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DISTRIBUTED ROBUST CONTROL OF KELLER-SEGEL EQUATIONS

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Abstract. We are concerned with the robust control problem for the Keller-Segel equations with the distributed control and disturbance. We consider the present problem as a differential game finding the best control which takes into account the worst disturbance. We prove the existence of solutions and the optimality conditions to a corresponding problem.

1. Introduction

In this paper we study the distributed robust control problem for the Keller-Segel equations with uncertain disturbance:

Problem (P) To find the saddle point $(\bar{u}, \bar{\lambda}) \in E \times G$ such that

$$J(\bar{u}, \lambda) \leq J(\bar{u}, \bar{\lambda}) \leq J(u, \bar{\lambda}).$$

The functional $J(u, \lambda)$ is of the form

$$J(u,\lambda) = \int_0^T \|y(u,\lambda) - y_d\|^2_{H^1(\Omega)} dt + \int_0^T \left[\gamma \|u\|^2_{H^\varepsilon(\Omega)} - l\|\lambda\|^2_{H^\varepsilon(\Omega)}\right] dt$$

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and $y = y(u, \lambda)$ is governed by the Keller-Segel equations(Keller and Segel [6])

$$(1.1)\qquad \begin{aligned} &\frac{\partial y}{\partial t} = a\Delta y - b\nabla\{y\nabla\rho\} \qquad \text{in } \Omega\times(0,T],\\ &\frac{\partial \rho}{\partial t} = d\Delta\rho + fy - g\rho + u + \lambda \qquad \text{in } \Omega\times(0,T],\\ &\frac{\partial y}{\partial n} = \frac{\partial \rho}{\partial n} = 0 \qquad \text{on } \partial\Omega\times(0,T],\\ &y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x) \qquad \text{in } \Omega. \end{aligned}$$

Here, Ω is a bounded region in $\mathbf{R}^2$ of C^3 class. a, b, d, f, $g > 0$ are given positive numbers. $u \geq 0$ and $\lambda \geq 0$ are a control and a disturbance varying in some bounded subsets E and G of $L^2(0,T;H^\varepsilon(\Omega))$, respectively. ε is some fixed exponent such that $0 < \varepsilon < 1/2$. $n = n(x)$ is the outer normal vector at a boundary point $x \in \partial\Omega$ and $\frac{\partial}{\partial n}$ denotes the differentiation along the vector n. $y_0(x)$ and $\rho_0(x)$ are nonnegative initial functions in $L^2(\Omega)$ and in $H^{1+\varepsilon}(\Omega)$, respectively. y, ρ are unknown functions of the Cauchy problem (1.1).

The Keller-Segel equations (1.1) was introduced by Keller and Segel [6] to describe the aggregating process of the cellular slime molds by chemical attraction. Unknown functions $y = y(x,t)$ and $\rho = \rho(x,t)$ denote the concentration of amoebae in Ω at time t and the concentration of the chemical substance in Ω at time t, respectively. The chemotactic term $-b\nabla\cdot\{y\nabla\rho\}$ indicates that the cells are sensitive to chemicals and are attracted by them, and the production term fy indicates that the chemical substance is itself emitted by cells.

Robustness, insensitivity of system properties in the environment and components, is essential for the operation of both man-made and biological system in the real world. Robust control theory, which generalizes optimal control theory, can be represented as a differential game between designer seeking the best control, simultaneously, nature seeking the maximally malevolent disturbance ([3]).

Optimal control and robust control problem associated to nonlinear equations have already studied by many authors ([1], [2], [3], [4], [7], [8], [9]). Recently, Ryu and Yagi [9] studied the distributed optimal control problem for the Keller-Segel equations of non-monotone type. The problem that we consider in this paper is different from [9]. We then obtain the existence and the optimality conditions by using the method presented in [3].

This paper is organized as follows: In Section 2, we recall some known results. Section 3 introduce the robust control problem and prove the existence of solution, and obtain the optimality conditions for the problem **(P)**.

Notations. $\mathbf{R}$ denotes the sets of real numbers. For a region $\Omega \subset \mathbf{R}^2$, the usual L^p space of real valued functions in Ω is denoted by $L^p(\Omega)$, $1 \le p \le \infty$. The real Sobolev space in Ω with an exponent $s \ge 0$ is denoted by $H^s(\Omega)$. Let I be an interval in $\mathbf{R}$. $L^p(I;\mathcal{H})$, $1 \le p \le \infty$, denotes the L^p space of measurable functions in I with values in a Hilbert space $\mathcal{H}$. $\mathcal{C}(I;\mathcal{H})$ denotes the space of continuous functions in I with values in $\mathcal{H}$. For simplicity, we shall use a universal constant C to denote various constants which are determined in each occurrence in a specific way by δ, M, N, and so forth. In a case when C depends also on some parameter, say θ, it will be denoted by C_θ.

2. The formulation of problem

Let us briefly recall the way how to formulate (1.1) as a semilinear abstract differential equation in a Hilbert space. Let $A_1 = -a\Delta + a$ and $A_2 = -d\Delta + g$ be the Laplace operators equipped with the Neumann boundary conditions. The part of A_i in $L^2(\Omega)$ is a positive definite self-adjoint operator in $L^2(\Omega)$ with the domain $\mathcal{D}(A_i) = H_n^2(\Omega) = \{y \in H^2(\Omega); \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega\}$. $\mathcal{D}(A_i^\theta) = H^{2\theta}(\Omega)$ for $0 \le \theta < \frac{3}{4}$, and $\mathcal{D}(A_i^\theta) = H_n^{2\theta}(\Omega)$ for $\frac{3}{4} < \theta \le \frac{3}{2}$ (see Triebel [10]).

We introduce two product Hilbert spaces $\mathcal{V} \subset \mathcal{H}$ as

$$\mathcal{V} = H^1(\Omega) \times \mathcal{D}(A_2^{1+\varepsilon/2}) \text{ and } \mathcal{H} = L^2(\Omega) \times \mathcal{D}(A_2^{(1+\varepsilon)/2}),$$

respectively, where ε is some fixed exponent $\varepsilon \in (0, \frac{1}{2})$. By the identification of $\mathcal{H}$ and its dual $\mathcal{H}'$, we have: $\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'$. It is then seen that $\mathcal{V}' = (H^1(\Omega))' \times \mathcal{D}(A_2^{\varepsilon/2})$ with the duality product

$$\langle \Phi, Y\rangle_{\mathcal{V}'\times\mathcal{V}} = \langle \zeta, y\rangle_{(H^1)'\times H^1} + \left(A_2^{\varepsilon/2}\varphi, A_2^{1+\varepsilon/2}\rho\right)_{L^2},$$
$$\Phi = \begin{pmatrix}\zeta\\ \varphi\end{pmatrix} \in \mathcal{V}',\ Y = \begin{pmatrix}y\\ \rho\end{pmatrix} \in \mathcal{V}.$$

In this paper, the norms of $\mathcal{V}$, $\mathcal{H}$, and $\mathcal{V}'$ are denoted by $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$, respectively. The duality product between $\mathcal{V}$ and $\mathcal{V}'$ is denoted by $\langle\cdot,\cdot\rangle$.

We set also a symmetric sesquilinear form on $\mathcal{V}\times\mathcal{V}$:

$$a(Y, \widetilde{Y}) = \left(A_1^{1/2}y, A_1^{1/2}\tilde{y}\right)_{L^2} + \left(A_2^{1+\varepsilon/2}\rho, A_2^{1+\varepsilon/2}\tilde{\rho}\right)_{L^2},$$
$$Y = \begin{pmatrix}y\\ \rho\end{pmatrix}, \widetilde{Y} = \begin{pmatrix}\tilde{y}\\ \tilde{\rho}\end{pmatrix} \in \mathcal{V}.$$

Obviously, the form satisfies

$$|a(Y, \widetilde{Y})| \le M\|Y\|\|\widetilde{Y}\|, \quad Y, \widetilde{Y} \in \mathcal{V}, \tag{2.1}$$

$$a(Y, Y) \ge \delta\|Y\|^2, \quad Y \in \mathcal{V} \tag{2.2}$$

with some δ and $M > 0$. This form then defines a linear isomorphism $A = \begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}$ from $\mathcal{V}$ to $\mathcal{V}'$, and the part of A in $\mathcal{H}$ is a positive definite self-adjoint operator in $\mathcal{H}$ with the domain $\mathcal{D}(A) = \mathcal{D}(A_1)\times\mathcal{D}(A_2^{(3+\varepsilon)/2})$.

(1.1) is, then, formulated as an abstract equation

$$(2.3)\qquad \begin{aligned} &\frac{dY}{dt} + AY = F(Y) + G(t), \quad 0 < t \leq T, \\ &Y(0) = Y_0 \end{aligned}$$

in the space $\mathcal{V}'$. Here, $F(\cdot) : \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F(Y) = \begin{pmatrix} -b\nabla\{y\nabla\rho\} + ay \\ fy \end{pmatrix}, \qquad Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in \mathcal{V}.$$

$G(t) = \binom{0}{u(t)+\lambda(t)}$ and $Y_0 = \binom{y_0}{\rho_0}$.

As verified in ([9, Sec. 2]), $F(\cdot)$ satisfies the following conditions:

(f.i) For each $\eta > 0$, there exists an increasing continuous function $\phi_\eta : [0, \infty) \to [0, \infty)$ such that

$$\|F(Y)\|_* \leq \eta\|Y\| + \phi_\eta(|Y|), \quad Y \in \mathcal{V};$$

(f.ii) For each $\eta > 0$, there exists an increasing continuous function $\psi_\eta : [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned} &\|F(\widetilde{Y}) - F(Y)\|_* \\ &\leq \eta\|Y - \widetilde{Y}\| + (\|\widetilde{Y}\| + \|Y\| + 1)\psi_\eta(|\widetilde{Y}| + |Y|)|\widetilde{Y} - Y|, \widetilde{Y}, Y \in \mathcal{V}. \end{aligned}$$

Furthermore, $F(Y)$ is first order Fréchet differentiable with the derivative

$$F'(Y)Z = \begin{pmatrix} -b\nabla\{y\nabla w\} - b\nabla\{z\nabla\rho\} + az \\ fz \end{pmatrix}.$$

$F'(\cdot)$ satisfies the following estimates ([9, Sec. 2]):

(f.iii) For each $\eta > 0$, there exists an increasing continuous functions $\mu_\eta, \nu : [0, \infty) \to [0, \infty)$ such that

$$|\langle F'(Y)Z, P\rangle| \leq \begin{cases} \eta\|Z\|\|P\| + (\|Y\| + 1)\mu_\eta(|Y|)|Z|\|P\|, & Y, Z, P \in \mathcal{V}, \\ \eta\|Z\|\|P\| + (\|Y\| + 1)\mu_\eta(|Y|)\|Z\||P|, & Y, Z, P \in \mathcal{V}, \\ \nu(|Y|)\|Z\|\|P\|, & Y, Z, P \in \mathcal{V}. \end{cases}$$

(f.iv) There exists $C > 0$ such that

$$\|F'(\widetilde{Y})Z - F'(Y)Z\|_* \leq C|\widetilde{Y} - Y|\|Z\|, \quad \widetilde{Y}, Y, Z \in \mathcal{V}.$$

We then obtain the following result (For the proof, see Ryu and Yagi [9]).

Theorem 2.1. *Let* (2.1), (2.2), (f.i), *and* (f.ii) *be satisfied. Then, for any* $G \in L^2(0,T;\mathcal{V}')$ *and* $Y_0 \in \mathcal{H}$, *there exists a unique weak solution*

$$Y \in H^1(0,T(Y_0,G);\mathcal{V}') \cap \mathcal{C}([0,T(Y_0,G)];\mathcal{H}) \cap L^2(0,T(Y_0,G);\mathcal{V})$$

to (2.3), *the number* $T(Y_0,G) > 0$ *is determined by the norms* $\|G\|_{L^2(0,T;\mathcal{V}')}$ *and* $|Y_0|$.

3. Distributed robust control

In this section, we obtain the existence of solution to the problem (**P**). Let $\mathcal{E} \subset L^2(0,S;\mathcal{V}')$ and $\mathcal{G} \subset L^2(0,S;\mathcal{V}')$ be closed, bounded, and convex subsets. Let $G(t)$ be decomposed into the control part $U = \binom{0}{u}$ and the disturbance part $\Lambda = \binom{0}{\lambda}$. Then, the problem (**P**) is obviously formulated as follows:

Problem ($\overline{\mathbf{P}}$) To find the saddle point $(\overline{U},\overline{\Lambda}) \in \mathcal{E} \times \mathcal{G}$ such that

$$J(\overline{U},\Lambda) \leq J(\overline{U},\overline{\Lambda}) \leq J(U,\overline{\Lambda}).$$

The cost functional $J(U,\Lambda)$ is of the form

$$J(U,\Lambda) = \int_0^S \|DY(U,\Lambda) - Y_d\|^2 dt + \int_0^S [\gamma\|U\|_*^2 - l\|\Lambda\|_*^2] dt.$$

Here, $Y = Y(U,\Lambda)$ is the weak solution of (2.3) and is assumed to exist on a fixed interval $[0,S]$. $D\binom{y}{\rho} = \binom{y}{0}$ is a bounded operator from $\mathcal{V}$ into $\mathcal{V}$ and $Y_d = \binom{y_d}{0}$ is a fixed element of $L^2(0,S;\mathcal{V})$. γ and l are positive constants.

Definition 3.1. *The control* $\overline{U}$ *and the disturbance* $\overline{\Lambda}$, *and the solution* $\overline{Y} = Y(\overline{U},\overline{\Lambda})$ *to* (2.3) *associated with* $(\overline{U},\overline{\Lambda})$ *are said to solve the robust* *control problem* $(\overline{\mathbf{P}})$ *when a saddle point* $(\overline{U}, \overline{\Lambda})$ *of the cost functional* J *is reached such that*

$$J(\overline{U}, \Lambda) \leq J(\overline{U}, \overline{\Lambda}) \leq J(U, \overline{\Lambda}) \quad \forall (U, \Lambda) \in \mathcal{E} \times \mathcal{G}.$$

To derive the existence of the saddle point for $(\overline{\mathbf{P}})$, second order Fréchet differentiable of the mapping $F(\cdot) : \mathcal{V} \to \mathcal{V}'$ is necessary. It is indeed observed by a direct calculation that

$$F''(Y)(Z, Z) = \begin{pmatrix} -2b\nabla\{z\nabla w\} \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y \\ \rho \end{pmatrix}, \ Z = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathcal{V}.$$

Then, we have the following estimate(cf. [9, Sec. 2]):

(f.v) There exists $N > 0$ such that

$$\|F''(Y)(Z, Z)\|_* \leq N|Z|\|Z\|, \qquad Y, Z \in \mathcal{V}.$$

Lemma 3.2. *For any fixed* $\Lambda \in \mathcal{G}$, *the mapping* $U \to Y(U, \Lambda)$ *from* $\mathcal{E}$ *into* $H^1(0, S; \mathcal{V}') \cap L^2(0, S; \mathcal{V})$ *is differentiable in the sense*

$$\frac{Y(U + h\widetilde{U}, \Lambda) - Y(U, \Lambda)}{h} \to Z \ \textit{in} \ H^1(0, S; \mathcal{V}') \cap L^2(0, S; \mathcal{V})$$

as $h \to 0$, *where* $U, \widetilde{U} \in \mathcal{E}$ *and* $U + h\widetilde{U} \in \mathcal{E}$. *Moreover,* $Z = Z(U, \Lambda; \widetilde{U}, 0)$ *satisfies the linear equation*

$$\text{(3.1)} \qquad \begin{aligned} &\frac{dZ}{dt} + AZ - F'(Y(U, \Lambda))Z = \widetilde{U}, \qquad 0 < t \leq S, \\ &Z(0) = 0. \end{aligned}$$

Proof. As the proof is similar to [9, Proposition 5.1], we will only sketch.

For any fixed $\Lambda \in \mathcal{G}$, let $U, \widetilde{U} \in \mathcal{E}$ and $0 \leq h \leq 1$. Let Y_h and Y be the solutions of (2.3) corresponding to $U + h\widetilde{U}$ and U, respectively.

Obviously, $W = Y_h - Y$ satisfies

$$\begin{aligned}(3.2)\qquad & \frac{dW}{dt} + AW - (F(Y_h(t)) - F(Y(t))) = h\widetilde{U}(t), \quad 0 < t \leq S,\\ & W(0) = 0.\end{aligned}$$

Taking the scalar product of the equation (3.2) with W and using (2.2) and (f.ii), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|W(t)|^2 + \delta\|W(t)\|^2\\ &\leq \frac{\delta}{2}\|W(t)\|^2 + \big(\|Y_h(t)\|^2 + \|Y(t)\|^2 + 1\big)\psi_{\frac{\delta}{4}}\big(|Y_h(t)| + |Y(t)|\big)^2|W(t)|^2\\ &+4h^2\delta^{-1}\|\widetilde{U}(t)\|_*^2.\end{aligned}$$

Using Gronwall's lemma, we obtain that

$$|W(t)|^2 \leq Ch^2\|\widetilde{U}\|^2_{L^2(0,S;\mathcal{V}')}e^{\int_0^S(\|Y_h(s)\|^2+\|Y(s)\|^2+1)\psi_{\frac{\delta}{4}}(|Y_h(s)|+|Y(s)|)^2ds}$$

for all $t \in [0, S]$. Hence, $Y_h \to Y$ strongly in $\mathcal{C}([0,S];\mathcal{H})$ as $h \to 0$.

On the other hand, we consider the linear problem (3.1). From (2.1), (2, 2), (f.i), (f.ii), and (f.iii), we can easily verify that (3.1) possesses a unique weak solution $Z \in H^1(0,S;\mathcal{V}')\cap\mathcal{C}([0,S];\mathcal{H})\cap L^2(0,S;\mathcal{V})$ on $[0,S]$ (cf. [5, Chap. XVIII, Theorem 2]). Define $F_h' = \int_0^1 F'\big(Y+\theta(Y_h-Y)\big)d\theta$. Then $\widetilde{W} = \frac{Y_h-Y}{h} - Z$ satisfies

$$\begin{aligned}(3.3)\qquad & \frac{d\widetilde{W}(t)}{dt} + A\widetilde{W}(t) - F_h'\widetilde{W}(t) = (F_h' - F_0')Z(t), \quad 0 < t \leq S,\\ & \widetilde{W}(0) = 0.\end{aligned}$$

Taking the scalar product of the equation of (3.3) with $\widetilde{W}$ and using (f.iii) and (f.iv), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\widetilde{W}(t)|^2 + \frac{\delta}{2}\|\widetilde{W}(t)\|^2\\ &\leq (\|Y(t)\|^2 + \|Y_h(t) - Y(t)\|^2 + 1)\tilde{\mu}(|Y_h|^2 + |Y|^2)|\widetilde{W}(t)|^2\\ &+C|Y_h(t) - Y(t)|^2\|Z(t)\|^2.\end{aligned}$$

where $\tilde{\mu} : [0, \infty) \to [0, \infty)$ is some increasing continuous function. Therefore,

$$\frac{1}{2}|\widetilde{W}(t)|^2 + \frac{\delta}{2}\int_0^t \|\widetilde{W}(s)\|^2 ds$$
$$\leq \int_0^t (\|Y(s)\|^2 + \|Y_h(s)\|^2 + 1)\tilde{\mu}(|Y_h|^2 + |Y|^2)|\widetilde{W}(s)|^2 ds$$
$$+C|Y_h - Y|^2_{\mathcal{C}([0,S];\mathcal{H})} \int_0^S \|Z(s)\|^2 ds.$$

Using Gronwall's lemma, we obtain that

$$|\widetilde{W}(t)|^2 + \int_0^t \|\widetilde{W}(s)\|^2 ds$$
$$\leq C|Y_h - Y|^2_{\mathcal{C}([0,S];\mathcal{H})}\|Z\|^2_{L^2(0,S;\mathcal{V})} e^{\int_0^S (\|Y(s)\|^2 + \|Y_h(s)\|^2 + 1)\tilde{\mu}(|Y_h|^2 + |Y|^2) ds}$$

for all $t \in [0, S]$. Since $Y_h \to Y$ in $\mathcal{C}([0, S]; \mathcal{H})$, we conclude that $\frac{Y_h - Y}{h}$ is strongly convergent to Z in $H^1(0, S; \mathcal{V}') \cap L^2(0, S; \mathcal{V})$. □

Lemma 3.3. *For any fixed $U \in \mathcal{E}$, the mapping $\Lambda \to Y(U, \Lambda)$ from $\mathcal{G}$ into $H^1(0, S; \mathcal{V}') \cap L^2(0, S; \mathcal{V})$ is differentiable in the sense*

$$\frac{Y(U, \Lambda + h\widetilde{\Lambda}) - Y(U, \Lambda)}{h} \to \widetilde{Z} \text{ in } H^1(0, S; \mathcal{V}') \cap L^2(0, S; \mathcal{V})$$

as $h \to 0$, for $\Lambda, \widetilde{\Lambda} \in \mathcal{G}$ and $\Lambda + h\widetilde{\Lambda} \in \mathcal{G}$. Moreover, $\widetilde{Z} = \widetilde{Z}(U, \Lambda; 0, \widetilde{\Lambda})$ satisfies the linear equation

$$\frac{d\widetilde{Z}}{dt} + A\widetilde{Z} - F'(Y(U, \Lambda))\widetilde{Z} = \widetilde{\Lambda}, \qquad 0 < t \leq S, \tag{3.4}$$
$$\widetilde{Z}(0) = 0.$$

Proof. The proof is similar to that of Lemma 3.2. □

We have similar results for second order derivatives of $Y(U, \Lambda)$ with respect to the control U and the disturbance Λ, respectively. The proof is similar to that of Lemma 3.2.

Lemma 3.4. *For any fixed $\Lambda \in \mathcal{G}$, the mapping $U \to Y(U,\Lambda)$ from $\mathcal{E}$ into $H^1(0,S;\mathcal{V}') \cap L^2(0,S;\mathcal{V})$ is second order differentiable in the sense*

$$\frac{Z(U+h\widehat{U},\Lambda;\widetilde{U},0) - Z(U,\Lambda;\widetilde{U},0)}{h} \to \Phi \text{ in } H^1(0,S;\mathcal{V}') \cap L^2(0,S;\mathcal{V})$$

as $h \to 0$, for $U, \widehat{U} \in \mathcal{E}$ and $U+h\widehat{U} \in \mathcal{E}$. Moreover, $\Phi = \Phi(U,\Lambda;\widetilde{U},0;\widehat{U},0)$ satisfies the linear equation

$$\frac{d\Phi}{dt} + A\Phi - F''(Y(U,\Lambda))(Z,Z) - F'(Y(U,\Lambda))\Phi = 0, \quad 0 < t \le S,$$
$$\Phi(0) = 0.$$

Here, Z is the solution of (3.1).

Lemma 3.5. *For any fixed $U \in \mathcal{E}$, the mapping $\Lambda \to Y(U,\Lambda)$ from $\mathcal{G}$ into $H^1(0,S;\mathcal{V}') \cap L^2(0,S;\mathcal{V})$ is second order differentiable in the sense*

$$\frac{\widetilde{Z}(U,\Lambda+h\widehat{\Lambda};0,\widetilde{\Lambda}) - \widetilde{Z}(U,\Lambda;0,\widetilde{\Lambda})}{h} \to \widetilde{\Phi} \text{ in } H^1(0,S;\mathcal{V}') \cap L^2(0,S;\mathcal{V})$$

as $h \to 0$, for $\Lambda, \widehat{\Lambda} \in \mathcal{G}$ and $\Lambda+h\widehat{\Lambda} \in \mathcal{G}$. Moreover, $\widetilde{\Phi} = \widetilde{\Phi}(U,\Lambda;0,\widetilde{\Lambda};0,\widehat{\Lambda})$ satisfies the linear equation

$$\frac{d\widetilde{\Phi}}{dt} + A\widetilde{\Phi} - F''(Y(U,\Lambda))(\widetilde{Z},\widetilde{Z}) - F'(Y(U,\Lambda))\widetilde{\Phi} = 0, \quad 0 < t \le S,$$
$$\widetilde{\Phi}(0) = 0.$$

Here, $\widetilde{Z}$ is the solution of (3.4).

Proposition 3.6. *There exist $\bar{\gamma}$ and $\bar{l}$ such that, for $\gamma > \bar{\gamma}$ and $l > \bar{l}$, we have*

1. *$\forall \Lambda \in \mathcal{G}$, $U \to J(U,\Lambda)$ is strictly convex lower semicontinuous,*
2. *$\forall U \in \mathcal{E}$, $\Lambda \to J(U,\Lambda)$ is strictly concave upper semicontinuous.*

Proof. First, we prove that $U \to J(U,\Lambda)$ is lower semicontinuous for all $\Lambda \in \mathcal{G}$, and $\Lambda \to J(U,\Lambda)$ is upper semicontinuous for all $U \in \mathcal{E}$.

Let U_n be a minimizing sequence of J, i.e. $\liminf_{n\to\infty} J(U_n,\Lambda) = \min_{U\in\mathcal{E}} J(U,\Lambda)(\forall\Lambda\in\mathcal{G})$. Since $\mathcal{E}$ is bounded, we can extract from U_n a subsequence also denoted by U_n such that $U_n \to \widetilde{U}$ weakly in $L^2(0,S;\mathcal{V}')$. Using the similar estimate of the solution $Y(U_n,\Lambda)$, we see as in the proof of [9, Theorem 2.1] that

$$\|Y(U_n,\Lambda)\|_{L^2(0,S;\mathcal{V})} \le C, \quad \left\|\frac{dY(U_n,\Lambda)}{dt}\right\|_{L^2(0,S;\mathcal{V}')} \le C.$$

Then we have

$$\begin{aligned} Y(U_n,\Lambda) &\to \widetilde{Y} \quad \text{weakly in} \quad L^2(0,T;\mathcal{V}),\\ Y(U_n,\Lambda) &\to \widetilde{Y} \quad \text{strongly in} \quad L^2(0,T;\mathcal{H}). \end{aligned}$$

Therefore, by the uniqueness of the solution, $\widetilde{Y} = Y(\widetilde{U},\Lambda)$. Since the norm is lower semicontinuous, we have that $U \to J(U,\Lambda)$ is lower semicontinuous for all $\Lambda\in\mathcal{G}$. By using the same technique we obtain that $\Lambda \to J(U,\Lambda)$ is upper semicontinuous for all $U\in\mathcal{E}$.

Now, we prove that $\Lambda \to J(U,\Lambda)$ is strictly concave for all $U\in\mathcal{E}$, and $U\to J(U,\Lambda)$ is strictly convex for all $\Lambda\in\mathcal{G}$.

As in [3], to prove the concavity, it is enough to prove that $g(h) = J(U,\Lambda+h\widetilde{\Lambda})$ is concave with respect to h near $h=0$, i.e., $g''(0)<0$. Denote $Y^h = Y(U,\Lambda+h\widetilde{\Lambda})$. First, we note that the derivative $g'(h)$ of h reads:

$$\int_0^S \langle DY^h - Y_d, D\widetilde{Z}^h\rangle dt - l\int_0^S \langle \Lambda + h\widetilde{\Lambda},\widetilde{\Lambda}\rangle dt.$$

Here, $\widetilde{Z}^h = Z(U,\Lambda+h\widetilde{\Lambda};0,\widetilde{\Lambda})$ satisfies

$$\begin{aligned} &\frac{d\widetilde{Z}^h}{dt} + A\widetilde{Z}^h - F'(Y^h)\widetilde{Z}^h = \widetilde{\Lambda}, \qquad 0<t\le S,\\ &\widetilde{Z}^h(0) = 0. \end{aligned} \tag{3.5}$$

Taking the scalar product with $\widetilde{Z}^h$ to (3.5) and using (f.iii), (f.iv), we have, for $0 < t < S$,

$$\begin{aligned}(3.6)\qquad &\frac{d}{dt}|\widetilde{Z}^h(t)|^2 + \delta\|\widetilde{Z}^h(t)\|^2 \\ &\le (\|Y^h\|^2+1)\tilde{\mu}(|Y^h|^2)|\widetilde{Z}^h(t)|^2 + \frac{8}{\delta}\|\widetilde{\Lambda}\|_*^2,\end{aligned}$$

where $\tilde{\mu} : [0,\infty) \to [0,\infty)$ is some increasing continuous function. Using Gronwall's inequality, we obtain

$$\begin{aligned}(3.7)\qquad |\widetilde{Z}^h(t)|^2 &\le 8\delta^{-1}\|\widetilde{\Lambda}\|^2_{L^2(0,S;\mathcal{V}')}e^{\int_0^S(\|Y^h\|^2+1)\tilde{\mu}(|Y^h|^2)ds} \\ &\le C_1\|\widetilde{\Lambda}\|^2_{L^2(0,S;\mathcal{V}')}.\end{aligned}$$

Using this result in (3.6) and integrating from 0 and t, we have

$$(3.8)\qquad \int_0^S \|\widetilde{Z}^h(t)\|^2 dt \le C_2\|\widetilde{\Lambda}\|^2_{L^2(0,S;\mathcal{V}')}.$$

To calculate $g''(h)$, we need second order derivative of Y with respect to the disturbance. By Lemma 3.5, we see that $\widetilde{\Phi}^h = \widetilde{\Phi}(U, \Lambda + h\widetilde{\Lambda}; 0, \widetilde{\Lambda}; 0, \widehat{\Lambda})$ satisfies

$$\begin{aligned}(3.9)\qquad &\frac{d\widetilde{\Phi}^h}{dt} + A\widetilde{\Phi}^h - F''(Y^h)(\widetilde{Z}^h, \widetilde{Z}^h) - F'(Y^h)\widetilde{\Phi}^h = 0, \quad 0 < t \le S, \\ &\widetilde{\Phi}^h(0) = 0.\end{aligned}$$

Taking the scalar product with $\widetilde{\Phi}^h$ to (3.9) and using (f.iii), (f.iv), (f.v), we have, for $0 < t < S$,

$$\begin{aligned}(3.10)\qquad &\frac{1}{2}\frac{d}{dt}|\widetilde{\Phi}^h(t)|^2 + \delta\|\widetilde{\Phi}^h(t)\|^2 \\ &\le \frac{\delta}{2}\|\widetilde{\Phi}^h\|^2 + \frac{4}{\delta}N^2\|\widetilde{Z}^h\|^2|\widetilde{Z}^h|^2 + (\|Y^h\|^2+1)\tilde{\mu}(|Y^h|^2)|\widetilde{\Phi}^h|^2,\end{aligned}$$

where $\tilde{\mu} : [0,\infty) \to [0,\infty)$ is some increasing continuous function. Therefore, by Gronwall's inequality, we obtain

$$|\widetilde{\Phi}^h(t)|^2 \le 8\delta^{-1}N^2\|\widetilde{Z}^h\|^2_{L^2(0,S;\mathcal{V})}\|\widetilde{Z}^h\|^2_{L^\infty(0,S;\mathcal{H})}e^{\int_0^S(\|Y^h\|^2+1)\tilde{\mu}(|Y^h|^2)ds}$$

and thus, by (3.7) and (3.8),

$$|\widetilde{\Phi}^h(t)|^2 \le C_3\|\widetilde{\Lambda}\|^4_{L^2(0,S;\mathcal{V}')}.$$

Using this result in (3.10) and integrating from 0 and t, we have

$$\int_0^S \|\widetilde{\Phi}^h(t)\|^2 dt \leq C_4 \|\widetilde{\Lambda}\|^4_{L^2(0,S;\mathcal{V}')}.$$

Therefore, we obtain

$$\begin{aligned}&\int_0^S \langle \widetilde{\Phi}^h, D^*\Lambda(DY^h - Y_d)\rangle dt \\ &\leq \Big(\int_0^S \|\widetilde{\Phi}\|^2 dt\Big)^{1/2} \|D^*\| \Big(\int_0^S \|DY^h - Y_d\|^2 dt\Big)^{1/2} \leq C_5 \|\widetilde{\Lambda}\|^2_{L^2(0,S;\mathcal{V}')}.\end{aligned}$$

For second order derivative, we have

$$\begin{aligned} g''(h) &= \int_0^S \langle \widetilde{\Phi}^h, D^*\Lambda(DY^h - Y_d)\rangle dt \\ &\quad + \int_0^S \langle D\widetilde{Z}^h, D\widetilde{Z}^h\rangle_{\mathcal{V}} dt - l \int_0^S \langle \widetilde{\Lambda}, \widetilde{\Lambda}\rangle_{\mathcal{V}'} dt. \end{aligned}$$

Thus, under assumption $l > \tilde{l} = \|D\|^2 C_2 + C_5$,

$$g''(0) \leq (\tilde{l} - l)\|\widetilde{\Lambda}\|^2_{L^2(0,S;\mathcal{V}')} < 0 \quad \forall \widetilde{\Lambda} \neq 0.$$

Therefore, $\Lambda \to J(U, \Lambda)$ is concave if $l > \tilde{l}$.

For the convexity, it is sufficient to show that $k(h) = J(U + h\widetilde{U}, \Lambda)$ is convex with respect to h near $h = 0$, i.e., $k''(0) > 0$. Denote $Y_h = Y(U + h\widetilde{U}, \Lambda)$. Similarly, we obtain that

$$k''(h) = \int_0^S \langle \Phi^h, D^*\Lambda(DY_h - Y_d)\rangle dt + \int_0^S \langle DZ^h, DZ^h\rangle_{\mathcal{V}} dt + \gamma \int_0^S \langle \widetilde{U}, \widetilde{U}\rangle_{\mathcal{V}'} dt.$$

Here, $Z^h = Z(U + h\widetilde{U}, \Lambda; \widetilde{U}, 0)$ satisfies

$$\begin{aligned} &\frac{dZ^h}{dt} + AZ^h - F'(Y_h)Z^h = \widetilde{U}, \qquad 0 < t \leq S, \\ &Z^h(0) = 0. \end{aligned}$$

and $\Phi^h = \Phi(U + h\widetilde{U}, \Lambda; \widetilde{U}, 0; \widehat{U}, 0)$ satisfies

$$\begin{aligned} &\frac{d\Phi^h}{dt} + A\Phi^h - F''(Y_h)(Z^h, Z^h) - F'(Y_h)\Phi^h = 0, \quad 0 < t \leq S, \\ &\Phi^h(0) = 0. \end{aligned}$$

Using similar a priori estimates as previously, we obtain that under assumption $\gamma > \tilde{\gamma} = C_5$,

$$k''(0) \geq (\gamma - \tilde{\gamma})\|\widetilde{U}\|^2_{L^2(0,S;\mathcal{V}')} > 0 \quad \forall \widetilde{U} \neq 0.$$

Therefore, $U \to J(U, \Lambda)$ is convex if $\gamma > \tilde{\gamma}$. □

From the general framework developed in [3], we have the following result.

Theorem 3.7. *Assume that $\mathcal{E}$ and $\mathcal{G}$ are non-empty, closed, bounded, convex subsets of $L^2(0, S; \mathcal{V}')$ and that $\gamma > \bar{\gamma}$ and $l > \bar{l}$. Then, there exists a saddle point $(\overline{U}, \overline{\Lambda})$ such that*

$$J(\overline{U}, \Lambda) \leq J(\overline{U}, \overline{\Lambda}) \leq J(U, \overline{\Lambda}) \quad \forall (U, \Lambda) \in \mathcal{E} \times \mathcal{G}. \tag{3.11}$$

Now we can give the optimality conditions for the robust control problem $(\overline{\mathbf{P}})$.

Theorem 3.8. *Let $(\overline{U}, \overline{\Lambda})$ be an solution of $(\overline{\mathbf{P}})$ and $\overline{Y} = Y(\overline{U}, \overline{\Lambda}) \in H^1(0, S; \mathcal{V}') \cap \mathcal{C}([0, S]; \mathcal{H}) \cap L^2(0, S; \mathcal{V})$ be the solution to (2.3) with the control $\overline{U}(t)$ and the disturbance $\overline{\Lambda}(t)$. Then, there exists a unique solution $P \in H^1(0, S; \mathcal{V}') \cap \mathcal{C}([0, S]; \mathcal{H}) \cap L^2(0, S; \mathcal{V})$ to the linear problem*

$$\begin{aligned} -\frac{dP}{dt} + AP - F'(\overline{Y})^* P &= D^* \mathcal{J}(D\overline{Y} - Y_d), \quad 0 \leq t < S, \\ P(S) &= 0 \end{aligned} \tag{3.12}$$

in $\mathcal{V}'$, where $\mathcal{J} : \mathcal{V} \to \mathcal{V}'$ is a canonical isomorphism; moreover,

$$\int_0^S \langle \mathcal{J}P + \gamma\overline{U}, U - \overline{U}\rangle_{\mathcal{V}'} dt \geq 0 \qquad \text{for all } U \in \mathcal{E}. \tag{3.13}$$

and

$$\int_0^S \langle \mathcal{J}P - l\overline{\Lambda}, \Lambda - \overline{\Lambda}\rangle_{\mathcal{V}'} dt \leq 0 \qquad \text{for all } \Lambda \in \mathcal{G}. \tag{3.14}$$

Proof. Let $(\overline{U}, \overline{\Lambda})$ be a saddle point for the problem $(\overline{\mathbf{P}})$. For any $U \in \mathcal{E}$, by the convexity of $\mathcal{E}$, $U_h = \overline{U} + h(U - \overline{U}) \in \mathcal{E}$ for $0 \le h \le 1$. By Theorem 2.1, (2.3) has a unique solution $Y(U_h, \overline{\Lambda})$ corresponding to U_h and $\overline{\Lambda}$.

Using the second inequality of (3.11), we have

$$
\begin{aligned}
(3.15) \qquad & \lim_{h\to 0} \frac{J(U_h, \overline{\Lambda}) - J(\overline{U}, \overline{\Lambda})}{h} \\
&= \int_0^S \langle D\overline{Y} - Y_d, DZ\rangle_{\mathcal{V}} dt \ + \ \gamma \int_0^S \langle \overline{U}, U - \overline{U}\rangle_{\mathcal{V}'} dt \ge 0
\end{aligned}
$$

with $Z = Z(\overline{U}, \overline{\Lambda}; U - \overline{U}, 0)$ satisfying

$$
\begin{aligned}
&\frac{dZ}{dt} + AZ - F'(\overline{Y})Z = U - \overline{U}, \qquad 0 < t \le S, \\
&Z(0) = 0.
\end{aligned}
$$

Let P be the unique solution of (3.12) in $H^1(0, S; \mathcal{V}') \cap \mathcal{C}([0, S]; \mathcal{H}) \cap L^2(0, S; \mathcal{V})$. From [5, Chap. XVIII, Theorem 2], we can guarantee that such a solution P exists. Thus, the first integral in the right hand side of (3.15) is shown to be

$$
\begin{aligned}
(3.16) \quad \int_0^S \langle D\overline{Y} - Y_d, DZ\rangle_{\mathcal{V}}\, dt &= \int_0^S \langle D^*\mathcal{J}(D\overline{Y} - Y_d), Z\rangle dt \\
&= \int_0^S \langle -\frac{dP}{dt} + AP - F'(\overline{Y})^* P, Z\rangle dt \\
&= \int_0^S \langle P, \frac{dZ}{dt} + AZ - F'(\overline{Y})Z\rangle dt \\
&= \int_0^S \langle \mathcal{J}P, U - \overline{U}\rangle_{\mathcal{V}'} dt.
\end{aligned}
$$

Hence,

$$
\int_0^S \langle \mathcal{J}P + \gamma\overline{U}, U - \overline{U}\rangle_{\mathcal{V}'}\, dt \ge 0, \qquad \text{for all } U \in \mathcal{E}.
$$

This prove the inequality (3.13).

Similarly, for any $\Lambda \in \mathcal{G}$, $\Lambda_h = \overline{\Lambda} + h(\Lambda - \overline{\Lambda}) \in \mathcal{G}$ for $0 \le h \le 1$. By Theorem 2.1, (2.3) has a unique solution $Y(\overline{U}, \Lambda_h)$ corresponding to $\overline{U}$ and Λ_h .

Using the first inequality of (3.11), we have

$$\begin{aligned}\lim_{h\to 0} \frac{J(\overline{U},\Lambda_h) - J(\overline{U},\overline{\Lambda})}{h} &= \int_0^S \langle D\overline{Y} - Y_d, D\widetilde{Z}\rangle_{\mathcal{V}} dt \\ &\quad -l\int_0^S \langle \overline{\Lambda}, \Lambda - \overline{\Lambda}\rangle_{\mathcal{V}'} dt \le 0\end{aligned}$$

with $\widetilde{Z} = \widetilde{Z}(\overline{U}, \overline{\Lambda}; 0, \Lambda - \overline{\Lambda})$ satisfying

$$\begin{aligned}&\frac{d\widetilde{Z}}{dt} + A\widetilde{Z} - F'(\overline{Y})\widetilde{Z} = \Lambda - \overline{\Lambda}, \qquad 0 < t \le S, \\ &\widetilde{Z}(0) = 0.\end{aligned}$$

Similarly, as in (3.16), we obtain

$$\int_0^S \langle \mathcal{J}P - l\overline{\Lambda}, \Lambda - \overline{\Lambda}\rangle_{\mathcal{V}'}\, dt \le 0, \qquad \text{for all } \Lambda \in \mathcal{G}. \quad \square$$

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