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REPRESENTATION OF INTEGRAL OPERATORS ON $W_2^2(\Omega)$ OF REPRODUCING KERNELS

DONG-MYUNG LEE, JEONG-GON LEE AND MING-GEN CUI

Abstract. We prove that if $\mathbb{K}^*$ is adjoint operator on $W_2^2(\Omega)$, then $\mathbb{K}^*v(t,\tau) =< v(\cdot,\cdot), m(\cdot,\cdot;t,\tau) >$, $v(x,y) \in W_2^2(\Omega)$; it is also related to the decomposition of solution of Fredholm equations.

1. Intorduction

In this paper we are concerned with representing the adjoint operator $\mathbb{K}^*$ on $W_2^2(\Omega) = \{u(x,y)|u(x,y)$ is absolutely continuous and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \partial y} \in L^2(\Omega)\}$ of reproducing kernels :

$$\mathbb{K}^*v(t,\tau) =< v(\cdot,\cdot), m(\cdot,\cdot;t,\tau) >, \; v(x,y) \in W_2^2(\Omega),$$

where $\Omega = [a,b] \times [c,d]$. In the case of $W_2^1(\mathbb{R})$, $m(\cdot,\cdot)$ is known (see [3] and the referenses given for that theorem).

We use Cui's approach [3] to seek a good component $m(\cdot,\cdot;t,\tau)$, regarded as an ingtegral of reproducing kernels on Ω and extend Cui's result of $W_2^1(\mathbb{R})$ to the case of $W_2^2(\Omega)$.

Throughout, $L^2(\Omega)$ denotes, as usual, the Hilbert space of all Lebesgue square integrable functions on Ω. The needed facts about reproducing kernels can be found in [9].

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2. The Result

We now define, for $u, v \in W_2^2(\Omega)$,

$$< u,v >= \int\int_\Omega (u(x,y)v(x,y) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y})d\sigma \quad (2)$$

as inner product on $W_2^2(\Omega)$ and norm $||u||_{W_2^2} =< u,u >^{\frac{1}{2}}$.

We recall, see [4], that any $u(x) \in W_2^1([a,b])$ can be represented in terms of a reproducing kernel such as $u(x) =< u(\cdot), R_x(\cdot) >$, where

$$R_x^{ab}(y) = \frac{1}{2sinh(b-a)}[cosh(y+x-a-b)+cosh(|y-x|+a-b)], y \in [a,b].$$

Our next result extends this formula to the case of $\Omega = [a,b] \times [c,d]$.

Lemma 1. $R_{xy}(\cdot,\cdot)$ is a reproducing kernel of space $W_2^2(\Omega)$,where $R_{xy}(\cdot,\cdot) = R_x^{ab}(\cdot)R_y^{cd}(\cdot)$.

Proof. For any $u \in W_2^2(\Omega)$,

$$\begin{aligned} < u(\cdot,\cdot), R_{xy}(\cdot,\cdot) > &= \int_c^d d\eta \int_a^b \{u(\xi,\eta)R_{xy}(\xi,\eta) + u'_\xi(\xi,\eta)R'_{xy}(\xi,\eta)_\xi \\ &\quad + u'_\eta(\xi,\eta)R'_{xy}(\xi,\eta)_\eta + u''_{\xi\eta}(\xi,\eta)R''_{xy}(\xi,\eta)_{\xi\eta}\}d\xi \\ &= \int_c^d d\eta\{R_y(\eta)\int_a^b (u(\xi,\eta)R_x(\xi) + u'_\xi(\xi,\eta)R'_x(\xi)_\xi)d\xi \\ &\quad + R'_y(\eta)_\eta \int_a^b (u'_\eta(\xi,\eta)R_x(\xi) + u''_{\xi\eta}R'_x(\xi)_\xi)d\xi\}. \end{aligned}$$

Applying the reproducing properties of $R_y(\xi)$, we have

$$\int_a^b u(\xi,\eta)R_x(\xi) + u'_\xi(\xi,\eta)R'_x(\xi)_\xi d\xi =< u(\cdot,\eta), R_x(\cdot) >= u(x,\eta).$$

$$\int_a^b u'_\eta(\xi,\eta)R_x(\xi) + \frac{\partial}{\partial \xi}(u'_\eta(\xi,\eta))R'_x(\xi)_\xi d\xi =< u'_\eta(\cdot,\eta), R'_x(\cdot) >= u'_\eta(x,\eta).$$

So that,

$$< u(\cdot,\cdot), R_{xy}(\cdot,\cdot) >= \int_c^d \{ u(x,\eta)R_y(\eta) + u'_\eta(x,\eta)R'_x(\eta) \}d\eta.$$

Finally, using the definition of reproducing kernels to get the conclusion.

From the preceding results and the classical Fredholm equation, we have the following application. □

Lemma 2. Let $(x, y; t, \tau) \in \Omega \times \Omega$ and let $k(x, y; t, \tau) \in L^2(\Omega)$ for each fixed $(x, y) \in \Omega$. If, for each fixed $(t, \tau), k(x, y; t, \tau)$ is absolutely continuous on Ω and $k, \frac{\partial k}{\partial x}, \frac{\partial k}{\partial y}, \frac{\partial^2 k}{\partial x \partial y} \in L^2(\Omega)$, then the integral operator $\mathbb{K}$ on $W_2^2(\Omega)$ defined by

$$\mathbb{K}u(x, y) := \int_a^b \int_c^d k(x, y; t, \tau) u(t, \tau) d\tau dt,$$

for $u(x, y) \in W_2^2(\Omega)$ is well-defined and bounded.

Proof. First note that, since k is absolutely continuous, we have

$$\begin{aligned} k(x, y; t, \tau) &= \int_a^x \int_c^y \frac{\partial^2}{\partial \xi \partial \eta} k(\xi, \eta; t, \tau) d\eta d\xi + \int_a^x \frac{\partial}{\partial \xi} k(\xi, c; t, \tau) d\xi \\ &\quad + \int_c^y \frac{\partial}{\partial \eta} k(a, \eta; t, \tau) d\eta + k(a, c; t, \tau). \end{aligned}$$

Then, the definition of Fredholm equations and an elementary advanced calculus argument allow us to compute

$$\begin{aligned} &||\mathbb{K}u(b_i, c) - \mathbb{K}u(a_i, c)|| \\ &= |\int_a^b \int_c^d \int_{a_i}^{b_i} \frac{\partial}{\partial \xi} k(\xi, c; t, \tau) u(t, \tau) d\xi d\tau dt| \\ &\leq \frac{1}{d-c} \int_a^b \int_c^d \int_c^d \int_{a_i}^{b_i} |\frac{\partial}{\partial \xi} k(\xi, y; t, \tau) u(t, \tau)| d\xi dy d\tau dt \\ &\quad + \frac{1}{d-c} \int_a^b \int_c^d \int_c^d \int_{a_i}^{b_i} | \int_c^y \frac{\partial^2}{\partial \xi \partial \eta} k(\xi, \eta; t, \tau) d\eta u(t, \tau)| d\xi dy d\tau dt \\ &\leq \frac{1}{d-c} \int_a^b \int_c^d \int_c^d \int_{a_i}^{b_i} |\frac{\partial}{\partial \xi} k(\xi, y; t, \tau) u(t, \tau)| d\xi dy d\tau dt \\ &\quad + \int_a^b \int_c^d \int_c^d \int_{a_i}^{b_i} |\frac{\partial^2}{\partial \xi \partial \eta} k(\xi, \eta; t, \tau) u(t, \tau)| d\eta d\xi d\tau dt. \end{aligned}$$

So that, by the absolute continuity; for $\epsilon > 0, \sum_{i=1}^{n}(b_i - a_i) < \frac{\delta}{c-d}$ implies that

$$\sum_{i=1}^{n} ||\mathbb{K}u(b_i, c) - \mathbb{K}u(a_i, c)||$$
$$\leq \frac{1}{d-c}\int_a^b\int_c^d\int_c^d\int_{\cup_i(a_i,b_i)} |\frac{\partial}{\partial\xi}k(\xi, y; t, \tau)u(t,\tau)|d\xi dy d\tau dt$$
$$+\int_a^b\int_c^d\int_{\cup_i(a_i,b_i)}\int_c^d |\frac{\partial^2}{\partial\xi\partial\eta}k(\xi,\eta;t,\tau)u(t,\tau)|d\eta d\xi d\tau dt < \epsilon.$$

which shows that $\mathbb{K}u(x, c)$ is absolutely continuous.

Using the same method, we obtain the absolute continuity of $\mathbb{K}u(a, y)$. Consequently, it follows easily that $\frac{\partial}{\partial x}\mathbb{K}u(x,y)$, $\frac{\partial}{\partial y}\mathbb{K}u(x,y)$, and $\frac{\partial^2}{\partial x\partial y}\mathbb{K}u(x,y) \in L^2(\Omega)$, which implies that $\mathbb{K}u(x,y) \in W_2^2(\Omega)$.

Next, to show the boundedness of $\mathbb{K}$, let $u(x,y) \in W_2^2(\Omega)$. Then we have

$$< \mathbb{K}u, \mathbb{K}u >= \int_a^b\int_c^d \{(\mathbb{K}u(x,y))^2 + (\frac{\partial}{\partial x}\mathbb{K}u(x,y))^2$$
$$+(\frac{\partial}{\partial y}\mathbb{K}u(x,y))^2 + (\frac{\partial^2}{\partial x\partial y}\mathbb{K}u(x,y))^2\}dxdy$$
$$\leq \int_a^b\int_c^d\int_a^b\int_c^d [(k(x,y;t,\tau))^2 + (\frac{\partial}{\partial x}k(x,y;t,\tau))^2$$
$$+(\frac{\partial}{\partial y}k(x,y;t,\tau))^2 + (\frac{\partial}{\partial x\partial y}k(x,y;t,\tau))^2]d\tau dt dy dx$$
$$\times\int_a^b\int_c^d (u(t,\tau))^2 d\tau dt$$
$$= M||u||^2_{L^2(\Omega)},$$

where $M = \int_{\Omega\times\Omega}[k^2 + (\frac{\partial}{\partial x}k)^2 + (\frac{\partial}{\partial y}k)^2 + (\frac{\partial^2}{\partial x\partial y}k)^2]d\sigma$, which completes the proof. □

Our next result shows that the reproducing kernels of $W_2^2(\Omega)$ actually characterize the adjoint operator $\mathbb{K}^*$.

Theorem 3. Let $\mathbb{K}^*$ be adjoint operator of $\mathbb{K}$ on $W_2^2(\Omega)$. Then for $v(x,y) \in W_2^2(\Omega)$,

$$\mathbb{K}^* v(t,\tau) = < v(\cdot,\cdot), m(\cdot,\cdot;t,\tau) >,$$

where

$$m(x,y;t,\tau) = \int\int_\Omega k(x,y;\xi,\eta) R_{\xi\eta}(t,\tau) d\xi d\eta. \tag{3}$$

Proof. First note that, for fixed $(x,y;t,\tau), k(x,y;\xi,\eta)R_{\xi\eta}(t,\tau)$ is integrable with respect to (ξ,η) and has partial derivatives for almost all $t \in [a,b]$ by the properties of differentiating Lebesgue integrals with respect to absolute continuity.
So that, for $u, v \in W_2^2(\Omega)$ we have

$$\begin{aligned}
< \mathbb{K}u, v > &= \int_a^b \int_c^d [\int_a^b \int_c^d (kv + \frac{\partial}{\partial x} k \frac{\partial}{\partial x} v + \frac{\partial}{\partial y} k \frac{\partial}{\partial y} v \\
&\quad + \frac{\partial^2}{\partial x \partial y} k \frac{\partial^2}{\partial x \partial y} v) dy dx] u(\xi,\eta) d\xi d\eta \\
&= \int_a^b \int_c^d [\int_a^b \int_c^d (kv + \frac{\partial}{\partial x} k \frac{\partial}{\partial x} v + \frac{\partial}{\partial y} k \frac{\partial}{\partial y} v \\
&\quad + \frac{\partial^2}{\partial x \partial y} k \frac{\partial^2}{\partial x \partial y} v) dy dx] < u(\xi,\eta), R_{\xi\eta}(t,\tau) > d\xi d\eta \\
&= \int_a^b \int_c^d [\int_a^b \int_c^d (kv + \frac{\partial}{\partial x} k \frac{\partial}{\partial x} v + \frac{\partial}{\partial y} k \frac{\partial}{\partial y} v \\
&\quad + \frac{\partial^2}{\partial x \partial y} k \frac{\partial^2}{\partial x \partial y} v) dy dx] \\
&\quad \cdot [\int_a^b \int_c^d u(t,\tau) R_{\xi\eta}(t,\tau) + \frac{\partial}{\partial t} u(t,\tau) \frac{\partial}{\partial t} R_{\xi\eta}(t,\tau) \\
&\quad + \frac{\partial}{\partial \tau} u(t,\tau) \frac{\partial}{\partial \tau} R_{\xi\eta}(t,\tau) + \frac{\partial^2}{\partial t \partial \tau} u(t,\tau) \frac{\partial^2}{\partial t \partial \tau} R_{\xi\eta}(t,\tau) dt d\tau] d\eta d\xi \\
&= < u(t,\tau), h(t,\tau) >, \qquad (4)
\end{aligned}$$

where

$$\begin{aligned} h(t,\tau) &= \int_a^b\int_c^d\int_a^b\int_c^d kR_{\xi\eta}(t,\tau)v(x,y) + \frac{\partial}{\partial x}kR_{\xi\eta}(t,\tau)\frac{\partial}{\partial x}v(x,y) \\ &+\frac{\partial}{\partial y}kR_{\xi\eta}(t,\tau)\frac{\partial}{\partial y}v(x,y) \\ &+\frac{\partial^2}{\partial x\partial y}kR_{\xi\eta}(t,\tau)\frac{\partial^2}{\partial x\partial y}v(x,y)d\eta d\xi dydx. \end{aligned} \tag{5}$$

Thus, setting $m(x,y;t,\tau) := \int_a^b\int_c^d k(x,y;\xi,\eta)R_{\xi\eta}(t,\tau)d\eta d\xi$ from (5) and Lebesgue integral properties give that

$$\frac{\partial}{\partial t}m(x,y;t,\tau) = \int_a^b\int_c^d k(x,y;\xi,\eta)\frac{\partial}{\partial t}R_{\xi,\eta}(t,\tau)d\eta d\xi.$$

$$\begin{aligned} \frac{\partial}{\partial \tau}m(x,y;t,\tau) &= \int_a^b\int_c^d k(x,y;\xi,\eta)\frac{\partial}{\partial \tau}R_{\xi\eta}(t,\tau)d\eta d\xi \\ \frac{\partial^2}{\partial t\partial \tau}m(x,y;t,\tau) &= \int_a^b\int_c^d k(x,y;\xi,\eta)\frac{\partial^2}{\partial t\partial \tau}R_{\xi\eta}(t,\tau)d\eta d\xi, \end{aligned}$$

and so it follows that

$$\begin{aligned} h(t,\tau) &= \int_a^b\int_c^d [m(x,y;t,\tau)v(x,y) + \frac{\partial}{\partial x}m(x,y;t,\tau)\frac{\partial}{\partial x}v(x,y) \\ &+\frac{\partial}{\partial y}m(x,y;t,\tau)\frac{\partial}{\partial y}v(x,y) + \frac{\partial^2}{\partial x\partial y}m(x,y;t,\tau)\frac{\partial}{\partial x\partial y}v(x,y)]dxdy \\ &= < v(\cdot,\cdot), m(\cdot,\cdot;t,\tau) > . \ (6) \end{aligned}$$

So that, from (4) and (6) we have

$$\mathbb{K}^* v(t,\tau) = < v(\cdot,\cdot), m(\cdot,\cdot;t,\tau) > .$$

This completes the proof. □

Remark. Let us consider the following classical Fredholm equation

$$u(x,y) - \lambda\int_a^b\int_c^d k(x,y;t,\tau)u(t,\tau)dtd\tau = f(x,y),$$

and let

$$(I-\lambda\mathbb{K})u = f, \tag{7}$$

where $I$is identity operator, $\lambda$is parameter,and$\mathbb{K}u = \int_a^b \int_c^d k(x,y;t,\tau)$ $u(t,\tau)\ dtd\tau$. Then, Lemma 4 above asserts that $\mathbb{K}$ is well-defined and bounded. In a recent paper [7] Lees and Cui have shown that, if (p_i) is dense in Ω, $((I - \lambda\mathbb{K})^*\phi_j(p_i))$ is complete if and only if $I - \lambda\mathbb{K}$ is one-to-one, where $\phi_j(p_i) = R_{p_j}(p_i)$. By using this result with related properties of reproducing kernels of $W_2^2(\Omega)$, it is shown that u is decomposed in terms of reproducing kernels such as $u = \sum \tilde{f}_k \tilde{\psi}_k(p)$, where $\tilde{\psi}_k(p = (p_i)) = \sum_{j=1}^k \beta_{kj}\psi_j(p)$, $\psi_j(p) = ((I - \lambda\mathbb{K})^*\phi_j(p))$, and $\tilde{f}_k = \sum_{j=1}^k \beta_{kj} f(p)$. □

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D. M. Lee, J. G. Lee
Department of Mathematics
Won Kwang University
344-2 Shinyongdong Ik-San
Chunbuk 570-749, Korea
E-mail : dmlee@wonkwang.ac.kr
E-mail : jukolee@wonkwang.ac.kr

M. G. Cui
Harbin Institute of Technology
(WEI HAI branch Institute)
264209 Wei Hai
Shandong, China(P.R.)
E-mail : cmgyfs@263.net