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COXETER ALGEBRAS AND PRE-COXETER ALGEBRAS IN SMARANDACHE SETTING

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Abstract. In this paper we introduce the notion of a (pre-)Coxeter algebra and show that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. Moreover, we prove that the class of Coxeter algebras and the class of B-algebras of odd order are Smarandache disjoint. Finally, we show that the class of pre-Coxeter algebras and the class of BCK-algebras are Smarandache disjoint.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([5, 6]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [3, 4] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Recently, Y. B. Jun, E. H. Roh and H. S. Kim ([7]) introduced a new notion, called a BH-algebra, i.e., (I), (II) and (V) $x * y = 0$ and $y * x = 0$ imply $x = y$, which is a generalization of $BCH/BCI/BCK$-algebras. They also defined the notions of ideals and boundedness in BH-algebras, and showed that there

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is a maximal ideal in bounded BH-algebras. J. Neggers and H. S. Kim ([10]) introduced and investigated a class of algebras which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras and which seems to have rather nice properties without being excessively complicated otherwise. Furthermore, they demonstrated a rather interesting connection between B-algebras and groups. P. J. Allen et al. ([1]) included several new families of Smarandache-type P-algebras and studied some of their properties in relation to the properties of previously defined Smarandache-types. In this paper we introduce the notion of a (pre-)Coxeter algebra and show that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. Moreover, we prove that the class of Coxeter algebras and the class of B-algebras of odd order are Smarandache disjoint. Finally, we show that the class of pre-Coxeter algebras and the class of BCK-algebras are Smarandache disjoint.

2. Coxeter algebras

A *Coxeter algebra* is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$.

(III) $(x * y) * z = x * (y * z)$

for any $x, y, z \in X$. Coxeter algebras are special types of semigroups. An example of a Coxeter algebra is a Klein 4-group (see Theorem 2.3).

Proposition 2.1. *If $(X; *, 0)$ is a Coxeter algebra, then $0 * x = x$ for any $x \in X$.*

Proof. For any $x \in X$, we obtain $x = x*0 = x*(x*x) = (x*x)*x = 0*x$.

Proposition 2.2. *If* $(X;*,0)$ *is a Coxeter algebra, then the cancellation laws hold.*

Proof. By Proposition 2.1 we have $y = 0*y = (x*x)*y = x*(x*y)$. Similarly, $z = x*(x*z)$. If $x*y = x*z$, then we obtain $y = z$ which shows that the left cancellation law holds. On the other hand, since $y = (y*x)*x$ and $z = (z*x)*x$ for any $x \in X$, it follows that the right cancellation law holds.

Theorem 2.3. *If* $(X;*,0)$ *is a Coxeter algebra, then it is an abelian group all of whose elements have order 2, i.e., a Boolean group, and conversely.*

Proof. First, we show that every element x of X has a right inverse. For any $x \in X$, let $y \in X$ such that $x*y = 0$. Since $x*x = 0$, we have $x*y = x*x$. By Proposition 2.2, we have $x = y$, i.e., every element of X has a self-inverse. Moreover, the axiom (I) means that the order of $x \in X$ is 2, and hence $(x*y)*(x*y) = 0$ for any $x, y \in X$. This means that

$$\begin{aligned}
y &= 0*y && \text{[Proposition2.1]} \\
&= [(x*y)*(x*y)]*y \\
&= (x*y)*[(x*y)*y] \\
&= (x*y)*[x*(y*y)] \\
&= (x*y)*(x*0) \\
&= (x*y)*x
\end{aligned}$$

Multiplying x to the right side, we have

$$\begin{aligned} y*x &= [(x*y)*x]*x \\ &= (x*y)*(x*x) \\ &= x*y, \end{aligned}$$

proving that $(X;*,0)$ is abelian. The converse is trivial, and we omit the proof.

3. Coxeter algebras and B-algebras

J. Neggers and H. S. Kim introduced and investigated a class of algebras, called a B-algebra, which is related to several classes of algebras such as $BCH/BCI/BCK$-algebras. A *B-algebra* ([10]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms: (I), (II) and (IV) $(x*y)*z = x*(z*(0*y))$, for any $x, y, z \in X$.

Proposition 3.1. *If $(X;*,0)$ is a Coxeter algebra, then it is a B-algebra.*

Proof. For any $x, y, z \in X$, we have

$$\begin{aligned} (x*y)*z &= x*(y*z) && [\text{(III)}] \\ &= x*(z*y) && [\text{Theorem 2.3}] \\ &= x*(z*(0*y)) && [\text{Proposition 2.1}] \end{aligned}$$

Theorem 3.2. ([10]) *Let $(X;*,0)$ be a B-algebra. If $(X;*,0) \to (X;\circ,0)$, i.e., if $x \circ y = x*(0*y)$, then $(X;\circ,0)$ is a group.*

Moreover, given a group $(X;\cdot,e)$, if we define $x*y := x\cdot y^{-1}$, then $(X;*,0=e)$ is a B-algebra. We define $x\circ y := x*(0*y)$, $x, y \in X$, and we denote

$$x^n = \underbrace{(((x \circ x) \circ x) \circ \cdots) \circ x}_{n}$$

Proposition 3.3. *Let $(X;*,0)$ be a Coxeter algebra. Then it cannot contain a B-algebra $(X;*,0)$ which contains an element of the prime order $p\,(\geq 3)$.*

Proof. Assume X contains a B-algebra $(Y;*,e)$. Then $e = x*x = 0$ for any $x \in X$. Let $x \in X$ be an element of the prime order $p\,(\geq 3)$. Then $(<x>, \circ)$ is a cyclic subgroup of the prime order $p\,(\geq 3)$ of the derived group $(Y;\circ,0)$, where $x \circ y = x * (0 * y)$. By applying Proposition 2.1 we obtain

$$x^n = \begin{cases} x, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Thus $0 = x^p = x$, a contradiction.

Corollary 3.4. *Let $(X;*,0)$ be a Coxeter algebra. Then it cannot contain a B-algebra $(X;\circ,e)$ such that $|X| = 2n+1$ is odd.*

Proof. Assume it has a B-algebra $(X;\circ,e)$ where $|X| = 2n+1$ is odd. Then, by Proposition 2.1 and Theorem 3.2, $x \circ y = x*(0*y) = x*y$. In particular, $x \circ x = x*x = 0$ for any $x \in X$. Hence the cyclic group $<x>$ of its derived group $(X;\circ,e)$ is of order 2. By the Lagrange theorem, $o(x) = 2 \,|\, 2n+1 = |\,X\,|$, a contradiction.

Theorem 3.5. *Let $(X;*,0)$ be a B-algebra and $|\,X\,| = 2n+1$ where n is a natural number. If $(C;*,e)$ is a Coxeter algebra with $C \subseteq X$, then $|\,C\,| = 1$.*

Proof. For any $x \in C$, $0 = x * x = e$, i.e., $0 = e$. If $x \neq 0$ then $x * x = 0$ and $x = x^{-1}$, by Lagrange theorem, $o(x) = 2 \mid |X| = 2n+1$, a contradiction. Hence $o(x) = 1$ and $x = 0$, i.e., $|C| = 1$.

Let $(X, *)$ be a binary system/algebra. Then $(X, *)$ is a *Smarandache-type P-algebra* if it contains a subalgebra $(Y, *)$, where Y is non-trivial, i.e., $|Y| \geq 2$, or Y contains at least two distinct elements, and $(Y, *)$ is itself of type P. Thus, we have *Smarandache-type semigroups* (the type P-algebra is a semigroup), *Smarandache-type groups* (the type P-algebra is a group), *Smarandache-type abelian groups* (the type P-algebra is an abelian group). A Smarandache semigroup in the sense of Kandasamy is in fact a Smarandache-type group (see [11]). Smarandache-type groups are of course a larger class than Kandasamy's Smarandache semigroups since they may include non-associative algebras as well.

Given algebra types $(X, *)$ (type-P_1) and $(X, \circ)$ (type-P_2), we shall consider them to be *Smarandache disjoint* ([1]) if the following two conditions hold:

(A) If $(X, *)$ is a type-P_1-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_2-algebra $(X, \circ)$;

(B) If $(X, \circ)$ is a type-P_2-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_1-algebra $(X, *)$.

Using Corollary 3.4 and Theorem 3.5 we obtain:

Theorem 3.6. *The class of Coxeter algebras and the class of B-algebras of odd order are Smarandache disjoint.*

A B-algebra X is said to be 0-*commutative* ([2]) if $x*(0*y) = y*(0*x)$ for any $x, y \in X$.

Proposition 3.7. ([10]) *If $(X;*,0)$ is a 0-commutative B-algebra, then $(0*x)*(0*y)=y*x$ for any $x,y\in X$.*

Lemma 3.8. ([10]) *Let $(X;*,0)$ be a B-algebra. Then $0*(0*x)=x$ for any $x\in X$.*

Proposition 3.9. *Let $(X;*,0)$ be a B-algebra. If $(0*y)*(0*x)=x*y$ for any $x,y\in X$, then $(X;*,0)$ is 0-commutative.*

Proof. For any $x,y\in X$,

$$\begin{aligned} x*(0*y) &= (0*(0*y))*(0*x) \\ &= y*(0*x), \end{aligned}$$

proving the proposition.

Theorem 3.10. *Let $(X;*,e)$ be an abelian group. If we define $x*y:=x\cdot y^{-1}, x,y\in X$, then $(X;*,0=e)$ is a 0-commutative B-algebra.*

Proof. It is shown that $(X;*,0=e)$ is a B-algebra and $e*y=y^{-1}$ and $x*y=x\cdot y^{-1}=y^{-1}(x^{-1})^{-1}=(e*y)*(e*x)$ for any $x,y\in X$. By Proposition 3.9, it is a 0-commutative B-algebra.

Proposition 3.11. *Let $(X;*,0)$ be a B-algebra with $x*y=y*x$, for any $x,y\in X$. Then it is a Coxeter algebra.*

Proof. For any $x,y,z\in X$, we have

$$\begin{aligned} (x*y)*z &= x*(z*(0*y) && [(IV)] \\ &= x*((0*y)*z) && [\text{commutative}] \\ &= x*((y*0)*z) && [\text{commutative}] \\ &= x*(y*z) && [(II)] \end{aligned}$$

Proposition 3.12. *Let $(X;*,0)$ be a Coxeter algebra. If $x*y=0$, $x,y\in X$, then $x=y$, i.e., the axiom (V) holds.*

Proof. If $x*y=0$, then by (I), $x*x=x*y$. By applying Proposition 2.2 we have $x=y$.

4. Pre-Coxeter algebras

An algebra $(X;*,0)$ is called a *pre-Coxeter algebra* if it satisfies the axioms (I), (II), (V), (VI) $x*y=y*x$ for any $x,y\in X$.

Example 4.1. Let $X:=[0,\infty)$. If we define $x*y:=|x-y|, x,y\in X$, then $(X;*,0)$ is a pre-Coxeter algebra, but not a Coxeter algebra, since $(1*2)*3=2$, but $1*(2*3)=0$.

Example 4.2. Let $X:=\{e,a,b,c\}$ be a set with the following table:

$*$	e	a	b	c
e	e	a	b	c
a	a	e	a	a
b	b	a	e	a
c	c	a	a	e

Then $X:=\{e,q,b,c\}$ is a pre-Coxeter algebra, but not a Coxeter algebra, since $(a*b)*c=a\neq e=a*(b*c)$.

Proposition 4.3. *Every Coxeter algebra is a pre-Coxeter algebra.*

Proof. It follows from Theorem 2.3 and Proposition 3.12.

Theorem 4.4. *The class of pre-Coxeter algebras and the class of BCK-algebras are Smarandache disjoint.*

Proof. Let $(X;*,0)$ be a BCK-algebra and $(Y;*,0)$ be a pre-Coxeter algebra with $Y \subseteq X, |Y| \geq 2$. Then $x = x*0 = 0*x = 0$ for any $x \in Y$, a contradiction.

Lemma 4.5. *Let $(X;*,0)$ be a pre-Coxeter algebra. If $x*y = 0, x, y \in X$, then $x = y$.*

Proof. Straightforward.

Proposition 4.6. *Let $(X;*,0)$ be a Coxeter algebra. Then $x*(x*y) = y$, for any $x, y \in Y$.*

Proof. For any $x, y \in X$, we have

$$
\begin{aligned}
(x*(x*y))*y &= ((x*x)*y)*y && [(III)] \\
&= (0*y)*y && [(I)] \\
&= y*y && [\text{Proposition2.1}] \\
&= 0 && [(II)]
\end{aligned}
$$

Since every Coxeter algebra is a pre-Coxeter algebra, by Lemma 4.5, we obtain $x*(x*y) = y$.

Note that $x*(x*y) = y$ does not hold for pre-Coxeter algebras in general.

Example 4.7. Let $X := \{0, 1, 2, 3\}$ be a set with

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	3
2	2	3	0	1
3	3	3	1	0

Then $(X; *, 0)$ is a pre-Coxeter algebra, but $(1*(1*2))*2 = (1*3)*2 = 3*2 = 1 \neq 0$.

Theorem 4.8. *Let $(X; *, 0)$ be a pre-Coxeter algebra with $(x*(x*y))*y = 0$, for any $x, y \in X$. Then the cancellation laws hold.*

Proof. Assume $x*a = x*b$, where $x, a, b \in X$. Then, by Lemma 4.5, $a = x*(x*a) = x*(x*b) = b$.

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