}^{n} E_iXf_i\|} : n \in \mathbb{N}, f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L}\right\} < \infty$ and $< Yf, Xf > \geq 0$ for all f in $\mathcal{H}$.

Proof. By Theorem 2 [13],

$\sup\left\{\frac{\|\sum_{i=1}^{n} E_iYf_i\|}{\|\sum_{i=1}^{n} E_iXf_i\|} : n \in \mathbb{N}, f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L}\right\} < \infty$. Since $AX = Y$,

$AXf = Yf$ for all f in $\mathcal{H}$. Since A is positive, $< Yf, Xf > \geq 0$ for all f in $\mathcal{H}$.

Theorem 3. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let X and Y be operators acting on $\mathcal{H}$.

If $\sup\left\{\frac{\|\sum_{i=1}^{n} E_iYf_i\|}{\|\sum_{i=1}^{n} E_iXf_i\|} : n \in \mathbb{N}, f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L}\right\} < \infty$

and $< \sum_{i=1}^{n} E_iYf_i, \sum_{i=1}^{n} E_iXf_i > \geq 0$, $n \in \mathbb{N}$, $E_i \in \mathcal{L}$ and $f_i \in \mathcal{H}$, then there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$, A is positive and every E in $\mathcal{L}$ reduces A.

Proof. By Theorem 2 [13], there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A. Under the second condition of hypothesis, $< Af, f > \geq 0$ for all f in $\mathcal{M}$, where $\mathcal{M} = \left\{\sum_{i=1}^{n} E_iXf_i : n \in \mathbb{N}, f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L}\right\}$ is the linear manifold defined in the proof of Theorem 2[13]. Since $Ag = 0$ for all g in $\overline{\mathcal{M}}^{\perp}$, $< Ag, g >= 0$. Hence A is positive.

If we summarize Theorems 2 and 3, we can get the following theorem.

Theorem 4. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let X and Y be operators acting on $\mathcal{H}$. Then the following statements are equivalent.

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$, A is positive and every E in $\mathcal{L}$ reduces A.

(2) $\sup\left\{\frac{\|\sum_{i=1}^{n} E_iYf_i\|}{\|\sum_{i=1}^{n} E_iXf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty$ and $< \sum_{i=1}^{n} E_iYf_i, \sum_{i=1}^{n} E_iXf_i > \geq 0$, $n \in \mathbb{N}$, $E_i \in \mathcal{L}$ and $f_i \in \mathcal{H}$.

Theorem 5. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n$ be operators acting on $\mathcal{H}$. If there is an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j$ $(j = 1, 2, \cdots, n)$, A is positive and every E in $\mathcal{L}$ reduces A,

then $\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i}\|} : l\leq n, m_i\in\mathbb{N}, E_{k,i}\in\mathcal{L} \text{ and } f_{k,i}\in\mathcal{H}\right\}$ $<\infty$

and $< \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}, \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i} > \geq 0$, $m_i \in \mathbb{N}$, $l \leq n$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$.

Proof. By Theorem 3 [13], we can get the first part of result. Since $AX_j = Y_j (j = 1, 2, \cdots, n)$, $A(\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i}) = \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i} Y_if_{k,i}$. So $< \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}, \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i} > \geq 0$, $m_i \in \mathbb{N}$, $l \leq n$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$.

Theorem 6. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ be operators acting on $\mathcal{H}$. Assume that
$\mathcal{N} = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$ is dense in $\mathcal{H}$.
If $\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i}\|} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$ $< \infty$ and $< \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}, \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i} > \geq 0$, $m_i \in \mathbb{N}$, $l \leq n$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$, then there exists an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j$ $(j = 1, 2, \cdots, n)$, A is positive and every E in $\mathcal{L}$ reduces A.

Proof. Let A be the operator defined in the proof of Theorem 4 [13]. Then $AX_j = Y_j (j = 1, 2, \cdots, n)$ and every E in $\mathcal{L}$ reduces A. Since $AX_j = Y_j (j = 1, 2, \cdots, n)$,
$A(\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i}) = \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}$. So $<\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_if_{k,i}, \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_if_{k,i} > \geq 0$, $m_i \in \mathbb{N}$, $l \leq n$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$. Since $\mathcal{M}$ is

dense in $\mathcal{H}$, $< Af, f > \geq 0$. Hence A is positive.

Corollary 7. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ be operators acting on $\mathcal{H}$. Assume that the range of one of X_p's is dense in $\mathcal{H}$, let it be X_1. If
$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$$
$< \infty$ and $< Y_1 f, X_1 f > \geq 0$ for all f in $\mathcal{H}$, then there exists an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j$ $(j = 1, 2, \cdots, n)$, A is positive and every E in $\mathcal{L}$ reduces A.

Theorem 8. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ be operators acting on $\mathcal{H}$. Assume that the range of one of X_i's is dense in $\mathcal{H}$, let it be X_1. Then the following statements are equivalent.

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j$ $(j = 1, 2, \cdots, n)$, A is positive and every E in $\mathcal{L}$ reduces A.

(2) $$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$$
$< \infty$ and $< Y_1 f, X_1 f > \geq 0$ for all f in $\mathcal{H}$.

Theorem 9. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n$ be operators acting on $\mathcal{H}$.

If $$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$$
$< \infty$ and $< \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}, \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i} > \geq 0$, $m_i \in \mathbb{N}$, $l \leq n$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$, then there is an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j$ $(j = 1, 2, \cdots, n)$, A is positive and every E in $\mathcal{L}$ reduces A.

Proof. By Theorem 5 [13], there is an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j$ $(j = 1, 2, \cdots, n)$ and every E in $\mathcal{L}$ reduces A. Let $\mathcal{N}$ be the linear manifold defined in the proof of Theorem 5 [13]. Then $< Af, f > \geq 0$ for all f in $\overline{\mathcal{N}}$ by the second condition of hypothesis. Let g be a vector of $\overline{\mathcal{N}}^{\perp}$. Since $Ag = 0$, $< Ag, g >= 0$. Hence A is positive.

From Theorems 5 and 9, we can get the following theorem.

Theorem 10. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n$ be operators acting on $\mathcal{H}$. Then the following statements are equivalent.

(1) There exists an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j$ $(j = 1, 2, \cdots, n)$, A is positive and every E in $\mathcal{L}$ reduces A.

(2) $\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : l \leq n, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$ $< \infty$ and $< \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}, \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i} > \geq 0$ for all $f_{k,i}$ in $\mathcal{H}$ and $E_{k,i}$ in $\mathcal{L}$.

If we modify the proofs of previous theorems a little bit, we can get the following theorems.

Theorem 11. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $\{X_n\}$ and $\{Y_n\}$ be two infinite sequences of operators acting on $\mathcal{H}$. Assume that the range of one of X_n's is dense in $\mathcal{H}$, let it be X_1. If $\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : l, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$ and $< Y_1 f, X_1 f > \geq 0$ for all $f \in \mathcal{H}$, then there is an operator A in Alg$\mathcal{L}$ such that $AX_n = Y_n$ $(n = 1, 2, \cdots)$, A is positive and every E in $\mathcal{L}$ reduces A.

Theorem 12. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $\{X_n\}$ and $\{Y_n\}$ be two infinite sequences of operators acting on

$\mathcal{H}$. If there is an operator A in Alg$\mathcal{L}$ such that $AX_n = Y_n$ $(n = 1, 2, \cdots)$, A is positive and every E in $\mathcal{L}$ reduces A, then

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : l, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$$

and $< Y_k f, X_k f > \geq 0$ for all f in $\mathcal{H}$ and all $k = 1, 2, \cdots, n$.

From Theorems 11 and 12, we can get the following theorem.

Theorem 13. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $\{X_n\}$ and $\{Y_n\}$ be two infinite sequences of operators acting on $\mathcal{H}$. Assume that the range of one of X_n's is dense in $\mathcal{H}$, let it be X_1. Then the following statements are equivalent.

(1) There is an operator A in Alg$\mathcal{L}$ such that $AX_n = Y_n$ $(n = 1, 2, \cdots)$, A is positive and every E in $\mathcal{L}$ reduces A.

(2) $$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : l, m_i \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$$
$< \infty$ and $< Y_1 f, X_1 f > \geq 0$ for all f in $\mathcal{H}$.

Theorem 14. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $\{X_n\}$ and $\{Y_n\}$ be two infinite sequences of operators acting on $\mathcal{H}$. If there is an operator A in Alg$\mathcal{L}$ such that $AX_n = Y_n$ for all $n = 1, 2, \cdots$, A is positive and every E in $\mathcal{L}$ reduces A, then

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$$

and $< \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}, \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i} > \geq 0$, $m_i, l \in \mathbb{N}$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$.

Theorem 15. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $\{X_n\}$ and $\{Y_n\}$ be two infinite sequences of operators acting on $\mathcal{H}$.

If $\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}Y_if_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}X_if_{k,i}\|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\} < \infty$ and $< \sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}Y_if_{k,i}, \sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}X_if_{k,i} > \geq 0$, $m_i, l \in \mathbb{N}$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$, then there is an operator A in Alg$\mathcal{L}$ such that $AX_j = Y_j$ $(n = 1, 2, \cdots)$, A is positive and every E in $\mathcal{L}$ reduces A.

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