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ON FUZZY INTEGRALS DEFINED BY MAX-MEASURES

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Abstract. In this paper, we consider fuzzy integrals defined by max-measures and discuss some properties of these fuzzy integrals of measurable functions.

1. Preliminaries and definitions

In this section, we shall assume that X is a nonempty classical set. Let $\Im$ be any σ-algebra of subsets of X. Then $(X, \Im)$ is called a measurable space. A function $f : X \to [0, \infty]$ is measurable if for any $\alpha \in \Re$, $\{x \in X | f(x) > \alpha\} \in \Im$. Let $\Re^+$ denote the interval $[0, \infty]$. A fuzzy measure on a measurable space $(X, \Im)$ is an extended real-valued set function $\mu : \Im \to \Re^+$ satisfying

(1)$\mu(\emptyset) = 0$,

(2)$\mu(A) \leq \mu(B)$, whenever $A, B \in \Im, A \subset B$.

Definition 1.1[2] A set function $m : \Im \to \Re^+$ is said to be a max-measure if

(1)$m(\emptyset) = 0$,

(2)$m(\cup_{n=1}^{\infty} A_n) = \vee_{n=1}^{\infty} m(A_n)$, whenever $A_n \in \Im$ for $n = 1, 2, \cdots$,

where $m(A) \vee m(B)$ is the maximum of $m(A)$ and $m(B)$ for any $A, B \in \Im$.

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From the Definition 1.1, we have the following proposition.

Proposition 1.2 If $E \subset F$ for any $E, F \in \Im$, then we have $m(E) \leq m(F)$.

We note that a max-measure is a fuzzy measure.

Proposition 1.3 If m is a max-measure and $\{A_n\}$ is an increasing sequence in $\Im$, then $\lim_{n\to\infty} m(A_n) = m(A)$.

Proof. Let $A = \cup_{k=1}^{\infty} A_k$. Sine $\cup_{k=1}^{n} A_k \subset \cup_{k=1}^{\infty} A_k$ for all positive integer n, by proposition 1.2, we have $m(A) = m(\cup_{k=1}^{\infty} A_k) \geq m(\cup_{k=1}^{n} A_k) = \vee_{k=1}^{n} m(A_k) = m(A_n)$ for all positive integer n. Thus

$$(1.1) \qquad m(A) \geq \ lim_{n\to\infty} m(A_n).$$

Let $a = \vee_{n=1}^{\infty} m(A_n)$. For each $\epsilon > 0$, there exists n_0 such that $m(A_{n_0}) \geq a - \epsilon$. By Proposition 1.2, $a - \epsilon \leq m(A_n)$ for all $n \geq n_0$. Thus $a - \epsilon \leq lim_{n\to\infty} m(A_n)$. Since ϵ is arbitrary, we obtain $a \leq \lim_{n=1}^{\infty} m(A_n)$. Thus we have

$$(1.2) \qquad m(A) = \vee_{k=1}^{\infty} m(A_k) = a \leq \ lim_{n\to\infty} m(A_n).$$

By (1.1) and (1.2), we have $m(A_n) \uparrow m(A)$.

Definition 1.4[3] A set function m is called null-additive if $m(E \cup F) = m(E)$ whenever $E, F \in \Im$ and $m(F) = 0$.

Definition 1.5[3] A set function m is called autocontinuous from above [resp. from below] if we have $m(A \cup B_n) \to m(A)[m(A - B_n) \to m(A)]$ whenever $A, B_n \in \Im$ for $n = 1, 2, \cdots$ and $m(B_n) \to 0$ as $n \to \infty$.

If a set function m satisfies both autocontinuous from above and autocontinuous from below, then it is said to be autocontinuous.

proposition 1.6 A max-measure m is null-additive if and only if $m(E-F) = m(E)$ for all $E, F \in \Im$ with $m(F) = 0$.

Proof. Suppose that m is null-additive. By Proposition 1.2, we have $0 \leq m(E \cap F) \leq m(F) = 0$, $m(E) = m((E - F) \cup (E \cap F)) = m(E-F) \vee m(E \cap F) = m(E-F) \vee 0 = m(E-F)$. Conversely, let E and F be any subsets of $\Im$ with $m(F) = 0$. Since m is a max-measure, we have $m(E \cup F) = m((E-F) \cup F) = m(E-F) \vee m(F) = m(E) \vee 0 = m(E)$. Thus m is null-additive.

2. Fuzzy integrals defined by max-measures

In this section, we discuss fuzzy integrals defined by max-measures and investigate some properties of these fuzzy integrals. Let $(X, \Im, m)$ be a max-measure space..

Definition 2.1 Let $f : X \to \Re^+$ be a measurable function and $E \in \Im$. The fuzzy integral $\int_E f dm$ of f over E with respect to a max-measure m is defined by

$$(2.1) \qquad \int_E f dm = \sup_{\alpha \in [0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap E)],$$

where $F_\alpha = \{x \in X | f(x) \geq \alpha\}$, simlpy $\{f \geq \alpha\}$, and $a \wedge b$ is the minimum of a and b.

Using the fuzzy integral (2.1), for every measurable function f, we consider a R^+-valued set function λ_f on $\Im$ defined by

$$\lambda_f(E) = \int_E f dm.$$

Now, we have some basic properties of set functions λ_f.

proposition 2.2 If $f, g : X \to \Re^+$ are measurable functions and $f \leq g$, then $\lambda_f(E) \leq \lambda_g(E)$ for all $E \in \Im$.

Proof. Let $E \in \Im$. Since $f \leq g$, we obtain $\{f \geq \alpha\} \subset \{g \geq \alpha\}$ for all $\alpha \in \Re$. Then we have

$$\begin{aligned}
\lambda_f(E) &= \sup_{\alpha \in [0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap E)] \\
&\leq \sup_{\alpha \in [0,\infty]} [\alpha \wedge m(\{g \geq \alpha\} \cap E)] \\
&= \lambda_g(E).
\end{aligned}$$

proposition 2.3 If $E \subset F$ for any $E, F \in \Im$, then $\lambda_f(E) \leq \lambda_f(F)$.

Proof. Since $E \subset F$, we have $(\{f \geq \alpha\} \cap E) \subset (\{f \geq \alpha\} \cap F)$ for all $\alpha \in \Re$.

Then we have

$$\begin{aligned}
\lambda_f(E) &= \int_E f dm \\
&= \sup_{\alpha \in [0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap E)] \\
&\leq \sup_{\alpha \in [0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap F)] \\
&= \int_F f dm \\
&= \lambda_f(F).
\end{aligned}$$

proposition 2.4 For any $E, F \in \Im$, we have $\lambda_f(E \cup F) = \lambda_f(E) \vee \lambda_f(F)$.

Proof. Since m is a max-measure and $m((F_\alpha \cap E) \cup (F_\alpha \cap F)) = m(F_\alpha \cap E) \vee m(F_\alpha \cap F)$ for all $\alpha \in \Re$, we have

$$\lambda_f(E \cup F) = \int_{E \cup F} f dm$$

$$
\begin{aligned}
&= \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(F_\alpha \cap (E \cup F))] \\
&= \sup_{\alpha\in[0,\infty]} [\alpha \wedge m((F_\alpha \cap E) \cup (F_\alpha \cap F))] \\
&= \sup_{\alpha\in[0,\infty]} [\alpha \wedge (m(F_\alpha \cap E) \vee m(F_\alpha \cap F))].
\end{aligned}
$$

Since $(\alpha \wedge m(F_\alpha \cap E)) \vee (\alpha \wedge m(F_\alpha \cap F)) \geq (\alpha \wedge m(F_\alpha \cap E))$ and $(\alpha \wedge m(F_\alpha \cap E)) \vee (\alpha \wedge m(F_\alpha \cap F)) \geq (\alpha \wedge m(F_\alpha \cap F))$, we have

$$
\sup_{\alpha\in[0,\infty]} [\alpha \wedge (m(F_\alpha \cap E) \vee m(F_\alpha \cap F))]
$$

$$
\geq \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(F_\alpha \cap E)] \vee \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(F_\alpha \cap F)]. \tag{2.2}
$$

If there is a real number α such that either $(\alpha \wedge m(F_\alpha \cap E))$ or $(\alpha \wedge m(F_\alpha \cap F))$ is infinite, then

$$
\sup_{\alpha\in[0,\infty]} [(\alpha \wedge m(F_\alpha \cap E)) \vee (\alpha \wedge m(F_\alpha \cap F))]
$$

$$
= \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(F_\alpha \cap E)] \vee \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(F_\alpha \cap F)] = \infty. \tag{2.3}
$$

If $(\alpha \wedge m(F_\alpha \cap E))$ and $(\alpha \wedge m(F_\alpha \cap F))$ are finite for all $\alpha \in \Re$, then

$$
\sup_{\alpha\in[0,\infty]} [(\alpha \wedge m(F_\alpha \cap E)) \vee (\alpha \wedge m(F_\alpha \cap F))]
$$

$$
\leq \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(F_\alpha \cap E)] \vee \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(F_\alpha \cap F)]. \tag{2.4}
$$

By (2,2),(2.3), and (2.4), we have $\lambda_f(E \cup F) = \lambda_f(E) \vee \lambda_f(F)$.

proposition 2.5 For any $E \in \Im$ with $m(E) = 0$, $\lambda_f(E) = 0$.

Proof. By Proposition 1.2, $\lambda_f(E) = \int_E f dm = \sup_{\alpha\in[0,\infty]}[\alpha \wedge m(F_\alpha \cap E)] \leq \sup_{\alpha\in[0,\infty]}[\alpha \wedge m(E)] = \sup_{\alpha\in[0,\infty]}[\alpha \wedge 0] = 0$. Thus $\lambda_f(E) = 0$.

The Proposition 2.5 means that the set function λ_f is absolutely continuous with respect to the max-measure or $\lambda_f \ll m$.

proposition 2.6 If m is null-additive and f is a measurable function, then λ_f is null-additive.

Proof. Let $E, F \in \Im$ with $\lambda_f(F) = 0$. Since $\lambda_f(F) = \int_F f dm = 0$ and $f > 0$ on $\Im$, we have $m(F) = 0$. By the null-additive of m, $m(\{f \geq \alpha\} \cap (E \cup F)) = m(\{f \geq \alpha\} \cap E) \cup (\{f \geq \alpha\} \cap F)) = m(\{f \geq \alpha\} \cap E) \vee m(\{f \geq \alpha\} \cap F) = m(\{f \geq \alpha\} \cap E)$.

Then we have

$$\begin{aligned}
\lambda_f(E \cup F) &= \int_{E \cup F} f dm \\
&= \sup_{\alpha \in [0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap (E \cup F))] \\
&= \sup_{\alpha \in [0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap E)] \\
&= \int_E f dm \\
&= \lambda_f(E).
\end{aligned}$$

proposition 2.7 If m is autocontinuous, then λ_f is autocontinuous.

Proof. Let $m(A \cup B_n) \to m(A)$ with $A, B_n \in \Im$, and $m(B_n) \to 0$ as $n \to \infty$.

Since m is autocontinuous from above and $m((\{f \geq \alpha\} \cap A) \cup (\{f \geq \alpha\} \cap B_n)) \to m(\{f \geq \alpha\} \cap A)$, we have

$$\begin{aligned}
\lambda_f(A \cup B_n) &= \int_{(A \cup B_n)} f dm \\
&= \sup_{\alpha \in [0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap (A \cup B_n))] \\
&= \sup_{\alpha \in [0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap A) \cup (\{f \geq \alpha\} \cap B_n))]
\end{aligned}$$

$$\to \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap A)] = \lambda_f(A).$$

Finally, let $m(A - B_n) \to m(A)$ with $A, B_n \in \Im$, and $m(B_n) \to 0$ as $n \to \infty$.

Since m is autocontinuous from below and $m(\{f \geq \alpha\} \cap (A - B_n)) = m(\{f \geq \alpha\} \cap (A \cap B_n{}^c)) = m(\{f \geq \alpha\} \cap A) - B_n) \to m(\{f \geq \alpha\} \cap A)$, then we have

$$\begin{aligned}\lambda_f(A - B_n) &= \int_{(A-B_n)} f dm \\ &= \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap (A - B_n))] \\ &\to \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap A)] = \lambda_f(A).\end{aligned}$$

Theorem 2.8 Let $A_n \uparrow A$, $A_n \in \Im$ for $n = 1, 2, \cdots$, and let m be autocontinuous from below. Then we have

$$\lambda_f(\cup_{n=1}^{\infty} A_n) = \vee_{n=1}^{\infty} \lambda_f(A_n).$$

Proof. Let $\{A_n\}$be an increasing sequence of measurable sets of $\Im$. Then we have $\cup_{n=1}^{l} A_n \subset \cup_{n=1}^{\infty} A_n$ for all $l = 1, 2, \cdots$. By Proposition 2.3 and 2.4, we have $\lambda_f(\cup_{n=1}^{\infty} A_n) \geq \lambda_f(\cup_{n=1}^{l} A_n) = \vee_{n=1}^{l} \lambda_f A_n)$ for any positive l. Since l is arbitrary positive integer, we have that

$$\lambda_f(\cup_{n=1}^{\infty} A_n) \geq \lambda_f(\cup_{n=1}^{l}(A_n)) \tag{2.5}$$

as $l \to \infty$.

Let $B_n = A_n - A_{n-1}, n = 1, 2, \cdots$, with $A_0 = \emptyset$ and let $H_n = \cup_{k=n+1}^{\infty} B_k, n = 1, 2, \cdots$. Then we have $A = \cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$, and $A_n = \cup_{k=1}^{n} B_k$. Since $m(H_n) \to 0$ as $n \to \infty$, $F_\alpha \cap (A - H_n) = (F_\alpha \cap A) - H_n$ for all $\alpha \in \Re$ and for all positive integer n, by Proposition 1.3, we obtain

$$
\begin{aligned}
\lambda_f(A_n) &= \lambda_f(A - H_n) \\
&= \int_{(A-H_n)} f dm \\
&= \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap (A - H_n))] \\
&= \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap A) - H_n)] \\
&\to \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap A)] = \lambda_f(A).
\end{aligned}
$$

For each $\epsilon > 0$, there exists l such that $\lambda_f(A) \leq \lambda_f(A - H_l) + \epsilon = \vee_{n=1}^{l} \lambda_f(A_n) + \epsilon \leq \vee_{n=1}^{\infty} \lambda_f(A_n) + \epsilon$.

Since ϵ is arbitray, we have

$$\lambda_f(A) = \lambda_f(\cup_{n=1}^{\infty} A_n) \leq \vee_{n=1}^{\infty} \lambda_f(A_n). \tag{2.6}$$

From (2.5) and (2.6), we have $\lambda_f(A) = \lambda_f(\cup_{n=1}^{\infty} A_n) = \vee_{n=1}^{\infty} \lambda_f(A_n)$.

Remark The Theorem 2.8 means that λ_f is not a max-measure but a lower semicontinuous fuzzy measure on $(X, \Im)$ by Proposition 1.3.

Definition 2.10 Let f and g be measurable functions on $(X, \Im)$.

(1) The maximum of f and g, in symbol, $f \vee g$ is defined by $(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in X$.

(2) The minimum of f and g, in symbol, $f \wedge g$ is defined by $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in X$.

Theorem 2.11 Let f and g be measurable functions on $(X, \Im)$.

(1) $\lambda_{f\vee g}(E) \leq \lambda_f(E) \vee \lambda_g(E)$ for all $E \in \Im$.

(2) $\lambda_{f\wedge g}(E) \geq \lambda_f(E) \wedge \lambda_g(E)$ for all $E \in \Im$.

Proof. (1) Let $E \in \Im$. Since $m(\{(f \vee g) \geq \alpha\}) = m(\{f \geq \alpha\}) \vee m(\{g \geq \alpha\})$, we have

$$\begin{aligned}
\lambda_{f\vee g}(E) &= \int_E (f \vee g)dm \\
&= \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{(f \vee g) \geq \alpha\} \cap E)] \\
&= \sup_{\alpha\in[0,\infty]} [\alpha \wedge [m(\{f \geq \alpha\} \cap E) \vee m(\{g \geq \alpha\} \cap E)]] \\
&= \sup_{\alpha\in[0,\infty]} [[\alpha \wedge m(\{f \geq \alpha\} \cap E)] \vee [\alpha \wedge m(\{g \geq \alpha\} \cap E)]] \\
&\leq \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap E)] \vee \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{g \geq \alpha\} \cap E)] \\
&= \int_E fdm \vee \int_E gdm \\
&= \lambda_f(E) \vee \lambda_g(E).
\end{aligned}$$

(2) Let $E \in \Im$. Since $m(\{(f \wedge g) \geq \alpha\}) = m(\{f \geq \alpha\}) \wedge m(\{g \geq \alpha\})$, we have

$$\begin{aligned}
\lambda_{f\wedge g}(E) &= \int_E (f \wedge g)dm \\
&= \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{(f \wedge g) \geq \alpha\} \cap E)] \\
&= \sup_{\alpha\in[0,\infty]} [\alpha \wedge [m(\{f \geq \alpha\} \cap E) \wedge m(\{g \geq \alpha\} \cap E)]] \\
&= \sup_{\alpha\in[0,\infty]} [[\alpha \wedge m(\{f \geq \alpha\} \cap E)] \wedge [\alpha \wedge m(\{g \geq \alpha\} \cap E)]] \\
&\geq \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{f \geq \alpha\} \cap E)] \wedge \sup_{\alpha\in[0,\infty]} [\alpha \wedge m(\{g \geq \alpha\} \cap E)] \\
&= \int_E fdm \wedge \int_E gdm \\
&= \lambda_f(E) \wedge \lambda_g(E).
\end{aligned}$$

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