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ON THE TWO SIDED IDEALS OF ORDERS IN A QUATERNION ALGEBRA

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Abstract. The orders in quaternion algebras play central role in the theory of Hecke operators. In this paper, we study the order of two sided ideal group in orders of a quaternion algebra.

1. Introduction

A quaternion algebra over a field k means a semi simple algebra of dimension 4 over k. It is known that there are three kinds of primitive orders in quaternion algebras over a local field. First, if A is a division algebra, an order of A is primitive if it contains the full ring of integers of a quadratic extension field of k. Second, if A is isomorphic to $\mathrm{Mat}_{2\times 2}(k)$, an order of A is primitive if it contains a subset which is isomorphic to $\mathfrak{o} \oplus \mathfrak{o}$, where $\mathfrak{o}$ is the ring of integers in k. Finally, if A is isomorphic to $\mathrm{Mat}_{2\times 2}(k)$, an order of A is also called primitive if it contains the full ring of integers in a quadratic extension field of k. The arithmetic properties of first two types of primitive orders were studied in [3], [7]. For the remaining type was studied in [5] only for the non dyadic local field case. In this paper we will study the arithmetic theory of the remaining type over a dyadic local field. As an application, the class number of primitive orders over a dyadic local field will be computed.

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2. Primitive orders

2.1. In this section, we summarize the arithmetic theory of a quaternion algebra and its order.

A lattice on A is a finitely generated $\mathbb{Z}$ module containing a base of A over $\mathbb{Q}$. An order of A is a lattice on A which is also a subring with 1. The analogous definitions hold for lattices and orders in $A_p = A \otimes \mathbb{Q}_p$ for a prime p.

Throughout this paper we assume that k is a dyadic local field, $\mathbb{Q}_2$. Let $\mathfrak{o}$ denote the ring of integers in k, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. By $\Delta(\alpha)$, we denote the discriminant of α.

$$\Delta(\alpha) = \mathrm{Tr}(\alpha)^2 - 4N(\alpha),$$

where Tr and N are the trace and norm of L over k respectively, where L is a quadratic extension field of k. If Γ is an $\mathfrak{o}$ algebra of rank 2 contained in L, then $\Gamma = \mathfrak{o} + \mathfrak{o}x$ and the discriminant of Γ is

$$\Delta(\Gamma) = \Delta(x) \mod U^2,$$

where U is the set of all units in $\mathfrak{o}$.

Let $\mathfrak{o}^2 - 4\mathfrak{o} = \{s^2 - 4n | s, n \in \mathfrak{o}\}$. Then we consider the set of all possible discriminants $(\mathfrak{o}^2 - 4\mathfrak{o})/U^2$.

2.2. Note that $\Delta_\sigma^* \neq \phi$ only if $\sigma = 2\rho, 0 \le \rho \le e$ or $\sigma = 2e+1$ where $e = \mathrm{ord}_k(2)$. Let

$$\Delta^* = \cup_{\sigma=0}^{\infty} \Delta_\sigma^* = \left(\cup_{\rho=0}^{e} \Delta_{2\rho}^*\right) \cup \Delta_{2e+1}^*.$$

Then we know Γ is a maximal order of a quadratic extension field of k if and only if $\Delta(\Gamma) \in \Delta^*$. If $e > 0$ and $1 \le \rho \le e$

$$\Delta_{2\rho}^* = \pi^{2\rho}(U^2 + \pi^{2e-2\rho+1}U)/U^2.$$

There is a bijective correspondence between elements of Δ^* and quadratic extension field of k given by $\Delta(\Gamma) \to \Gamma \otimes \mathfrak{o}_k$ for $\Delta(\Gamma)$ an element of Δ^*.

Thus we can classify all quadratic extension fields of a dyadic local field k as follows: Δ_0^* contains one point which corresponds to a unique unramified quadratic extension of k and

$$\Delta_{2e+1}^* = \pi^{2e+1}U/U^2$$

contains $2q^2$ points representatives where $q = |\mathfrak{o}/\mathfrak{p}|$.

Definition 1. Let L be a quadratic extension of k. We define

$$t = t(L) = \mathrm{ord}_k(\Delta(L)) - 1.$$

Remark. Note that if L is an unramified extension field of k, then $t = -1$. On the other hand, if L is a ramified extension field of a dyadic field k, then $t > 0$ (See 1.3 in [4]).

2.3. Let A be a rational quaternion algebra ramified precisely at the odd prime q and ∞. That is, $A_q = A \times \mathbb{Q}_q$ and $A_\infty = A \times R$ are division algebras. Otherwise, $A_p = A \times \mathbb{Q}_p$ is isomorphic to $M_2(\mathbb{Q}_p)$ for a finite prime $p \neq q$ (See [5]).

Fix a prime $p \neq q$ and let L be a quadratic extension field of $\mathbb{Q}_p$. It is known that $\left\{ \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in L \right\}$ is a quaternion algebra over $\mathbb{Q}_p$.

Let $\left\{ \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in L \right\} = L + \xi L$, where $\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\xi\alpha = \overline{\alpha}\xi, \xi^2 = 1$ and $\overline{\xi} = -\xi$.

Hence, we can define the norm of an element in A as its determinant.

2.4. Let P_L be the prime ideal of $\mathcal{O}_L$ which is the ring of integers in L. In [6], we have computed that the possibilities of an order, R of A_2

containing $\mathcal{O}_L$. We state the results in the following theorem.

Theorem 2.1. Let the notations be as in 2.3 and 2.4. If an order R of A_2 contains $\mathcal{O}_L$, then the possibilities of R are one of the followings.

(i) If p is a unramified prime in L, $R = \mathcal{O}_L + \xi P_L^{\nu}$.
(ii) If p is a ramified prime in L, $R = \mathcal{O}_L + (1+\xi)P_L^{\nu-t-1}$ or $\overline{R} = \mathcal{O}_L + (1-\xi)P_L^{\nu-t-1}$.

for some nonnegative integer ν with some $\xi \in A_2$.

Proof. See [6].

Remark. Since $\mathrm{ord}_k(2) = e$, if L is a ramified quadratic extension field of k, then $\mathrm{ord}_L(2) = 2e$. Hence, in the above definition, if $t(L) < 2e$, then $(1+\xi)P_L^{-t-1} = (1-\xi)P_L^{-t-1}$. That is $R_0(L) = \overline{R_o(L)}$. On the other hand, if $t(L) = 2e$, then there are two different maximal orders $R_0(L)$ and $\overline{R_0(L)}$. However, $\mathcal{O}_L + (1+\xi)P_L^{-t} = \mathcal{O}_L + (1-\xi)P_L^{-t}$ by the same reasoning of the $t(L) < 2e$ case. i.e. $R_1(L) = R_0(L) \cap \overline{R_0(L)}$.

We now define the level of order M of A.

Definition 2. Let A be an quaternion algebra over a number field K and let L be a quadratic extension field of K. An order M of A is called primitive if M contains the ring of integers of L.

Remark. A primitive order was studied in Eicher's thesis [2]. Over a local field, this primitive order is divided into three types of orders. Since $A_p = A \otimes K_p$ is either a division algebra or a 2×2 matrix algebra over K_p and $L \otimes K_p$ is either $K_p \otimes K_p$ or a quadratic extension field of K_p, we are able to classify the primitive orders M_p of A_p as follows.

1. If A_p is a division algebra, then $L \otimes K_p$ is a quadratic extension field of K_p. Hence, M_p contains the ring of integers of $L \otimes K_p$.

2. If A_p is a 2×2 matrix algebra over K_p, then M_p contains $\mathfrak{o}_K \otimes \mathfrak{o}_K$, where $\mathfrak{o}_K$ is the ring of integers in K_p.
3. If A_p is a 2×2 matrix algebra over K_p, then M_p contains the ring of integers of $L \otimes K_p$.

In this paper we study third type of orders over a dyadic local field.

Definition 3. Let A be a rational quaternion algebra ramified precisely at one finite prime q and ∞. For finite primes, $p_1, p_2, \cdots p_d \neq q$, an order M has level $(q; L(p_1), \nu(p_1); \cdots; L(p_d), \nu(p_d))$ if

(i) M_q is the maximal order of A_q.

(ii) for a prime $p \neq q$, there exists a quadratic extension field $L(p)$ of $\mathbb{Q}_p$ and nonnegative integer $\nu(p)$ (which is even if $L(p)$ is unramified) such that $M_p = R_{\nu(p)}(L(p))$,

(iii) $\nu(p_i) > 0$ for $i = 1, 2, \cdots, d$ and $\nu(p) = 0$ for $p \neq q, p_1 \cdots, p_d$. (i.e. M_p is a maximal order of A_p if $p \neq p_1, p_2, \cdots, p_d$).

2.5. In the rest of this paper, let A be a rational quaternion algebra ramified precisely at the odd prime q and ∞ and we will restrict ourselves with the primitive orders $\mathcal{O}$ in a quaternion algebra which has level $N' = (q; L(p_1), \nu(p_1); \cdots; L(p_d), \nu(p_d))$ with $\nu(p_i) > 1$ for $i = 1, \cdots d$. If $L(p)$ is the unramified extension field of Q_p, $\nu(p)$ is always even number.

Definition 4. Let $\mathcal{O}$ be an order of level N' in A. A left $\mathcal{O}$ ideal I is a lattice on A such that $I_p = \mathcal{O}_p a_p$ (for some $a_p \in A_p^\times$) for all $p < \infty$. Two left $\mathcal{O}$ ideals I and J are said to belong to the same class if $I = Ja$ for some $a \in A^\times$. One has the analogous definition for right $\mathcal{O}$ ideals.

Definition 5. The norm of an ideal, denoted by $N(I)$, is the positive rational number which generates the fractional ideal of $\mathbb{Q}$ generated by $\{N(a) | a \in I\}$. The conjugate of an ideal I, denoted by $\bar{I}$, is given by $\bar{I} = \{\bar{a} | a \in I\}$. The inverse of an ideal, denoted by I^{-1}, is given by

$I^{-1} = \{a \in A | IaI \subset I\}$.

Definition 6. The class number of left ideals for any order $\mathcal{O}$ of level N' is the number of distinct classes of such ideals. We denote this by $H(N')$.

Remark. Let A be a quaternion algebra and let M be any order of A. The idele group of J_A of A is

$$J_A = \{\tilde{a} = (a_p) \in \prod_p A_p^\times | a_p \in U(M_p) \text{ for almost all } p\ \},$$

where $U(M_p)$ is the set of all units in M_p.

Here the product is over all primes, finite and infinite. Note that since for two orders M and N of A, $M_p = N_p$ for almost all p, J_A is independent of the particular orders used in this definition. J_A is a locally compact group with the topology induced by the product topology on the open set $\prod_{p \in S} A_p^\times \prod_{p \notin S} U(M_p)$, where S ranges over all finite subset of primes containing ∞. If $\tilde{a} \in J_A$, we define the volume of $\tilde{a}$ as $\mathrm{vol}(\tilde{a}) = \prod_p |N(a_p)|_p$ where $|\cdot|_p$ is normalized such that $|p|_p = \frac{1}{p}$ for $p < \infty$ and $|\cdot|_\infty$ is the ordinary absolute value in $\mathbb{R}$. Let $J_A^1 = \{\tilde{a} \in J_A | \mathrm{vol}(\tilde{a}) = 1\}$ and embed $A^\times \subset J_A^1$ along the diagonal. Finally, if M is an any order of A, let $\mathfrak{U}(M) = \{\tilde{a} \in J_A^1 | a_p \in U(M_p) \text{ for all } p < \infty\}$.

Proposition 2.2. Let $\mathcal{O}$ be any order of level N' in A. Then

(1) $A^\times$ is a discrete subgroup of J_A^1.
(2) $J_A^1/A^\times$ is compact.
(3) $\mathfrak{U}(\mathcal{O})$ is an open compact subgroup of J_A^1.

Proof. See Weil [11].

Proposition 2.3. The double cosets $\mathfrak{U}(\mathcal{O})\backslash J_A^1/A^\times$ are in 1-1 correspondence with the ideal classes of left $\mathcal{O}$ ideals.

Proof. If $J_A^1 = \cup_{i=1}^H \mathfrak{U}(\mathcal{O})\tilde{a}_i A^\times$, then $\mathcal{O}\tilde{a}_i, \quad i = 1, \cdots, H$ represent the distinct left $\mathcal{O}$ ideal classes.

Proposition 2.4. J_A^1 acts transitively (by conjugation) on orders of level N' in A.

Proof. The action is for $\tilde{a} \in J_A^1$ and $\mathcal{O}$ an order of level N': $\mathcal{O} \leftrightarrow \{\mathcal{O}_p\} \mapsto \{a_p^{-1}\mathcal{O}_p a_p\} \leftrightarrow \mathcal{O}'$ and we write $\mathcal{O}' = \tilde{a}^{-1}\mathcal{O}\tilde{a}$. The action is obviously transitive.

Definition 7. Let I be a left $\mathcal{O}$-ideal for some order of level N'. The left order of $I = \{a \in A | aI \subseteq I\}$. If $I = \mathcal{O}\tilde{a}$, then the left order of I is $\mathcal{O}$ and the right order is $\tilde{a}^{-1}\mathcal{O}\tilde{a}$. Thus if I is an ideal of an order of level N', its left and right orders also have level N.

From the above definition, we are able to define two sided ideals. A left $\mathcal{O}$-ideal is said to be two sided if its right order is also $\mathcal{O}$, i.e. if it is also a right $\mathcal{O}$-ideal. More explicitly, we define two sided ideals as follows.

Definition 8. Let $I = \mathcal{O}\tilde{a}$ for some order $\mathcal{O}$ of level N' and $\tilde{a} \in J_A^1$. Then I is called a two sided ideal if $\tilde{a}^{-1}\mathcal{O}\tilde{a} = \mathcal{O}$.

3. The Normalizer of orders

It is clear that if we fix $\mathcal{O}$, the set of two sided ideals form a group. If I and J are two sided $\mathcal{O}$ ideals and $I = Ja$ for $a \in A^\times$, then $a^{-1}\mathcal{O}a = \mathcal{O}$ as I and J have the same right order and thus $\mathcal{O}a$ is also a two sided $\mathcal{O}$ ideal. Hence we can consider the ideal class group of two sided $\mathcal{O}$ ideals.

The order of this group is called the class number of two sided $\mathcal{O}$ ideals. This group is important to study the action of the canonical involution acting on the certain spaces of modular forms.

Definition 9. We define the normalizer of an order $\mathfrak{O}$ of a quaternion algebra A as

$$\mathfrak{N}(\mathfrak{O}) = \{\tilde{a} \in J_A^1 | \tilde{a}^{-1}\mathfrak{O}_p\tilde{a} = \mathfrak{O}\},$$

locally $\mathcal{N}(\mathfrak{O}_p) = \{a_p \in A^\times | a_p^{-1}\mathfrak{O}_p a_p = \mathfrak{O}_p \text{ for all } p < \infty\}$.

In order to compute the normalizer of orders, we first compute the normalizer of orders locally. If $p \neq 2$, the normalizer of orders were computed by several authors in [3], [5], [7]. Hence we will compute only for dyadic local field case, i.e. $p = 2$ case.

Recall the definition of orders, $R_\nu = \mathcal{O}_L + \xi P_L^{\nu-t-1}$. For the computational convenience, we introduce a new notation : $M(R_\nu) = \{x \in R_0(L)^\times | x^{-1}R_\nu x = R_\nu\}$.

Theorem 3.1. Let L be a unramified quadratic extension field of k and $k = \mathbb{Q}_2$. Then for an order of $A_2 = A \otimes k$, $R_\nu(L)$, we have

$$M(R_\nu) = \begin{cases} R_0^\times \\ R_\nu^\times \cup \xi R_\nu(L)^\times \text{ for } \nu > 0. \end{cases}$$

Proof. $\nu = 0$ case is trivial. Hence assume that $\nu > 0$. Let $\alpha + \xi\beta \in R_\nu(L) = \mathcal{O}_L + \xi P_L^\nu$ and $g \in R_0^\times = (\mathcal{O}_L + \xi\mathcal{O}_L)^\times$.

$$\begin{aligned} g(\alpha + \xi\beta)\overline{g} &= (\gamma + \xi\delta)\cdot(\alpha + \xi\beta)\cdot(\overline{\gamma + \xi\delta}) \\ &= (\alpha\gamma + \beta\overline{\delta} + \xi(\alpha\delta + \beta\overline{\gamma}))\cdot(\overline{\gamma} - \xi\delta) \\ &= \alpha\gamma\overline{\gamma} + \beta\overline{\gamma\delta} - \overline{\alpha\delta}\delta - \overline{\beta}\gamma\delta + \xi(\alpha\overline{\gamma}\delta + \beta\overline{\gamma}^2 - \overline{\alpha\gamma}\delta - \beta\delta^2) \\ &\in \mathcal{O}_L + \xi P_L^\nu. \end{aligned}$$

$\alpha\overline{\gamma}\delta+\beta\overline{\gamma}^2-\overline{\alpha\gamma}\delta-\beta\delta^2 \in P_L^\nu$ implies that $\mathrm{ord}_k((\alpha-\overline{\alpha})\overline{\gamma}\delta) \geq \nu$. Hence either $\mathrm{ord}_L(\delta) \geq \nu$ and $\gamma \in \mathcal{O}_L^\times$, or $\mathrm{ord}_L(\gamma) \geq \nu$ and $\delta \in \mathcal{O}_L^\times$. This means $M(R_\nu(L)) = R_\nu(L)^\times \cup \xi R_\nu(L)^\times$.

Theorem 3.2. Let L be a ramified quadratic extension field of k and $k = \mathbb{Q}_2$. Then for an order of $A_2 = A \otimes k$, $R_\nu(L)$, we have

$$M(R_\nu) = \begin{cases} R_\nu^\times & \text{if } \nu = 0 \\ R^\times_{[\frac{1}{2}(\nu+1)]} & \text{if } 0 < \nu \leq 2t+2 \\ R^\times_{\nu-t-1} \cup \xi R^\times_{\nu-t-1} & \text{if } 2t+2 < \nu, \end{cases}$$

where $[x]$ is the largest integer not greater than x.

Proof. If $\nu = 0$, R_0 is a maximal order. $M(R_0) = R_0^\times$ clear from the definition.

Now assume that L is ramified. we divide this into two cases. That is, $t = 2e$ case and $t < 2e$ case.

We first consider $t = 2e$ case.

There are two different maximal orders which contain $\mathcal{O}_L$, i.e. $R_0(L) = \mathcal{O}_L + (1+\xi)P_L^{-t-1}$ and $\overline{R_0(L)} = \mathcal{O}_L + (1-\xi)P_L^{-t-1}$. Since $t = 2e$, $\mathcal{O}_L + (1-\xi)P_L^{-t} = \mathcal{O}_L + (-1-\xi+2)P_L^{-t} = \mathcal{O}_L + (1+\xi)P_L^{-t}$. $R_0(L) \cap \overline{R_0(L)} = R_1(L)$. Thus, by Hijikata's theorem in [3], $R_1 \simeq \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ 2\mathfrak{o} & \mathfrak{o} \end{pmatrix}$, where $\mathfrak{o}$ is the ring of integers in k. $M(R_1) = R_1^\times$ was computed in [3].

If $1 < \nu \leq 2t+2$, then $R_\nu = \mathcal{O}_L + (1+\xi)P_L^{\nu-t-1}$. Let $g \in M(R_1)$. Then gR_1g^{-1} contains R_ν and gR_1g^{-1} is the second largest order containing R_ν, which implies $gR_\nu g^{-1} = R_\nu$. Without loss of generality, we assume that $M(R_\nu) \subset M(R_1) = R_1^\times$. Let $g = c + d + \xi d \in R_1^\times$ and

$a+b+\xi b \in R_\nu = \mathcal{O}_L + (1+\xi)P_L^{\nu-t-1}$.

$$\begin{aligned}
g(\alpha+\xi\beta)\overline{g} &= (c+d+\xi d)\cdot(a+b+\xi b)\cdot(\overline{c+d+\xi d}) \\
&= (c+d+\xi d)\cdot(a+b+\xi b)\cdot(\overline{c+d}-\xi d) \\
&= ((c+d)(a+b)+b\overline{d}+\xi((a+b)d+b(\overline{c+d}))\cdot(\overline{c+d}-\xi d) \\
&= N(c+d)(a+b)+b\overline{d}(\overline{c+d})-(\overline{a+b})\overline{d}d-\overline{b}(c+d)d \\
&\quad +\xi((a+b)(\overline{c+d})d+b(\overline{c+d})^2-\overline{(c+d)(a+b)}d-\overline{b}d^2) \\
&\in \mathcal{O}_L+(1+\xi)P_L^{\nu-t-1}.
\end{aligned}$$

Thus we need two conditions, $(a+b)(\overline{c+d})d+b(\overline{c+d})^2-\overline{(c+d)(a+b)}d-\overline{b}d^2 \in P_L^{\nu-t-1}$ and $N(c+d)(a+b)+b\overline{d}(\overline{c+d})-(\overline{a+b})\overline{d}d-\overline{b}(c+d)d-\{(a+b)(\overline{c+d})d+b(\overline{c+d})^2-\overline{(c+d)(a+b)}d-\overline{b}d^2\} \in \mathcal{O}_L$. For the first one, we have the followings.

$$\begin{aligned}
&(a+b)(\overline{c+d})d+b(\overline{c+d})^2-\overline{(c+d)(a+b)}d-\overline{b}d^2 \\
=\ &((a+b)-(\overline{a+b}))(\overline{c+d})d+b(\overline{c+d})^2-\overline{b}d^2 \\
=\ &((a-\overline{a})(\overline{c+d})d+(b-\overline{b})(\overline{c+d})d+b\overline{c}^2+2b\overline{c}\overline{d}+b\overline{d}^2-\overline{b}d^2 \\
=\ &((a-\overline{a})(\overline{c+d})d+(b-\overline{b})\overline{c}d+b\overline{c}^2+2b\overline{c}\overline{d}+b\overline{d}^2-\overline{b}d^2+(b-\overline{b})d\overline{d} \\
=\ &((a-\overline{a})(\overline{c+d})d+(b-\overline{b})\overline{c}d+b\overline{c}^2+2b\overline{c}\overline{d}+(b\overline{d}-\overline{b}d)(d+\overline{d}).
\end{aligned}$$

Since $d \in P_L^{-t}$, $\mathrm{Tr}(d) = d+\overline{d} \in \mathcal{O}_L$. Hence, $b \in P^{\nu-t-1}$ implies that $\mathrm{ord}_L((a-\overline{a})(\overline{c+d})d) = t+1+2\mathrm{ord}_L(d) \geq \nu-t-1$ is needed. That is, $\mathrm{ord}_L(d) \geq \frac{1}{2}\nu-t-1$ and the second condition is easily satisfied with $\mathrm{ord}_L(d) \geq \frac{1}{2}\nu-t-1$. Thus $M(R_\nu(L)) = R_{[\frac{1}{2}(\nu+1)]}(L)$ for $1 \leq \nu \leq 2t+2$, where $[x]$ is the largest integer not greater than x.

Next, if $2t+2 \leq \nu$, then $R_\nu = \mathcal{O}_L + \xi P_L^{\nu-t-1}$. Let $\alpha + \xi\beta \in R_\nu(L)$ and $g \in R_{t+1}^\times = (\mathcal{O}_L + \xi\mathcal{O}_L)^\times$.

$$\begin{aligned} g(\alpha+\xi\beta)\overline{g} &= (\gamma+\xi\delta)\cdot(\alpha+\xi\beta)\cdot\overline{(\gamma+\xi\delta)} \\ &= (\alpha\gamma+\beta\overline{\delta}+\xi(\alpha\delta+\beta\overline{\gamma}))\cdot(\overline{\gamma}-\xi\delta) \\ &= \alpha\gamma\overline{\gamma}+\beta\overline{\gamma\delta}-\overline{\alpha\delta}\delta-\overline{\beta}\gamma\delta+\xi(\alpha\overline{\gamma}\delta+\beta\overline{\gamma}^2-\overline{\alpha}\overline{\gamma}\delta-\beta\delta^2) \\ &\in \mathcal{O}_L+\xi P_L^{\nu-t-1}. \end{aligned}$$

$\mathrm{ord}_k((\alpha-\overline{\alpha})\overline{\gamma}\delta) \geq \nu-t-1 \Rightarrow \mathrm{ord}_L(\delta) \geq \nu-2t-2$ and $\nu \geq 0$. Finally, it is easy to see $\xi R_\nu \xi^{-1} = R_\nu$. Thus $M(R_\nu(L)) = R_{\nu-t-1}(L)^\times \cup \xi R_{\nu-t-1}(L)^\times$ for $2t+2 < \nu$.

$$M(R_\nu) = \begin{cases} R^\times_{[\frac{1}{2}(\nu+1)]} & \text{if } t+1 \leq 2t+2 \\ R_{\nu-t-1}(L)^\times \cup \xi R_{\nu-t-1}(L)^\times & \text{if } 2t+2 \leq \nu. \end{cases}$$

Finally, $t < 2e$, The computation of this case is exactly same manner as in the case $t = 2e$.

Theorem 3.3. Let R_ν be an order of A_2 over a dyadic local field k. Then

$$\mathcal{N}(R_\nu)/R_\nu^\times k^\times \approx \begin{cases} \{1\} \text{ if } \nu = 0 \\ R^\times_{[\frac{1}{2}(\nu+1)]}/R_\nu^\times \\ \qquad\qquad \text{if } 0 < \nu \leq 2t+2 \text{ and } L \text{ is ramified} \\ R^\times_{\nu-t-1}/R_\nu^\times \cup \xi R^\times_{\nu-t-1}/R_\nu^\times \\ \qquad\qquad \text{if } 2t+2 < \nu \text{ and } L \text{ is ramified} \\ \{1,\xi\} \\ \qquad\qquad \text{if } 0 < \nu \text{ and } L \text{ is unramified,} \end{cases}$$

where $\approx$ is a set theoritical bijective relation.

Proof. From the facts that $\mathcal{N}(R_\nu) = k^\times M(R_\nu)$, this is immediate from Theorem 3.1 and 3.2.

Corollary 3.4. Let the notations be as in Theorem 3.3. Then

$$|\mathcal{N}(R_\nu)/R_\nu^\times k^\times| = \begin{cases} 1 & \text{if } \nu = 0 \\ 2^{\nu - [\frac{1}{2}(\nu+1)]} & \text{if } 0 < \nu \leq 2t+2 \text{ and } L \text{ is ramified} \\ 2^{t+1} & \text{if } 2t+2 < \nu \text{ and } L \text{ is ramified} \\ 2 & \text{if } 0 < \nu \text{ and } L \text{ is unramified.} \end{cases}$$

Proof. By Theorem 3.3, this is immediately given.

From Definition 8, the classes of two sided ideal correspond to $\mathfrak{N}(\mathcal{O})/\mathfrak{U}(\mathcal{O})J_{\mathbb{Q}}^1$ where $\mathfrak{U}(\mathcal{O}) = \{\tilde{a} | a_p \in \mathcal{O}_p^\times \text{ for all } p < \infty\}$. We are now finally able to find the general formula for class number of the two sided $\mathcal{O}$ ideal classes. For $p = 2$ we have computed the normalizer in this paper and for the other primes, we refer to [5].

Theorem 3.5. Let $\mathcal{O}$ be an order of level $N' = (q; L(2), \nu(2); \cdots ; L(p_d), \nu(p_d))$ in A. Then

$$H(N') = 2^{d_1 \cdot (\nu - [\frac{1}{2}(\nu+1)])} \cdot 2^{d_2(t+1)} \cdot 2^{d_3},$$

where d_1 is the number of ramified prime p_i with $0 < \nu(p_i) \leq 2t+2$, d_2 is the number of ramified primes with $2t+2 < \nu(p_i)$ and d_3 is the number of unramified primes.

Proof. By Theorem 3.3 and Corollary 3.4, we have

$$\begin{aligned} H(N') &= |\mathfrak{N}(\mathcal{O})/\mathfrak{U}(\mathcal{O})J_{\mathbb{Q}}^1| \\ &= \prod_p |\mathcal{N}(\mathcal{O}_p)/\mathcal{O}_p^\times k^\times| \\ &= 2^{d_1 \cdot (\nu - [\frac{1}{2}(\nu+1)])} \cdot 2^{d_2(t+1)} \cdot 2^{d_3}. \end{aligned}$$

where d_1 is the number of ramified prime p_i with $0 < \nu(p_i) \leq 2t + 2$, d_2 is the number of ramified primes with $2t + 2 < \nu(p_i)$ and d_3 is the number of unramified primes.

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