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C^*-ALGEBRAS OF SOME SEMIGROUPS

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Abstract. In this paper the left regular representation and the reduced C^*-algebra for a commutative separative semigroup is defined. The universal representation, the reduced C^*-algebra and the full C^*-algebra for the additive semigroup N^+ are given. Also it is proved that $C_r^*(N^+) \ncong C^*(N^+)$.

0. Introduction

Much work has been done on the C^*-algebras of groups, but much less on the C^*-algebras of semigroups. The idea of the left regular representation of a non-involutive semigroup $\sum$ which is the first step in defining the reduced C^*-algebra of $\sum$, has not been considered yet. This paper studies those for a commutative separative semigroup (C.S.S.).

Perhaps it is worth noticing that, the existence of an equivalence relation " $\sim$" on a commutative semigroup $\sum$, and the fact that $\frac{\sum}{\approx}$ is the maximal semilattice homomorphic image of $\sum$ ([1, Theorem 4.12]) first appeared in [11] from which it follows easily that: any commutative semigroup $\sum$ (separative or not) is uniquely expressible as a semilattice of "archimedean" semigroups ([1, Theorem 4.13]). Except for the uniqueness, this fact is also essentially due to [10], who showed that:

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any commutative semigroup $\sum$ is a disjoint union of archimedean semigroups. Later on, Hewitt and zuckerman in [7] used this fact and showed that, the characters of a commutative semigroup $\sum$ separate the elements of $\sum$ if and only if $\sum$ is separative. Based on the above fact Dunkl and Ramirez in [4] initiated a kind of plancherel theorem for inverse semigroups.

In section 1 of this paper a commutative semigroup will be written as a *disjointunion* of its *cancellative* subsemigroups. Based on this fact we will define the left regular representation of a C.S.S., and show that this representation is faithful. At the end of this section concrete examples of C.S.S.'s, their decompositions and their left regular representations are given.

Section 2 is devoted to the reduced C^*-algebra of N^+ and its minimal ideals.

In section 3 we will define representations, the universal representation, and the full C^*-algebra of N^+. Also, it will be shown that, the C^*-algebra $C^*(S \oplus S^*)$ can have the identity element.

Section 4 deals with the determination of the minimal ideals of $C^*(S \oplus S^*)$. At the end of this section by a comparison between the minimal ideals of the C^*-algebras $C_r^*(N^+)$ and $C^*(S \oplus S^*)$ we conclude that

$$C_r^*(N^+) \neq C^*(N^+).$$

1. The left regular representation of a commutative separative semigroup

A semigroup $\sum$ is called separative if for every s, t in $\sum$

$$s^2 = st = t^2$$

implies $s = t$.

Throughout this section, unless otherwise specified, $\sum$ will denote a commutative separative semigroup (C.S.S.).

In this section we will define an equivalence relation on $\sum$ and show that each equivalence class is a cancellative subsemigroup of $\sum$. By the decomposition of $\sum$ under this equivalence relation we will define the left regular representation of $\sum$ and it will be shown that this representation is faithful.

We begin with a basic definition.

Definition 1.1. For each s in $\sum$, the set

$h_s = \{t \in \sum : t^n = su$ and $s^m = vt$, for some u, v in $\sum$ and m, n in $N\}$ is called an archimedean component of $\sum$.

Now we state an important theorem on which this section is based. Note that other versions of this theorem can be seen in [1] and [7].

Theorem 1.2 (a) The relation $s \sim t$ if and only if $t \in h_s$ is an equivalence relation on $\sum$ [4, propsition 3.3];

(b) If $s \in \sum$, then h_s is a subsemigroup of $\sum$ [4, proposition 3.4];

(c) Each h_s is a cancellative subsemigroup of $\sum$ [4, Theorem 3.5].

Note that by part (a) of the above theorem, for every $\sum$ we can write

$$\sum = \bigcup_{S \in \sum} h_s.$$

Moreover, cancellative property of each h_s is among the necessary tools in considering the properties of the left regular representation of $\sum$.

At this stage we proceed to introduce a candidate for the left regular representation of $\sum$.

Let $\{\delta_t : t \in \sum\}$ be the standard orthonormal basis for $\ell^2(\sum)$. To each $s \in \sum$ we associate the linear operator $\lambda(s)$ on $\ell^2(\sum)$ such that

$$(\Gamma) \quad \lambda(s)\delta_t = \begin{cases} \delta_{st} & \text{if } st \in h_t \\ 0 & \text{otherwise.} \end{cases}$$

We consider the properties of λ in the following lemmas.

Lemma 1.3. If we correspond to each $s \in \sum$, the linear operator $\lambda(s)$ such that (Γ) holds, then

$$\lambda(rs) = \lambda(r)\lambda(s)$$

for every r, s in $\sum$.

Proof. If $t \in \sum$, then

$$\lambda(rs)\delta_t = \delta_{rst}$$

if and only if $rst \in h_t$.
And,

$$\lambda(r)\lambda(s)\delta_t = \lambda(r)\delta_{st} = \delta_{rst}$$

if and only if

$$st \in h_t \text{ and } rst \in h_{st}.$$

Therefore in order to show that

$$\lambda(rs) = \lambda(r)\lambda(s)$$

it is enough to prove that,

$$rst \in h_t \text{ if and only if } st \in h_t \text{ and } rst \in h_{st}.$$

To see this, from $st \in h_t$ and $rst \in h_{st}$, we have

$$t \sim st \text{ and } st \sim rst.$$

Now since $\sim$ is an equivalence relation on $\sum$ we have

$$t \sim rst \text{ or } rst \in h_t.$$

Conversely let $rst \in h_t$. Hence

$$(rst)^n = ut \text{ and } t^m = v(rst)$$

for some u, v in $\sum$ and m, n in N. Since

(1) $$t^m = v(rst) = (vr)(st)$$

we have

(2) $$(ts)^m = t^m s^m = s^m v(rst) = (s^{m+1}vr)t.$$

From (1) and (2) we see that

$$st \in h_t.$$

Now from $rst \in h_t$ and $st \in h_t$ it is easily seen that

$$rst \in h_{st}.\Box$$

The following lemma shows that each $\lambda(s)$ is a partial isometry. (Partial isometry and its properties are considered in [9, ch.6]).

Lemma 1.4. For $s \in \sum$, the linear operator $\lambda(s)$ which is defined in (Γ) is a partial isometry.

Proof. Let $s \in \sum$ and

$$D_s = \{t \in \sum : st \in h_t\}.$$

We will prove that each $\lambda(s)$ is a partial isometry with the initial space $\ell^2(D_s)$. It suffices to show that the map

$$t \to st$$

is injective on D_s.

Let $t_1, t_2 \in D_s$ and $st_1 = st_2$. Since $st_1 \in h_{t_1}$ and $st_2 \in h_{t_2}$, from

$$t_1 \sim st_1 = st_2 \sim t_2$$

we have $t_1 \in h_{t_2}$. Therefore $h_{t_1} = h_{t_2}$. Hence

$$st_1 = st_2 \in h_{t_1}.$$

Since h_{t_1} is a cancellative semigroup form $st_1 = st_2$ we have $t_1 = t_2$.$\Box$

By Lemmas 1.3 and 1.4 we have the following definition.

Definition 1.5. Let $\sum$ be a commutative separative semigroup. For each $s \in \sum$, the linear operator $\lambda(s)$ on $\ell^2(\sum)$ defined by,

$$\lambda(s)\delta_t = \begin{cases} \delta_{st} & \text{if } st \in h_t \\ 0 & \text{otherwise,} \end{cases}$$

is the left regular representation of $\sum$.

Examples 1.6.

(a) Let $\sum$ be the additive semigroup

$$Z^+ = \{0, 1, 2, 3, \cdots\}.$$

obviously Z^+ is separative, $h_0 = \{0\}, h_1 = \{1, 2, 3, \cdots\}$ and

$$Z^+ = h_0 \cup h_1.$$

$\lambda : Z^+ \to PI(\ell^2(Z^+))$ by

$$\lambda(m)\delta_n = \begin{cases} \delta_{m+n} & \text{if } m + n \in h_n \\ 0 & \text{otherwise} \end{cases}$$

is the left regular representation of Z^+. It is easily seen that $\lambda(0) = I, \lambda(1) = S'$ is a partial isometry with the initial space $\ell^2(N^+)$, and

$$\lambda(m) = \lambda(1)^m = (S')^m$$

(b) Let $\sum$ be the additive semigroup $N^+ = \{1, 2, 3, \cdots\}$. obviously $h_1 = N$ and,
$\lambda : N^+ \to PI(\ell^2(N^+))$ defined by

$$\lambda(m)\delta_n = \delta_{m+n}$$

is the left regular representation of N^+. Clearly $\lambda(1)$ is a shift of multiplicity one and

$$\lambda(m) = \lambda(1)^m = S_1^m$$

We close this section by proving that, the left regular representation of a commutative separative semigroup is faithful.

Theorem 1.7. If $\sum$ is a commutative separative semigroup and λ its left regular representation, then λ is faithful.

Proof. Let $s_1, s_2 \in \sum$ and $\lambda(s_1) = \lambda(s_2)$. Since each $\lambda(s)$ is a partial isometry with the initial space $\ell^2(D_s)$ where

$$D_s = \{t \in \sum : st \in h_t\},$$

from $\lambda(s_1) = \lambda(s_2)$ we have $D_{s_1} = D_{s_2}$.
Since $s_1^2 \in h_{s_1}$ we see that

$$s_1 \in D_{s_1} = D_{s_2}.$$

From $s_1 \in D_{s_2}$ we have

$$s_2 s_1 \in h_{s_1}.$$

Therefore

$$\lambda(s_1)\delta_{s_1} = \lambda(s_2)\delta_{s_1}$$

or,

$$s_1^2 = s_1 s_2.$$

Since h_{s_1} is a cancellative semigroup and

$$s_1^2 = s_1 s_2 \in h_{s_1}$$

we have $s_1 = s_2$ i.e., λ is faithful. □

Conclusion 1.8.
(a) For a given commutative separative semigroup $\sum$ we can define its reduced C^*-algebra, $C_r^*(\sum)$ as follows:

$$C_r^*(\sum) = C^*(\{\lambda(s), \lambda(s)^* : s \in \sum\})$$

where λ is the left regular representation of $\sum$.
(b) If D is a commutative cancellative semigroup, it is obviously a separative semigroup, therefore we can consider its reduced C^*-algebra, $C_r^*(D)$.

2. The reduced C^*-algebra of $N^+, C_r^*(N^+)$

The additive semigroups Z^+ and N^+ are commutative and separative. In this section, first we will introduce $C_r^*(Z^+)$, then we consider $C_r^*(N^+)$ and its minimal ideals.

As we showed in section 1, λ the left regular representation of Z^+ is defined by

$$\lambda(m)\delta_n = \begin{cases} \delta_{m+n} & \text{if } m+n \in h_n \\ 0 & \text{otherwise} \end{cases}$$

where $m \in Z^+$ and $\{\delta_n : n \in Z^+\}$ is the standard orthonormal basis for the separable Hilbert space $H = \ell^2(Z^+)$. It is obvious that $\lambda(0) = I$ is the identity operator on H and $\lambda(1) = S'$ is a partial isometry with the initial space $H_1 = \ell^2(N^+)$; that is, with $H_0 = H \Theta H_1$, we have $S'|_{H_0} = 0$ and $S'|_{H_1}$ is a unilateral shift. Thus $S'(\delta_n) = \begin{cases} 0 & \text{if } \ n = 0 \\ \delta_{n+1} & \text{if } \ n > 0 \end{cases}$; and by 1.8

$$C_r^*(Z^+) = C^*(\{I, \lambda(1)^n, (\lambda(1)^*)^n : n \in N\}).$$

Here we will consider the reduced C^*-algebra of $N^+, C_r^*(N^+)$. By 1.6(b), λ the left regular representation on N^+ is given by $\lambda(m)\delta_n = \delta_{m+n}$ where $m \in N^+$ and $\{\delta_n : n \in N\}$ is the standard orthonormal basis for the separable Hilbert space $H_1 = \ell^2(N^+).\lambda(1) = S_1$ is a unilateral shift on H_1 and by 1.8

$$C_r^*(N^+) = C^*(\{\lambda(1)^n, (\lambda(1)^*)^n : n \in N\})$$

is nothing but the C^*-algebra generated by S_1 and S_1^*.

In order to have a better understanding of $C_r^*(N^+)$, let us introduce another C^*-algebra which is isometrically *-isomorphic to $C_r^*(N^*)$.

Let S be the unilateral shift operator. Also, let $C^*(S)$ be the C^*-algebra generated by S and S^*. In fact if $p(x, y)$ is a polynomial in two non-commuting variables x, y with complex coefficients i.e.,

$$p(x,y) = \sum_{\text{finite}} a_{i_1 i_2 \cdots i_k} x^{i_1} y^{i_2} \cdots y^{i_k}$$

then $C^*(S)$ is the norm closure of all polynomials $p(S, S^*)$.

Now we will show that $C^*(S) \cong C_r^*(N^+)$.

Let $H = \ell^2(Z^+)$ and $H_1 = \ell^2(N^+)$. The operator $T : H_1 \to H$ defined by $T\delta_n = \delta_{n-1}, (n \in N)$, is a unitary; and since $T^*ST(\sum_{j=1}^{\infty} \xi_j \delta_j) = (0, \xi_1, \xi_2, \cdots) = S_1(\sum_{j=1}^{\infty} \xi_j \delta_j)$ the mapping

$$S \longleftrightarrow T^*ST$$

induces an isometric *-isomorphism between $C^*(S)$ and $C_r^*(N^+)$.

Remark. By the above argument we may assume that (up to isomorphism), $C_r^*(N^+)$ is generated by $\{S, S^*\}$. Note that this remark will be used in sections 3 and 4 without any comment.

Since one of the main objectives of sections 3 and 4 is the comparison of $C_r^*(N^+)$ and $C^*(N^+)$, and the easiest way to achieve this goal seems to be the determination of their minimal ideals, the rest of this section will be devoted to the determination of minimal ideals of $C_r^*(N^+)$.

We start by sorting out some notation.

Let $H = \ell^2(N^+)$. The orthogonal projection onto the one dimensional subspace of H spanned by δ_n is denoted by P_n. For $x, y \in H$, the linear operator $T_{y,z} : H \rightarrow H$, is defined by

$$T_{y,z}(x) = < x, y > z$$

where $< .,. >$ is the inner product in H, is a rank one operator. If T is a finite rank operator on H, then there are orthonormal vectors $\delta_1, \delta_2, \cdots, \delta_n$ and vectors $y_1, y_2, \cdots, y_n$ such that

$$Tx = \sum_{i=1}^{n} < x, \delta_i > y_i$$

for all $x \in H$.

If

$$I = \{T \in B(H) : T \text{ is a finite rank operator}\}$$

then I is a two-sided ideal in $B(H)$. It is well known tht every non-zero two-sided ideal in $B(H)$ contains I; and k, the ideal of all compact operators on H is the norm closure of the ideal I in $B(H)$.

An operator T on a Hilbert space H has a cyclic vector e if the vectors $e, Te, T^2e, \cdots$ span a dense subspace of H. Equivalently e is a cyclic vector for T in case the set of all vectors of the form $p(T)e$, where p varies over all polynomials, is dense in H. The unilateral shift operator S restricted to $\ell^2(N^+)$ has a cyclic vector i.e., δ_1. By [5, prob. 160] S^* has cyclic vector.

CONVENTION. In this section, by S we mean, the restriction of S to $\ell^2(N^+)$, or S_1, or the unilateral shift operator on $\ell^2(N^+)$.

The following lemma shows that, there exists a nontrivial closed two-sided ideal in $C_r^*(N^+)$.

Lemma 2.1. $C_r^*(N^+)$ has nontrivial closed two-sided ideal.

Proof. Since $C_r^*(N^+)$ is generated by S, we have

$$S^*S - SS^* = I - SS^* \in C_r^*(N^+).$$

From

$$(I - SS^*)(\xi_1, \xi_2, \xi_3, \cdots) = (\xi_1, \xi_2, \cdots) - S(\xi_2, \xi_3, \cdots)$$

$$= (\xi_1, \xi_2, \cdots) - (0, \xi_2, \xi_3, \cdots) = (\xi_1, 0, 0, \cdots)$$

$$= P_1(\xi_1, \xi_2, \cdots),$$

we see that

$$I - SS^* = P_1.$$

Since $P_1 \in k$ we conclude that

$$P_1 = S^*S - SS^* \in k \cap C_r^*(N^+).$$

Therefore at least $k \cap C_r^*(N^+)$ is a nontrivial closed two-sided ideal in $C_r^*(N^+)$. □

Lemma 2.2. If J is a nontrivial closed two-sided ideal in $C_r^*(N^+)$, then $P_1 \in J$.

Proof. Since J is a nontrivial ideal it has a non-zero element, say A. So there exists $m \in N^+$ such that

$$||A\delta_m|| \neq 0.$$

From,

$$S^{m-1}P_1(S^*)^{m-1}(\xi_1, \xi_2, \cdots) = S^{m-1}P_1(\xi_m, \xi_{m+1}, \cdots)$$

$$= S^{m-1}(\xi_m, 0, 0, \cdots) = (0, 0, \cdots 0, \xi_m, 0, 0, \cdots)$$

we see that

$$S^{m-1}P_1(S^*)^{m-1} = P_m.$$

Since J is an ideal and $P_m \in C_r^*(N^+)$, we have

$$P_m A^* A P_m \in J.$$

Now for any $x = (\xi_1, \xi_2, \xi_3, \cdots) \in \ell^2(N^+)$, we have

$$P_mA^*AP_mx = \sum_{n=1}^{\infty} < P_mA^*AP_mx, \delta_n > \delta_n =< P_mA^*AP_mx, \delta_m > \delta_m$$

$$=< AP_mx, A\delta_m > \delta_m = \xi_m < A\delta_m, A\delta_m > \delta_m = ||A\delta_m||^2 P_mx$$

i.e.,

$$P_mA^*AP_m = ||A\delta_m||^2 P_m.$$

Therefore $P_m \in J$ and consequently

$$(S^*)^{m-1} P_m S^{m-1} = P_1 \in J.\square$$

In the light of the following lemma we can detremine the minimal ideals of $C_r^*(N^+)$.

Lemma 2.3. Any nontrivial closed two-sided ideal in $C_r^*(N^+)$ contains I, where I is the ideal of all finite rank operators.

Proof. Let J be a nontrivial closed two-sided ideal in $C_r^*(N^+)$. It sufices to prove that

$$T_{y,z} \in J$$

for all $y, z \in \ell^2(N^+)$. Since δ_1 is a cyclic vector for S for given y, z in $\ell^2(N^+)$ and $\varepsilon > 0$ there are polynomials p, q such that

$$||p(S)\delta_1 - y|| < \varepsilon \text{ and } ||q(S)\delta_1 - z|| < \varepsilon.$$

Now

$$\begin{aligned}
||P_1[p(S)]^* - T_{y,\delta_1}|| &= \sup\{||P_1[p(S)]^*x - < x, y > \delta_1|| : ||x|| \leq 1\} \\
&= \sup\{|| < p(S)^*x, \delta_1 > \delta_1 - < x, y > \delta_1|| : ||x|| \leq 1\} \\
&= \sup\{|| < x, p(S)\delta_1 > \delta_1 - < x, y > \delta_1|| : ||x|| \leq 1\} \\
&= \sup\{|| < x, p(S)\delta_1 - y > \delta_1|| : ||x|| \leq 1\} \\
&\leq ||p(S)\delta_1 - y|| < \varepsilon.
\end{aligned}$$

By lemma 2.2, $P_1 \varepsilon J$ therefore $P_1[p(S)]^* \in J$.
From

$$||P_1[p(S)]^* - T_{y,\delta_1}|| < \varepsilon$$

we see that

$$T_{y,\delta_1} \in J.$$

Since

$$\begin{aligned}||q(S)T_{y,\delta_1} - T_{y,z}|| &= \sup\{||q(S) < x, y > \delta_1 - < x, y > z|| : ||x|| \leq 1\} \\ &= \sup\{|| < x, y > (q(S)\delta_1 - z)|| : ||x|| \leq 1\} \\ &\leq ||q(S)\delta_1 - z|| \, ||y|| < \varepsilon ||y||\end{aligned}$$

from $T_{y,\delta_1} \varepsilon J$ and $||q(S)T_{y,\delta_1} - T_{y,z}|| < \varepsilon ||y||$
we see that

$$T_{y,z} \in J. \square$$

The following theorem determines the minimal ideals of $C_r^*(N^+)$.

Theorem 2.4. k, the ideal of all compact operators on $\ell^2(N^+)$ is the unique minimal closed two-sided ideal in $C_r^*(N^+)$.

Proof. Let J be any nontrivial closed two-sided ideal in $C_r^*(N^+)$. By lemma 2.3 we have $I \subseteq J$ where I is the ideal of all finite rank operators on $\ell^2(N^+)$.

Since $k = \overline{I}$ and $\overline{I} \subseteq J$ we have

$$k \subseteq J.$$

Therefore k is minimal. $\square$

3. The full C^*-algebra of $N^+, C^*(N^+)$

In this section, first we define the universal representation of N^+ then consider its full C^*-algebra, $C^*(N^+)$, and show that the C^*-algebra generated by $S^\alpha \oplus (S^*)^\beta$ has the identity element.

The existence and properties of p.p.i.'s (power partial isometries) were discussed in [6]. Therefore let V be a p.p.i. on H_V. For given m, n in N, V^m, V^n, and $V^m V^n = V^{m+n}$ are partial isometries. Thus $\sum_V = \{V^n : n \in N\}$ is a semigroup of partial isometries generated by V. The additive semigroup N^+ is generated by 1. The mapping $n \to V^n$ is a one-to-one correspondence between the semigroups N^+ and $\sum\limits_V$. These observations show that we can represent N^+ as a semigroup of partial isometries on some Hilbert space H_V. Inspired by the above facts we begin to define the represenations of N^+.

Let V be a p.p.i. on H_V. We define $\pi_V : N^+ \to PI(H_V)$ by $\pi_V(n) = V^n$. Obviously, π_V represents N^+ as a cyclic semigroup of partial isometries generated by V on H_V, or

$$\sum_V = \{V^n : n \in N^+\}.$$

We call (π_V, H_V) a <u>representation</u> of N^+. Thus for each p.p.i., say V, on H_V, we have a representation (π_V, H_V) for N^+.

Here we wish to prepare the grounds to define the C^*-algebra of N^+. We start with a lemma.

Lemma 3.1. Every unitary operator is the direct sum of unitary operators acting on separable Hilbert spaces.

Proof. Let U be a unitary on an infinite-dimensional Hilbert space H. Choose $\xi \in H$ such that $||\xi|| = 1$. With $B_0 = \{U^n \xi : n \in Z\}$, take $H_0 = \overline{span} B_0$ (The closed linear span of B_0). The Hilbert space H_0 is separable; and leaves U, U^* invariant (H_0 reduces U). Thus we have $H = H_0 \oplus H_0^{\perp}$; and with $U_0 = U|_{H_0}$, we can write $U = U_0 \oplus U'$. Now let M be the collection of all X such that X is a set of mutually orthogonal separable subspaces of H which reduce U (i.e., each element of X reduces U). Set inclusion defines a partial ordering on M. Every linearly ordered subset Y of M has an upper bound, namely, the union of elements of Y. By Zorn's lemma, M has a maximal element, say $\{H_\gamma\}$, where γ runs over some index set. Now the closed space K spanned by $\bigcup_\gamma H_\gamma$ is H. If not, we have $K^\perp = \cap H_\gamma^\perp \neq \{0\}$. Take a unit vector η in $K^\perp$ and apply the construction at the start of the proof; hence there would exist a separable Hilbert subspace in $K^\perp$, that could be added to the collection $\{H_\gamma\}$, contradicting the maximality of $\{H_\gamma\}$. Thus $K^\perp = \{0\}$; and the closed linear space generated by $\{H_\gamma\}$ is H. That is

$$H = \oplus_\gamma H_\gamma; \text{ and } U = \oplus_\gamma U_\gamma. \square$$

Remark. Let $W = \{w : w$ is a unitary operator on a separable Hilbert space$\}$, and $V = \bigoplus_{w \in W} w$. Each U_γ of the preceding lemma is an element of W. Hence $U_\gamma \approx V|_{\text{subspace} \cong H_\gamma}$; and $\oplus_\gamma U \approx \oplus_\gamma V|_{\text{subspace}} = V^\delta|_{\text{subspace} \cong \oplus_\delta H_\gamma}$ where δ is the cardinal number of the set of $\gamma's$.

The following lemma is quite useful for our purpose.

Lemma 3.2. Let A be an index set. If for $v \in A, T_v \in B(H_v)$, then $\oplus T_v^{\alpha_v} \approx \oplus_\mu(\oplus T_v)|_{\text{suitable subspace}}$, where $T_v^{\alpha_v}$ denotes T_v acting with multiplicity α_v on $\oplus_{\alpha_v} H_v$, and $\mu = \max\{\alpha_v : v \in A\}$.

Proof. Let $H = \oplus_v(\oplus_{\alpha_v} H_v), K = \oplus_\mu(\oplus_v H_v)$. With $\xi^{(v)} \in H_v$, and $(\xi_j^{(v)})_{\alpha_v} \in \oplus_{\alpha_v} H_v$, define $U : H \to K$ by letting $U((\xi_j^{(v)})_{\alpha_v})_v = ((\xi_j^{(v)})_v)_\mu$, where in $((\xi_j^{(v)})_v)_\mu$ we take $\xi_j^{(v)} = 0$ for $j \in (\alpha_v, \mu]$. Clearly U is a unitary from H onto M=range U, where
$M = \oplus_\mu(\oplus_v H_v)$ in which $H_v = \{0\}$ on $(\alpha_v, \mu]$.

If $A = \oplus_v T_v^{\alpha_v}$, and $B = \oplus_\mu(\oplus_v T_v)$, then
$U^{-1}BU((\xi_j^{(v)})_{\alpha_v})_v = U^{-1}B((\xi_j^{(v)})_v)_\mu = U^{-1}((T_v\xi_j^{(v)})_v)_\mu = ((T_v\xi_j^{(v)})_{\alpha_v})_v = A((\xi_j^{(v)})_{\alpha_v})_v$; and M is the required suitable subspace. $\square$

Now the grounds are ready for defining the $C^*(N^+)$.

Let T be a p.p.i. By the theorem of [6] and lemma 3.1

$$T \approx \oplus_\gamma U_\gamma \oplus S^\alpha \oplus (S^*)^\beta \oplus \oplus_{k=2}^\infty N_k^{\alpha_k}$$

where $U = \oplus U_\gamma$ is the unitary summand of T. By lemma 3.2 and its preceding remark

$$T \approx \oplus_\mu (V \oplus S \oplus S^* \oplus \oplus_{k=2}^\infty N_k)|_{\text{suitable subspace}}$$

in which $\mu = \sup\{\delta, \alpha, \beta, \alpha_{k's}\}$ (compare with the decomposition of a non-degenerate representation of a C^*-algebra as a direct sum of cyclic representations). We take $V \oplus S \oplus S^* \oplus \oplus_{k=2}^\infty N_k$ as the building block for representations of N^+; (in analogy with cyclic representations for C^*-algebras) Define a representation π of N^+ by

$$\pi(1) = V \oplus S \oplus S^* \oplus \oplus_{k=2}^\infty N_k.$$

we call π the universal representation of N^+. (Note that any representation of N^+ can be identified with a subrepresentation of a direct sum of copies of the universal representation). We define the C^*-algebra of N^+ as follows:

$$C^*(N^+) = C^*(V \oplus S \oplus S^* \oplus \oplus_{k=2}^\infty N_k).$$

One of the most interesting results in the amenability theory is that for an amenable group G the reduced C^*-algebra of $G, C_r^*(G)$ which is generated by the left regular representation of G is canonically isomorphic to the full C^*-algebra of $G, C^*(G)$ (see [2, Th. VII.2.8]). But in the theory of the amenable semigroups such theorem does not exist. obviously because it seems impossible to define the left regular representation for a general semigroup.

For the additive semigroup N^+, we defined the left regular representation and we constructed the C^*-algebras $C_r^*(N^+)$ and $C^*(N^+)$. By [3, sec. 4, (H)], N^+ is an amenable semigroup. Therefore the following question arises.

Is the reduced C^*-algebra of N^+ isomorphic to the full C^*-algebra of N^+?

The answer is no. To see this let us compare $C_r^*(N^+)$ with $C^*(S^\alpha \oplus (S^*)^\beta)$.

We know that $C_r^*(N^+)$ is a C^*-algebra with the identity element, simply because

$$I = S^*S \in C_r^*(N^+).$$

Therefore in comparing $C_r^*(N^+)$ with $C^*(S^\alpha \oplus (S^*)^\beta)$ the first question is:

Is $C^*(S^\alpha \oplus (S^*)^\beta)$ a C^*-algebra with the identity element?

The answer is yes, and we will prove it in the rest of this section.

First of all we require the following lemma wich may be of some intrinsic interest.

Lemma 3.3. $C^*(S^\alpha \oplus (S^*)^\beta) = C^*(S \oplus S^*)$.

Proof. Let $T = (S^*)^\beta$. The mapping

$$S^\alpha \oplus T \to S \oplus T$$

extends to the polynomials in $(S^\alpha \oplus T), (S^\alpha \oplus T)^*$ with complex coefficients, i.e., to

$p((S^\alpha \oplus T), (S^\alpha \oplus T)^*) = \sum_{\text{finite}} a_{i_1 i_2 \cdots i_k} (S^\alpha \oplus T)^{i_1} ((S^\alpha \oplus T)^*)^{i_2} \cdots ((S^\alpha \oplus T)^*)^{i_k}$.

by [8, sec. 2.6] we have

$\sum a_{i_1 i_2 \cdots i_k} (S^\alpha \oplus T)^{i_1} ((S^\alpha \oplus T)^*)^{i_2} \cdots ((S^\alpha \oplus T^*)^{i_k} =$
$(\sum a_{i_1 i_2 \cdots i_k} (S^\alpha)^{i_1} ((S^*)^\alpha)^{i_2} \cdots ((S^*)^\alpha)^{i_k}) \oplus (\sum a_{i_1 i_2 \cdots i_k} T^{i_1} (T^*)^{i_2} \cdots (T^*)^{i_k})$

$$= p(S^\alpha, (S^*)^\alpha) \oplus p(T, T^*),$$

and similarly

$$p((S \oplus T), (S \oplus T)^*) = p(S, S^*) \oplus p(T, T^*),$$

we see that

$$p(S^\alpha, (S^*)^\alpha) \oplus p(T, T^*) \to p(S, S^*) \oplus p(T, T^*).$$

Therefore the extension is an *-homomorphism. Since

$$||p(S^\alpha,(S^*)^\alpha)\oplus p(T,T^*)|| = ||\bigoplus_{\alpha\ copies} p(S,S^*)\oplus p(T,T^*)||$$

$$= ||p(S,S^*)\oplus p(T,T^*)||.$$

The mapping extends to an isometry from $C^*(S^\alpha\oplus(S^*)^\beta)$ onto $C^*(S\oplus(S^*)^\beta)$.

Similarly the mapping

$$S\oplus(S^*)^\beta \to S\oplus S^*$$

extends to an isometry from $C^*(S\oplus(S^*)^\beta))$ onto $C^*(S\oplus S^*)$. Therefore

$$C^*(S^\alpha\oplus(S^*)^\beta)\cong C^*(S\oplus(S^*)^\beta)\cong C^*(S\oplus S^*).$$

This completes the proof. □

Now we will show that $C^*(S\oplus S^*)$ has the identity element.

So far, we have seen that

$$m((S\oplus S^*),(S\oplus S^*)^*) = m(S,S^*)\oplus m(S^*,S)$$

and

$$p((S\oplus S^*),(S\oplus S^*)^*) = p(S,S^*)\oplus p(S^*,S))$$

where $m(x,y)$ is a monomial in two free variables, x,y.

Inspired by the preceding fact, the following definition is made.

Definition 3.4. We call $p(S,S^*)$, $[p(S^*,S)]$ the <u>twain</u> of $p(S^*,S)$, $[p(S,S^*)]$ in $p((S\oplus S^*),(S\oplus S^*)^*)$ and similarly $m(S,S^*)$, $[m(S^*,S)]$ is called the <u>twain</u> of $m(S^*,S)$, $[m(S,S^*)]$ in $m((S\oplus S^*),(S\oplus S^*)^*)$.

The next lemma is quite useful.

Lemma 3.5. Let $m(x,y)$ be a monomial in two free variables x,y. If $m(x,y)$ ends with y, then $m(S,S^*)$ cannot be the identity operator.

Proof. Suppose

$$m(S, S^*) = S^{i_1}(S^*)^{i_2}S^{i_3}\cdots(S^*)^{i_k}$$

where $i_k \geq 1$. Take

$$\xi = (\xi_0, \xi_1, \xi_2, \xi_3, \cdots, \xi_{i_k-1}, 0, 0, \cdots)$$

in which at least one of the $\xi_0, \xi_1, \cdots, \xi_{i_k-1}$ is different from zero. Then

$$m(S, S^*)\xi = S^{i_1}(S^*)^{i_2}S^{i_3}\cdots S^{i_k}(S^*)^{i_k}(\xi_0, \xi_1, \cdots \xi_{i_k-1}, 0, 0, \cdots)$$

$$= S^{i_1}(S^*)^{i_2}S^{i_3}\cdots S^{i_k-1}(0, 0, 0, \cdots, 0, 0, \cdots) = 0.$$

This shows that $m(S, S^*)$ is not the identity operator. Hence the lemma is proved. □

The next result, presented with proof, is an easy consequence of the above lemma.

Corollary 3.6. For every monomial $m(x, y)$ in two free variables x, y, we have

$$m(S, S^*) \oplus m(S^*, S) \neq I \oplus I.$$

Proof. If $m(S, S^*)$ ends with S^*, by 3.5 it is not the identity operator. If $m(S, S^*)$ ends with S, then its twain $m(S^*, S)$ will end with S^* and consequently is not the identity operator. Thus in any case

$$m(S, S^*) \oplus m(S^*, S) \neq I \oplus I.$$

and this completes the proof. □

Now S^*S is the identity operator, but its twain i.e., SS^* is not. Therefore if we want the twain of a monomial to become an identity at least it must end with S^*. For example SS^*SS^* is a monomial whose twain i.e., S^*SS^*S is the identity operator.

It is obvious that the polynomial

$$S^*S + SS^*SS^*$$

is not the identity operator, but a part of its twain i.e., S^*SS^*S is the identity. Thus if we can add a monomial $m(S, S^*)$ to $S^*S + SS^*SS^*$ such that

(a) $S^*S + SS^*SS^* + m(S, S^*) = S^*S$

and

(b) $SS^* + S^*SS^*S + m(S^*, S) = S^*SS^*S$

then we will have

$$(S^*S + SS^*SS^* + m(S, S^*)) \oplus (SS^* + S^*SS^*S + m(S^*, S)) = I \oplus I.$$

By (a) and (b) we must find $m(S, S^*)$ such that

$$\begin{cases} SS^*SS^* + m(S, S^*) = 0 \\ SS^* + m(S^*, S) = 0 \end{cases}$$

Since

$$SS^*SS^*(\xi_0, \xi_1, \xi_2, \cdots) = (0, \xi_1, \xi_2, \xi_3, \cdots)$$

and

$$S(S^*)^2S(\xi_0, \xi_1, \xi_2, \cdots) = (0, \xi_1, \xi_2, \xi_3, \cdots)$$

if we take $m(S, S^*) = -S(S^*)^2S$ then

$$SS^*SS^* - S(S^*)^2S = 0$$

and

$$SS^* - S^*S^2S^* = 0$$

This shows that

$$(S^*S + SS^*SS^* - S(S^*)^2S) \oplus (SS^* + S^*SS^*S - S^*S^2S^*) = I \oplus I.$$

We can summarize our results in the following theorem.

Theorem 3.7. The C^*-algebra $C^*(S \oplus S^*)$ has the identity element.

4. The minimal ideals of $C^*(S^\alpha \oplus (S^*)^\beta)$

In section 3 we defined the full C^*-algebra of $N^+, C^*(N^+)$; and that is the C^*-algebra generated by

$$V \oplus S \oplus S^* \oplus \oplus_{k=2}^{\infty} N_k.$$

In this section we will determine the minimal ideals of $C^*(S \oplus S^*)$, and will prove that

$$C_r^*(N^+) \neq C^*(N^+).$$

Remember that we continue to assume that S is the unilateral shift operator and S^*, the adjoint of S is the backward shift operator.

We start with the following lemma.

Lemma 4.1. The C^*-algebra $C^*(S \oplus S^*)$ has a nontrivial closed two-sided ideal.

Proof. The following simple calculations show that

$$(k \oplus k) \cap C^*(S \oplus S^*)$$

is non empty.

$$(S \oplus S^*)^*(S \oplus S^*) - (S \oplus S^*)(S \oplus S^*)^* \in C^*(S \oplus S^*).$$

On the other hand by [8, sec. 2.6] we have

$$(S \oplus S^*)^*(S \oplus S^*) - (S \oplus S^*)(S \oplus S^*)^* = (S^*S \oplus SS^*) - (SS^* \oplus S^*S)$$
$$= (S^*S - SS^*) \oplus (SS^* - S^*S),$$

since

$$(S^*S - SS^*) \oplus (SS^* - S^*S) = P_0 \oplus (-P_0) \in k \oplus k$$

where k is the ideal of all compact operators on $\ell^2(Z^+)$, we see that

$$P_0 \oplus (-P_0) \in (k \oplus k) \cap C^*(S \oplus S^*).$$

Therefore at least $(k \oplus k) \cap C^*(S \oplus S^*)$ is a nontrivial closed two-sided ideal in $C^*(S \oplus S^*)$. □

The following lemma is quite useful for our purpose.

Lemma 4.2. If J is a nontrivial closed two-sided ideal in $C^*(S \oplus S^*)$ then

$$P_0 \oplus 0 \in J \text{ or } 0 \oplus P_0 \in J.$$

Proof. For $m \geq 0$ by [8, sec. 2.6] we have

$$(S \oplus S^*)^m (P_0 \oplus (-P_0))((S \oplus S^*)^*)^m = S^m P_0 (S^*)^m \oplus (S^*)^m (-P_0) S^m$$

$$= P_m \oplus 0 \in C^*(S \oplus S^*)$$

and

$$((S \oplus S^*)^*)^m (-P_0 \oplus P_0)(S \oplus S^*)^m = (S^*)^m (-P_0) S^m \oplus S^m P_0 (S^*)^m$$

$$= 0 \oplus P_m \in C^*(S \oplus S^*).$$

Now let J be a nontrivial closed two-sided ideal in $C^*(S \oplus S^*)$. Hence there exists a non-zero element, say C, in J. Let $C = A \oplus B$. If $A \neq 0$, then for some $N \geq 0$ we have $A\delta_N \neq 0$. Since $A \oplus B \in J, P_N \oplus 0 \in C^*(S \oplus S^*)$ and J is an ideal we see that

$$(P_N \oplus 0)(A \oplus B)^*(A \oplus B)(P_N \oplus 0) = P_N A^* A P_N \oplus 0 \in J.$$

By the similar argument used in the proof of lemma 2.2, we have

$$P_N A^* A P_N = ||A\delta_N||^2 P_N.$$

Hence

$$P_N \oplus 0 = ||A\delta_N||^{-2} P_N A^* A P_N \oplus 0 \in J$$

Thus

$$((S \oplus S^*)^*)^N (P_N \oplus 0)(S \oplus S^*)^N = (S^*)^N P_N S^N \oplus 0 = P_0 \oplus 0 \in J.$$

If $B \neq 0$ similar argument shows tht

$$0 \oplus P_0 \in J.$$

This completes the proof. □

The following theorem determines the minimal ideals of $C^*(S \oplus S^*)$.

Theorem 4.3. $k \oplus 0$ and $0 \oplus k$ are two minimal closed two-sided ideals in $C^*(S \oplus S^*)$.

Proof. Let J be a nontrivial closed two-sided ideal in $C^*(S \oplus S^*)$. Since S and S^* have cyclic vectors by an argument similar to the proof of lemma 2.3, we see that J contains either all operators of the form $T_{y,z} \oplus 0$ or all operators of the form

$$0 \oplus T_{y,z}$$

where $y, z \in \ell^2(Z^+)$, and

$$T_{y,z}(x) =< x, y > z.$$

Since k is the norm closure of the ideal of all finite rank operators, we see that
either $k \oplus 0 \subseteq J$ or $0 \oplus k \subseteq J$. Since J was arbitrary, the theorem is proved. □

Now it is time for making a comparison between the reduced C^*-algebra of N^+ and the full C^*-algebra of N^+ and get an important conclusion.

Corollary 4.4. $C_r^*(N^+) \neq C^*(N^+)$.

Proof. $C^*(N^+) = C^*(V \oplus S \oplus S^* \oplus \oplus_{k=2}^{\infty} N_k)$. Hence $C^*(N^+)$ is a C^*-subalgebra of $B(H)$ where $H = H_V \oplus H_S \oplus H_{S^*} \oplus \oplus_{k=2}^{\infty} H_{N_k}$. The inclusion mapping from $C^*(N^+)$ into $B(H)$ is a faithful representation of $C^*(N^+)$ on H. For each $A \in C^*(N^+)$, the mapping

$$A \to A|_{H_S \oplus H_{S^*}} : C^*(N^+) \to B(H_S \oplus H_{S^*})$$

is a representation of $C^*(N^+)$ on $H_S \oplus H_{S^*}$. By this representation the generator of $C^*(N^+)$ is mapped to $S \oplus S^*$. Hence

$$(*) \qquad C^*(S \oplus S^*) \cong \text{ a quotient of } C^*(N^+).$$

By 4.3 the left hand side of (*), and consequently $C^*(N^+)$ has at least two minimal ideals. Since by 2.4 $C_r^*(N^+)$ has a unique minimal ideal, the proof is complete. □

We can summarize the result of this section in the following conclusion.

Conclusion 4.5. The well known theorem ([12, Th. 6.3.3]) which says that, if G is an amenable group, then

$$C_r^*(G) \cong C^*(G)$$

does not hold for the amenable semigroup N^+.

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