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INTUITIONISTIC FUZZY NORMAL SUBGROUPS AND INTUITIONISTIC FUZZY COSETS

KUL HUR [a*], SU YEON JANG[a] AND HEE WON KANG [b]

Abstract. We study some properties of intuitionistic fuzzy normal subgroups of a group. In particular, we obtain two characterizations of intuitionistic fuzzy normal subgroups. Moreover, we introduce the concept of an intuitionistic fuzzy coset and obtain several results which are analogous of some basic theorems of group theory.

0. Introduction

The concept of a fuzzy set was introduced by Zadeh in [17], and since then these has been a tremendous interest in the subject due to its diverse applications ranging from engineering and computer science to social behavior studies. In particular, several researchers [6, 7, 14, 15, 16] applied the notion of a fuzzy set to group theory.

In 1986, Atanassov[1] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. After that time, Çoker and his colleagues [4, 5, 8], and Lee and Lee[13] applied the notion of intuitionistic fuzzy sets to topology. Also, several researchers [2, 3, 10, 11, 12] applied one to algebra.

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* Corresponding auther.

In this paper, we investigate some properties of intuitionistic fuzzy normal subgroups of a group. In particular, we obtain two characterizations of intuitionistic fuzzy normal subgroups. Moreover, we introduce the concept of intuitionistic fuzzy cosets and obtain several results which are analogs of some basic theorems of group theory.

1. Preliminaries

We will list some concepts and results needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I.

Definition 1.1[1]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) on X if $\mu_A + \nu_A \leq 1$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2[1]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\,]A = (\mu_A, 1 - \mu_A)$, $< > A = (1 - \nu_A, \nu_A)$.

Definition 1.3[1]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.

(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4[4]. $0_{\sim} = (0, 1)$ and $1_{\sim} = (1, 0)$.

Definition 1.5[4]. Let X and Y be nonempty sets and let $f : X \to Y$ be a mapping. Let $A = (\mu_A, \nu_A)$ be an IFS in X and $B = (\mu_B, \nu_B)$ be an IFS in Y. Then

(1) the *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by:

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\nu_B)),$$

where $f^{-1}(\mu_B) = \mu_B \circ f$ and $f^{-1}(\nu_B) = \nu_B \circ f$.

(2) the *image* of A under f, denoted by $f(A)$, is the IFS in Y defined by:

$$f(A) = (f(\mu_A), f(\nu_A)),$$

where for each $y \in Y$

$$f(\mu_A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu_A(x) & if f^{-1}(y) \neq \emptyset, \\ 0 & if f^{-1}(y) = \emptyset \end{cases}$$

and

$$f(\nu_A)(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \nu_A(x) & if f^{-1}(y) \neq \emptyset, \\ 1 & if f^{-1}(y) = \emptyset. \end{cases}$$

Definition 1.6[11]. Let $(G, \cdot)$ be a groupoid and let $A \in IFS(X)$. Then A is called an *intuitionistic fuzzy subgroupoid* (in short, $IFGP$) of G if for any $x, y \in G$, $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

We will denote the set of all IFGPs of G as IFGP(G).

Definition 1.7[12]. Let G be a group and let $A \in IFGP(G)$. Then A is called an *intuitionistic fuzzy subgroup* (in short, IFG) of G if $A(x^{-1}) \geq A(x)$, i.e., $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$, for each $x \in G$.

We will denote the set of all IFGs of G as IFG(G).

Result 1.A[12, Proposition 2.6]. Let $A \in$ IFG(G). Then $A(x^{-1}) = A(x)$, i.e., $\mu_A(x^{-1}) = \mu_A(x), \nu_A(x^{-1}) = \nu_A(x)$ and $A(x) \leq A(e)$, i.e., $\mu_A(x) \leq \mu_A(e), \nu_A(x) \geq \nu_A(e)$ for each $x \in G$, where e is the identity element of G.

Result 1.B[12, Proposition 2.8]. Let $A \in$ IFG(G). If $A(xy^{-1}) = A(e)$, for any $x, y \in G$, then $A(x) = A(y)$.

Definition 1.8. Let A be an IFS in a groupoid G. Then A is said to have *the sup property* if for any $T \in P(G)$, there exists a $t_0 \in T$ such that $A(t_0) = \bigcup_{t \in T} A(t)$, i.e., $\mu_A(t_0) = \bigvee_{t \in T} \mu_A(t)$ and $\nu_A(t_0) = \bigwedge_{t \in T} \nu_A(t)$, where $P(G)$ denotes the power set of G.

Result 1.C[12, Proposition 2.13]. Let $f : G \to G'$ be a group epimorphism, let $A \in$ IFG(G) and let $B \in$ IFG(G′). Then the followings hold:

(1) If A has the sup property, then $f(A) \in$ IFG(G′).

(2) $f^{-1}(B) \in$ IFG(G).

Definition 1.9[11]. Let A be an IFS in a set X and let $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$. Then the set $A^{(\lambda,\mu)} = \{x \in X : \mu_A(x) \geq \lambda \text{ and } \nu_A(x) \leq \mu\}$ is called a (λ, μ)-*level subset* of A.

Result 1.D[12, Proposition 2.18 and Proposition 2.19]. Let A be an IFS in a group G. Then $A \in$ IFG(G) if and only if for each $(\lambda, \mu) \in \mathrm{Im}A$ with $\lambda \leq \mu_A(e)$ and $\mu \geq \nu_A(e)$, $A^{(\lambda,\mu)}$ is a subgroup of G.

Let A be an IFG of a group G. Then for each $(t, s) \in I \times I$ with $A(e) \geq (t, s)$, i.e., $\mu_A(e) \geq t$ and $\nu_A(e) \leq s$, the level set $A^{(t,s)}$ is a subgroup of G. If $\mathrm{Im}A = \{(t_0, s_0), (t_1, s_1), \cdots, (t_n, s_n)\}$, the family of level subgroups

$$\{A^{(t_i,s_i)} : 0 \leq i \leq n\}$$

consititutes the complete list of level subgroups of A.If the image set of the IFG A of a finite group G consists of $\{(t_0, s_0), (t_1, s_1), \cdots (t_n, s_n)\}$, where $t_0 > t_1 > \cdots > t_n$ and $s_0 < s_1 < \cdots < s_n$, then the level subgroups of A form a chain :

$$A^{(t_0,s_0)} \subset A^{(t_1,s_1)} \subset \cdots \subset A^{(t_n,s_n)} = G$$

where $A(e) = (t_0, s_0)$.

Notation. $N \lhd G$ denotes that N is a normal subgroup of a group G.

2. Intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets

Lemma 2.1. If A is an IFGP of a finite group G, then A is an IFG of G.

Proof. Let $x \in G$. Since G is finite, x has finite order, say n. Then $x^n = e$, where e is the identity of G. Thus $x^{-1} = x^{n-1}$.Since A is an IFGP of G,

$$\mu_A(x^{-1}) = \mu_A(x^{n-1}) = \mu_A(x^{n-2}x) \geq \mu_A(x)$$

and

$$\nu_A(x^{-1}) = \nu_A(x^{n-1}) = \nu_A(x^{n-2}x) \leq \nu_A(x).$$

Hence A is an IFG of G.

Lemma 2.2. Let A be an IFG of a group G and let $x \in G$. Then $A(xy) = A(y)$ for each $y \in G$ if and only if $A(x) = A(e)$.

Proof. ($\Rightarrow$): Suppose $A(xy) = A(y)$ for each $y \in G$. Then clearly $A(x) = A(e)$.

($\Leftarrow$): Suppose $A(x) = A(e)$ and let $y \in G$. Then, by Result 1.A, $\mu_A(y) \leq \mu_A(x)$ and $\nu_A(y) \geq \nu_A(x)$. Since $A \in IFG(G)$, $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$. Thus $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$. On the other hand, by Result 1.A,

$$\mu_A(y) = \mu_A(x^{-1}xy) \geq \mu_A(x) \wedge \mu_A(xy) = \mu_A(xy)$$

and

$$\nu_A(y) = \nu_A(x^{-1}xy) \leq \nu_A(x) \vee \nu_A(xy) = \nu_A(xy).$$

Hence $\mu_A(xy) = \mu_A(y)$ and $\nu_A(xy) = \nu_A(y)$ for each $y \in G$.

Remark 2.3. It is easy to see that if $A(x) = A(e)$, then $A(xy) = A(yx)$ for each $y \in G$.

Definition 2.4. Let A be an IFG of a group G and let $x \in G$. We define a complex mappings

$Ax = (\mu_{Ax}, \nu_{Ax}) : G \to I \times I$

and

$xA = (\mu_{xA}, \nu_{xA}) : G \to I \times I$

as follows respectively: for each $g \in G$,

$$Ax(g) = A(gx^{-1}) \text{ and } xA(g) = A(x^{-1}g).$$

Then Ax [resp. xA] is called the *intuitionistic fuzzy right* [resp. *left*] *coset* of G determined by x and A.

Remark 2.5. Definition 2.4 extends in a natural way the usual definition of a "coset" of a group. This is seen as follows : Let H be a subgroup of a group G and let $A = (\chi_H,\ \chi_{H^c})$, where χ_H is the characteristic function of H. Let $x, g \in G$. Then $Ax = (\chi_{Hx},\ \chi_{Hx^c})$.

Suppose $g \in H$. Then

$$\begin{aligned} Ax(gx) &= (\chi_{Hx}(gx),\ \chi_{Hx^c}(gx)) \\ &= (\chi_H(gxx^{-1}),\ \chi_{H^c}(gxx^{-1})) \\ &= (\chi_H(g),\ \chi_{H^c}(g)) \\ &= (1,\ 0) \\ &= 1_\sim. \end{aligned}$$

Suppose $g \notin H$. Then :

$$\begin{aligned} Ax(gx) &= (\chi_{H_x}(gx),\ \chi_{Hx^c}(gx)) \\ &= (\chi_H(gxx^{-1}),\ \chi_{H^c}(gxx^{-1})) \\ &= (\chi_H(g),\ \chi_{H^c}(g)) \\ &= (0,\ 1) \\ &= 0_\sim. \end{aligned}$$

So, it follows that $Ax : G \to I \times I$ is a complex mapping such that $Ax\mid_{Hx} = 1_\sim$ and $Ax \mid_{Hx^c} = 0_\sim$. Hence $Ax = (\chi_{Hx},\ \chi_{Hx^c})$.

The following is the immediate result of Definition 2.4:

Proposition 2.6. Let A be an IFG of a group G. Then:

(1) $(xy)A = x(yA)$.

(2) $A(xy) = (Ax)y$.

(3) $xA = A$ if $A(x) = 1_\sim$.

We know that any two left [resp. right] cosets of a subgroup H of a group G are equal or disjoint. However this fact is not valid in the intuitionistic fuzzy case as shown in the following example.

Example 2.7. Let $G = \{e = a, b, c, d\}$ be the Klein's four group and let A be the IFG of G defined by:

$A(a) = 1_\sim,\ A(b) = (t_1, 1-t_1),\ A(c) = A(d) = (t_2, 1-t_2)$

where $1 > t_1 \leq t_2$ ($t_2 \neq 0$). Then $bA \neq cA$.

Definition 2.8[11]. Let A be an IFG of a group G. Then A is called an *intuitionistic fuzzy normal subgroup* (in short, $IFNG$) if $A(xy) = A(yx)$ for any $x, y \in G$.

We will denote the set of all IFNGs of G as IFNG(G).

The following is the immediate result of Definition 2.4, Definition 2.8:

Proposition 2.9. Let A be an IFG of a group G. Then the followings are equivalent:

(1) $\mu_A(xyx^{-1}) \geq \mu_A(y)$ and $\nu_A(xyx^{-1}) \leq \nu_A(y)$ for any $x, y \in G$.

(2) $A(xyx^{-1}) = A(y)$ for any $x, y \in G$.

(3) $A \in$ IFNG(G).

(4) $xA = Ax$ for each $x \in G$.

(5) $xAx^{-1} = A$ for each $x \in G$.

Remark 2.10. Let G be a group.

(1) If μ_A is a fuzzy normal subgroup of G, then $A = (\mu_A, \mu_A^c) \in$ IFNG(G).

(2) If $A \in$ IFNG(G), then μ_A and ν_A^c are fuzzy normal subgroups of G.

(3) If $A \in$ IFNG(G), then $[\ ]A, <\ > A \in$ IFNG(G).

Let G be a group and $a, b \in G$. We say that a is *conjugate* to b if there exists $x \in G$ such that $b = x^{-1}ax$. It is well-known that conjugacy is an equivalence relation on G. The equivalence classes in G under the relation of conjugacy are called *conjugate classes* [8].

Theorem 2.11. Let A be an IFG of a group G. Then $A \in$ IFNG(G) if and only if A is constant on the conjugate classes of G.

Proof. ($\Rightarrow$): Suppose $A \in$ IFNG(G) and let $x, y \in G$. Then $A(y^{-1}xy) = A(xyy^{-1}) = A(x)$. Hence A is constant on the conjugate classes.

($\Leftarrow$): Suppose the necessary condition holds and let $x, y \in G$. Then $A(xy) = A(xyxx^{-1}) = A(x(yx)x^{-1}) = A(yx)$. Hence $A \in$ IFNG(G).

Let G be a group and let $x, y \in G$. Then the element $x^{-1}y^{-1}xy$ is usually denoted by $[x, y]$ and called the *commutator* of x and y. It is clear that if x and y commute with each other, then clearly $[x, y] = e$.

Let H and K be two subgroups of a group G. Then the subgroup $[H, K]$ is defined as the subgroup generated by the elements

$$\{[x, y] : x \in H, y \in K\}.$$

It is well-known that $N \lhd G$ if and only if $[N, G] \leq N$.

The following is the generalization of the above result using intuitionistic fuzzy sets.

Theorem 2.12. Let A be an IFG of a group G. Then $A \in$ IFNG(G) if and only if $\mu_A([x,y]) \geq \mu_A(x)$ and $\nu_A([x,y]) \leq \nu_A(x)$ for any $x, y \in G$.

Proof. ($\Rightarrow$): Suppose $A \in$ IFNG(G) and let $x, y \in G$. Then:

$$\begin{aligned}
\mu_A([x,y]) &= \mu_A(x^{-1}y^{-1}xy) \\
&= \mu_A(y^{-1}xyx^{-1}) \text{ (By the hypothesis)} \\
&\geq \mu_A(y^{-1}xy) \wedge \mu_A(x^{-1}) \text{ (Since } A \in \text{IFG(G))} \\
&= \mu_A(x) \wedge \mu_A(x) \text{ (By Theorem 2.11 and Result 1.A)} \\
&= \mu_A(x)
\end{aligned}$$

and

$$\begin{aligned}
\nu_A([x,y]) &= \nu_A(x^{-1}y^{-1}xy) \\
&= \nu_A(y^{-1}xyx^{-1}) \\
&\leq \nu_A(y^{-1}xy) \vee \nu_A(x^{-1}) \\
&= \nu_A(x) \vee \nu_A(x) \\
&= \nu_A(x).
\end{aligned}$$

Hence the necessary conditions hold.

($\Leftarrow$): Suppose the necessary conditions hold and let $x, z \in G$. Then:

$$\begin{aligned}
\mu_A(x^{-1}zx) &= \mu_A(zz^{-1}x^{-1}zx) \\
&\geq \mu_A(z) \wedge \mu_A([z,x]) \text{ (Since } A \in \text{IFG(G))} \\
&= \mu_A(z) \text{ (By the hypothesis)}
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(x^{-1}zx) &= \nu_A(zz^{-1}x^{-1}zx) \\
&\leq \nu_A(z) \vee \nu_A([z,x]) \\
&= \nu_A(z).
\end{aligned}$$

On the other hand,

$$\begin{aligned}\mu_A(z) &= \mu_A(xx^{-1}zxx^{-1}) \\ &\geq \mu_A(x) \wedge \mu_A(x^{-1}zx) \wedge \mu_A(x^{-1}) \text{ (Since } A \in \text{IFG(G))} \\ &= \mu_A(x) \wedge \mu_A(x^{-1}zx) \text{ (By Result 1.A)}\end{aligned}$$

and

$$\begin{aligned}\nu_A(z) &= \nu_A(xx^{-1}zxx^{-1}) \\ &\leq \nu_A(x) \vee \nu_A(x^{-1}zx) \vee \nu_A(x^{-1}) \\ &\leq \nu_A(x) \vee \nu_A(x^{-1}zx).\end{aligned}$$

Case (i): Suppose $\mu_A(x) \wedge \mu_A(x^{-1}zx) = \mu_A(x)$ and $\nu_A(x) \vee \nu_A(x^{-1}zx) = \nu_A(x)$. Then $\mu_A(z) \geq \mu_A(x)$ and $\nu_A(z) \leq \nu_A(x)$ for any $x, z \in G$. Thus A is a constant mapping. So $A(xy) = A(yx)$ for any $x, y \in G$, i.e., $A \in$ IFNG(G).

Case (ii): Suppose $\mu_A(x) \wedge \mu_A(x^{-1}zx) = \mu_A(x^{-1}zx)$ and $\nu_A(x) \vee \nu_A(x^{-1}zx) = \nu_A(x^{-1}zx)$. Then $\mu_A(z) \geq \mu_A(x^{-1}zx)$ and $\nu_A(z) \leq \nu_A(x^{-1}zx)$ for any $x, z \in G$. Thus $\mu_A(x^{-1}zx) = \mu_A(z)$ and $\nu_A(x^{-1}zx) = \nu_A(z)$, i.e., $A(x^{-1}zx) = A(z)$ for any $x, z \in G$. So A is constant on the conjugate classes. By Theorem 2.11, $A \in$ IFNG(G). Hence, in either cases, $A \in$ IFNG(G). This completes the proof.

Proposition 2.13. Let A be an IFNG of a group G and let $(\lambda, \mu) \in I \times I$ such that $\lambda \leq \mu_A(e), \mu \geq \nu_A(e)$ and $\lambda + \mu \leq 1$, where e denotes the identity of G. Then $A^{(\lambda,\mu)} \lhd G$.

Proof. By Result 1.D, $A^{(\lambda,\mu)}$ is a subgroup of G. Let $x \in A^{(\lambda,\mu)}$ and let $z \in G$. Since $A \in$ IFNG(G), by Proposition 2.9(2), $A(z^{-1}xz) = A(x)$. Since $x \in A^{(\lambda,\mu)}$, $\mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \mu$. Thus $\mu_A(z^{-1}xz) \geq \lambda$ and $\nu_A(z^{-1}xz) \leq \mu$. So $z^{-1}xz \in A^{(\lambda,\mu)}$. Hence $A^{(\lambda,\mu)} \lhd G$.

Let A be an IFNG of a finite group G with Im$A = \{(t_0, s_0), (t_1, s_1), \cdots, (t_r, s_r)\}$, where $t_0 > t_1 > \cdots > t_r$ and $s_0 < s_1 < \cdots < s_r$. Then it follows from Theorem 2.7 that the level subgroups of A form a chain of normal subgroups:

$$A^{(t_0,s_0)} \subset A^{(t_1,s_1)} \subset \cdots \subset A^{(t_r,s_r)} = G. \ (*)$$

The following is the immediate result of Proposition 2.13:

Corollary 2.13[11, Proposition 3.5]. Let A be an IFNG of a group G with identity e. Then $G_A \lhd G$, where $G_A = \{x \in G : A(x) = A(e)\}$.

The following is the converse of Proposition 2.13:

Proposition 2.14. If A is an IFG of a finite group G such that all the level subgroups of A are normal in G, then $A \in$ IFNG(G).

Proof. Let Im$A = \{(t_0, s_0), (t_1, s_1), \cdots, (t_r, s_r)\}$, where $t_0 > t_1 > \cdots > t_r$ and $s_0 < s_1 < \cdots < s_r$. Then the family $\{A^{(t_i,s_i)} : 0 \leq i \leq r\}$ is the complete set of level subgroups of G. By the hypothesis, $A^{(t_i,s_i)} \lhd G$ for each $0 \leq i \leq r$. From the definition of the level subgroup, it is clear that

$$A^{(t_i,s_i)} \setminus A^{(t_{i-1},s_{i-1})} = \{x \in G : A(x) = (t_i, s_i)\}.$$

Since a normal subgroup of a group is a complete union of conjugate classes, it follows that in the given chain $(*)$ of normal subgroups, each $A^{(t_i,s_i)} \setminus A^{(t_{i-1},s_{i-1})}$ is a union of some conjugate classes. Since A is constant on $A^{(t_i,s_i)} \setminus A^{(t_{i-1},s_{i-1})}$, it follows that A must be constant on each conjugate class of G. Hence, by Theorem 2.11, $A \in$ IFNG(G).

Example 2.15. Let G be the group of all symmetries of a square. Then G is a group of order 8 generated by a rotation through $\pi/2$ and a reflection along a diagonal of the square. Let us denote the elements of G by $\{1, 2, 3, 4, 5, 6, 7, 8\}$, where 1 is the identity, 2 is rotation through

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	1	6	7	8	5
3	3	4	1	2	7	8	5	6
4	4	1	2	3	8	5	6	7
5	5	8	7	6	1	4	3	2
6	6	5	8	7	2	1	4	3
7	7	6	5	8	3	2	1	4
8	8	7	6	5	4	3	2	1

Table 1

$\pi/2$, and 5 is a reflection along a diagonal: the multiplication table of G is as shown in Table 1.

We can easily see that the conjugate classes of G are

$$\{1\}, \{3\}, \{5, 7\}, \{6, 8\}, \{2, 4\}.$$

Let $H = \{1, 3\}$ and let $K = \{1, 2, 3, 4\}$. Then clearly, $H \lhd G$ and $K \lhd G$ (in fact, H is the center of G). Thus we have a chain of normal subgroups given by

$$1 \subset H \subset K \subset G. \ (**)$$

Now we will construct an IFG of G whose level subgroups are precisely the members of the chain $(**)$. Let $(t_i, s_i) \in I \times I$, $0 \leq i \leq 3$ such that $t_i + s_i \leq 1$, $t_0 > t_1 > t_2 > t_3$ and $s_0 < s_1 < s_2 < s_3$.

Define a complex mapping $A = (\mu_A, \nu_A) : G \longrightarrow I \times I$ as follows:

$$A(1) = (t_0, s_0), A(H \backslash \{1\}) = (t_1, s_1), A(K \backslash H) = (t_2, s_2), A(G \backslash K) = (t_3, s_3).$$

From the definition of A, it is clear that $A(x) = A(x^{-1})$ for each $x \in G$. Also, we can easily check that for any $x, y \in G$

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \text{ and } \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y).$$

Furthermore, it is clear that A is constant on the conjugate classes. Hence, by Theorem 2.11, $A \in$ IFNG(G).

Remark 2.16. Example 2.15 is the generalization of Example 3.10 in [14] using intuitionistic fuzzy sets.

The following can be easily proved and the proof is omitted:

Lemma 2.17. Let A be an IFG of a group and let $x \in G$. Then $A(x) = (\lambda, \mu)$ if and only if $x \in A^{(\lambda,\mu)}$ and $x \notin A^{(t,s)}$ for each $(t, s) \in I \times I$ such that $t + s \leq 1, t > \lambda$ and $s < \mu$.

It is well-known that if N is a normal subgroup of a group G, then $xy \in N$ if and only if $yx \in N$ for any $x, y \in G$.

The following result is the generalization of Proposition 2.14:

Proposition 2.18. Let A be an IFG of a group G. If $A^{(\lambda,\mu)}$, $(\lambda, \mu) \in$ ImA, is a normal subgroup of G, then $A \in$ IFNG(G).

Proof. For any $x, y \in G$, let $A(x, y) = (\lambda, \mu)$ and let $A(xy) = (t, s)$ be such that $t > \lambda$ and $s < \mu$. Then, by Lemma 2.17, $xy \in A^{(\lambda,\mu)}$ and $xy \notin A^{(t,s)}$. Thus $yx \in A^{(\lambda,\mu)}$ and $yx \notin A^{(t,s)}$. So $A(yx) = (\lambda, \mu)$, i.e.,$A(xy) = A(yx)$. Hence $A \in$ IFNG(G).

Proposition 2.19. Let $f : X \to Y$ be a groupoid homomorphism. If $A \in$ IFGP(X), then $f(A) \in$ IFGP(Y).

Proof. For each $y \in Y$, let $X_y = f^{-1}(y)$. Since f is a homomorphism, it is clear that

$$X_y X_y' \subset X_y y' \text{ for any } y, y' \in Y. \ (***)$$

Let $y, y' \in Y$.

Case (i): Suppose $yy' \in f(X)$. Then clearly $f(A)(yy') = 0_{\sim}$. Since $yy' \notin f(X), X_{yy'} = \emptyset$. By $(***)$, $X_y = \emptyset$ or $X'_y = \emptyset$. Thus $f(A)(y) = 0_{\sim}$ or $f(A)(y') = 0_{\sim}$. So

$$f(A)(yy') = 0_{\sim} = (\mu_{f(A)}(y) \wedge \mu_{f(A)}(y'), \nu_{f(A)}(y) \vee \nu_{f(A)}(y')).$$

Case(ii): Suppose $yy' \in f(X)$. Then $X_{yy'} \neq \emptyset$. If $X_y = \emptyset$ or $X'_y = \emptyset$, then $f(A)(y) = 0_{\sim}$ and $f(A)(y') = 0_{\sim}$. Thus

$$\mu_{f(A)}(yy') \geq \mu_{f(A)}(y) \wedge \mu_{f(A)}(y') \text{ and } \nu_{f(A)}(yy') \leq \nu_{f(A)}(y) \vee \nu_{f(A)}(y').$$

If $X_y \neq \emptyset$ and $X'_y \neq \emptyset$, then, by $(***)$,

$$\begin{aligned}
\mu_{f(A)}(yy') &= \bigvee_{z \in X_y y'} \mu_A(z) \geq \bigvee_{z \in X_y X'_y} \mu_A(z) = \bigvee_{z \in X_y, x' \in X'_y} \mu_A(xx') \\
&\geq \bigvee_{x \in X_y, x' \in X'_y} [\mu_A(x) \wedge \mu_A(x')] \text{ (Since } A \in \text{IFGP(X))} \\
&= (\bigvee_{x \in X_y} \mu_A(x)) \wedge (\bigvee_{x' \in X'_y} \mu_A(x')) \\
&= \mu_{f(A)}(y) \wedge \mu_{f(A)}(y')
\end{aligned}$$

and

$$\begin{aligned}
\nu_{f(A)}(yy') &= \bigwedge_{z \in X_y y'} \nu_A(z) \leq \bigwedge_{z \in X_y X'_y} \nu_A(z) = \bigwedge_{z \in X_y, x' \in X'_y} \nu_A(xx') \\
&\leq \bigwedge_{x \in X_y, x' \in X'_y} [\nu_A(x) \vee \nu_A(x')] \\
&= (\bigwedge_{x \in X_y} \nu_A(x)) \vee (\bigwedge_{x' \in X'_y} \nu_A(x')) \\
&= \nu_{f(A)}(y) \vee \nu_{f(A)}(y').
\end{aligned}$$

Consequently, $\mu_{f(A)}(yy') \geq \mu_{f(A)}(y) \wedge \mu_{f(A)}(y')$ and $\nu_{f(A)}(yy') \leq \nu_{f(A)}(y) \vee \nu_{f(A)}(y')$. Hence $f(A) \in$ IFGP(Y).

Proposition 2.20. Let $f : X \to Y$ be a group [resp. ring, algebra and field] homomorphism. If $A \in$ IFG(X) [resp. IFR(X), IFA(X) and IFF(X)], then $f(A) \in$ IFG(Y) [resp. IFR(Y), IFA(Y) and IFF(Y)], where IFG(X) [resp. IFR(X), IFA(X) and IFF(X)] denotes the set of all intuitionistic fuzzy subgroups [resp. subrings, subslgebras and subfields] of a group [resp. ring, algebra and field] X.

Proof. Suppose $f : X \to Y$ is a group homomorphism and let $A \in$ IFG(X). Then, by Proposition 2.15, we need only to show that $\mu_{f(A)}(y^{-1}) \geq \mu_{f(A)}(y)$ and $\nu_{f(A)}(y^{-1}) \leq \nu_{f(A)}(y)$ for each $y \in Y$. Let $y \in Y$.

Case(i): Suppose $y^{-1} \notin f(X)$. Then $y \notin f(X)$. Thus

$$f(A)(y^{-1}) = 0_{\sim} = f(A)(y).$$

Case(ii): Suppose $y^{-1} \in f(X)$. Then $y \in f(X)$. Thus

$$\mu_{f(A)}(y^{-1}) = \bigvee_{t^{-1} \in f^{-1}(y^{-1})} \mu_A(t^{-1}) \geq \bigvee_{t \in f^{-1}(y)} \mu_A(t) = \mu_{f(A)}(y)$$

and

$$\nu_{f(A)}(y^{-1}) = \bigwedge_{t^{-1} \in f^{-1}(y^{-1})} \nu_A(t^{-1}) \leq \bigwedge_{t \in f^{-1}(y)} \nu_A(t) = \nu_{f(A)}(y).$$

Hence $f(A) \in$ IFG(Y). The proofs of the rest are omitted. This completes the proof.

Another proof : Let $(\lambda, \mu) \in \mathrm{Im} f(A)$. Then there exists a $y \in Y$ such that

$$f(A)(y) = (\bigvee_{x \in f^{-1}(y} \mu_A(x), \bigwedge_{x \in f^{-1}(y} \nu_A(x)) = (\lambda, \mu).$$

Since $A \in$ IFG(X), by Result 1.A, $\lambda \leq \mu_A(e)$ and $\mu \geq \nu_A(e)$.

Case(i): Suppose $(\lambda, \mu) = 0_{\sim}$. Then clearly $(f(A))^{(\lambda,\mu)} = Y$.
So, by Result 1.D, $f(A) \in$ IFG(Y).

Case(ii): Suppose $\lambda > 0$ and $\mu < 1$. Then:

$z \in (f(A))^{(\lambda,\mu)}$

if and only if $\mu_{f(A)}(z) \geq \lambda$ and $\nu_{f(A)}(z) \leq \mu$

if and only if $\bigvee_{x \in f^{-1}(z)} \mu_A(x) \geq \lambda$ and $\bigvee_{x \in f^{-1}(z)} \nu_A(x) \leq \mu$

if and only if there exists an $x \in X$ such that $f(x) = z$, $\mu_A(x) \geq \lambda$ and

$\nu_A(x) \leq \mu$

if and only if $z \in (f(A^{(\lambda,\mu)})$.

Thus $(f(A))^{(\lambda,\mu)} = f(A^{(\lambda,\mu)})$. Since f is a homomorphism and $A^{(\lambda,\mu)}$ is a subgroup of X, $f(A^{(\lambda,\mu)})$ is a subgroup of Y. So, by Result 1.D, $f(A) \in$ IFG(X). Hence, in all, $f(A) \in$ IFG(X).

Remark 2.21. In Result 1.C, A has the sup property but in Proposition 2.20, there is no any restriction on A.

Proposition 2.22. Let $f : G \to G'$ be a group homomorphism, let $A \in$ IFNG(G) and let $B \in$ IFNG(G′). Then the followings hold:

(1) If f is surjective, then $f(A) \in$ IFNG(G′).

(2) $f^{-1}(B) \in$ IFNG(G).

Proof. (1) By Proposition 2.20, $f(A) \in$ IFG(G′). Let $(\lambda, \mu) \in \text{Im} f(A)$. From the Process of the another proof of Proposition 2.20, it is clear that $\lambda \leq \mu_A(e), \mu \geq \nu_A(e)$ and $(f(A))^{(\lambda,\mu)} = f(A^{(\lambda,\mu)})$. Since $A \in$ IFNG(G), by Proposition 2.13, $A^{(\lambda,\mu)} \lhd G$. Since f is an epimorphism, $(f(A))^{(\lambda,\mu)} = f(A^{(\lambda,\mu)}) \lhd G'$. Hence, by Proposition 2.18, $f(A) \in$ IFNG(G′).

(2) By Result 1.C (2), $f^{-1}(B) \in$ IFG(G). Let $x, y \in G$. Then:

$$\begin{aligned} f^{-1}(B)(xy) &= (f^{-1}(\mu_B)(xy), f^{-1}(\nu_B)(xy)) \\ &= (\mu_B(f(xy)), \nu_B(f(xy))) \\ &= (\mu_B(f(x)f(y)), \nu_B(f(x)f(y))) \\ &\quad \text{(Since } f \text{ is a homomorphism)} \\ &= (\mu_B(f(y)f(x)), \nu_B(f(y)f(x))) \text{ (Since } B \in \text{IFNG}(f(G))) \\ &= (\mu_B(f(yx)), \nu_B(f(yx))) \text{ (Since } f \text{ is a homomorphism)} \\ &= (f^{-1}(\mu_B)(yx), f^{-1}(\nu_B)(yx)) \\ &= f^{-1}(B)(yx). \end{aligned}$$

Hence $f^{-1}(B) \in$ IFNG(G).

The following is the immediate result of Proposition 2.20, Result 1.C (2) and Proposition 2.22:

Theorem 2.23. Let $f : G \to G'$ be a group homomorphism. Then the mapping $A \to f(A)$ defines a one-to-one correspondence between the set of all IF-invariant IFGs [resp. IFNGs] of G and IFG(G′) [resp. IFNG(G′)], provided that f is surjective in the latter case.

Theorem 2.24. Let A be an IFNG of a group G with identity e. We define a complex mapping $\hat{A} = (\mu_{\hat{A}}, \nu_{\hat{A}}) : G/G_A \longrightarrow I \times I$ as follows : for each $x \in G$,

$$\hat{A}(G_A x) = A(x).$$

Then $\hat{A} \in$ IFNG(G/G_A). Conversely, if $N \lhd G$ and $\hat{B} \in$ IFNG(G/N) such that $\hat{B}(Ng) = \hat{B}(N)$ only when $g \in N$, then there exists an $A \in$ IFNG(G) such that $G_A = N$ and $\hat{A} = \hat{B}$.

Proof. It is clear that $\hat{A} \in \text{IFS}(G/G_A)$ from the definition of $\hat{A}$. Suppose $G_Ax = G_Ay$ for some $x, y \in G$. Then, by Corollary 2.13, $xy^{-1} \in G_A$. Thus $A(xy^{-1}) = A(e)$. By Result 1.B, $A(x) = A(y)$. So $\hat{A}(G_Ax) = \hat{A}(G_Ay)$. Hence $\hat{A}$ is well- defined. Furthermore, it is easy to see that $\hat{A} \in \text{IFG}(G/G_A)$. Let $x, y \in G$. Then

$$\begin{aligned} \hat{A}(G_AxG_Ay) &= \hat{A}(G_Axy) \\ &= A(xy) \\ &= A(yx) \quad (\text{Since } A \in \text{IFNG}(G)) \\ &= \hat{A}(G_AyG_Ax). \end{aligned}$$

Hence $\hat{A} \in \text{IFNG}(G/G_A)$.

Now let $N \triangleleft G$ and let $\hat{B} \in \text{IFNG}(G/G_A)$ such that $\hat{B}(Ng) = \hat{B}(N)$ only when $g \in N$. We define a complex mapping $A = (\mu_A, \nu_A) : G \to I \times I$ as follows : for each $x \in G$, $A(x) = \hat{B}(Nx)$. Then we can easily see that A is well-defined and $A \in \text{IFG}(G)$. Let $x, y \in G$. Then

$$\begin{aligned} A(y^{-1}xy) &= \hat{B}(Ny^{-1}xy) \\ &= \hat{B}(Ny^{-1}NxNy) \\ &= \hat{B}(Nx) \quad (\text{Since } \hat{B} \in \text{IFNG}(G/N)) \\ &= A(x). \end{aligned}$$

Thus A is constant on the conjugate classes of G. So, by Theorem 2.11, $A \in \text{IFNG}(G)$.

Now let $g \in N$. Then $A(g) = \hat{B}(Ng) = \hat{B}(N) = A(e)$. Thus $g \in G_A$. So $N \subset G_A$. Let $x \in G_A$. Then $A(x) = A(e)$. Thus $\hat{B}(Nx) = \hat{B}(N)$. So $x \in N$, i.e., $G_A \subset N$. Hence $N = G_A$. Furthermore, $\hat{A} = \hat{B}$. This completes the proof.

3. Intuitionistic fuzzy Lagrange's Theorem

Let A be an IFS in a group G and for each $x \in G$, ${}_xf : G \to G$ [resp. $f_x : G \to G$] be a mapping defined as follows, respectively: for each $g \in G$,

$$ {}_xf(g) = xg \ \ [\text{resp. } f_x(g) = gx]. $$

Proposition 3.1. Let A be an IFG of a group G. Then

$$ {}_xf(A) = xA \ [\text{resp. } f_x(A) = Ax] \text{ for each} x \in G. $$

Proof. Let $g \in G$. Then:

$$ \mu_{f_x(A)}(g) = \bigvee_{g' \in f_x^{-1}(g)} \mu_A(g') = \bigvee_{g'x=g} \mu_A(g') = \mu_A(gx^{-1}) $$

and

$$ \nu_{f_x(A)}(g) = \bigwedge_{g' \in f_x^{-1}(g)} \nu_A(g') = \bigwedge_{g'x=g} \nu_A(g') = \nu_A(gx^{-1}). $$

Hence, $f_x(A) = Ax$. Similarly, we can see that ${}_xf(A) = xA$.

Proposition 3.2. Let A be a IFG of a group G and let $g_1, g_2 \in G$. Then $g_1A = g_2A$[resp. $Ag_1 = Ag_2$] if and only if $A(g_1^{-1}g_2) = A(g_2^{-1}g_1) = A(e)$ [resp. $A(g_1g_2^{-1}) = A(g_2g_1^{-1}) = A(e)$].

Proof. ($\Rightarrow$): Suppose $g_1A = g_2A$. Then $g_1A(g_1) = g_2A(g_1)$ and $g_1A(g_2) = g_2A(g_2)$. $A(g_2^{-1}g_1) = A(e)$ and $A(g_1^{-1}g_2) = A(e)$. Hence $A(g_2^{-1}g_1) = A(g_1^{-1}g_2) = A(e)$.

($\Leftarrow$): Suppose $A(g_1^{-1}g_2) = A(g_2^{-1}g_1) = A(e)$. Let $x \in G$. Then $g_1A(x) = A(g_1^{-1}x) = A(g_1^{-1}g_2g_2^{-1}x)$. Since A is a IFG(G),

$$ \begin{aligned} \mu_A(g_1^{-1}x) = \mu_A(g_1^{-1}g_2g_2^{-1}x) &\geq \mu_A(g_1^{-1}g_2) \wedge \mu_A(g_2^{-1}x) \\ &= \mu_A(e) \wedge \mu_A(g_2^{-1}x) \\ &= \mu_A(g_2^{-1}x) \text{ (By Result 1.A)} \end{aligned} $$

and

$$\begin{aligned}\nu_A(g_1^{-1}x) = \nu_A(g_1^{-1}g_2g_2^{-1}x) &\leq \nu_A(g_1^{-1}g_2) \vee \nu_A(g_2^{-1}x) \\ &= \nu_A(e) \vee \nu_A(g_2^{-1}x) \\ &= \nu_A(g_2^{-1}x).\end{aligned}$$

Thus $g_2A \subset g_1A$. Similarly, we have that $g_1A \subset g_2A$. Hence $g_1A = g_2A$. This completes the proof.

Proposition 3.3. Let A be an IFG of a group G. If $Ag_1 = Ag_2$ for any $g_1, g_2 \in G$, then $A(g_1) = A(g_2)$.

Proof. Suppose $Ag_1 = Ag_2$ for any $g_1, g_2 \in G$. Then $Ag_1(g_2) = Ag_2(g_2)$. Thus $A(g_2g_1^{-1}) = A(e)$. Hence, by Result 1.B, $A(g_1) = A(g_2)$.

Proposition 3.4. Let A be an IFG of a group G. If $A^{(\lambda,\mu)}x = A^{(\lambda,\mu)}y$ for any $x, y \in G \setminus A^{(\lambda,\mu)}$ and each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$, then $A(x) = A(y)$.

Proof. Suppose $A^{(\lambda,\mu)}x = A^{(\lambda,\mu)}y$ for any $x, y \in G \setminus A^{(\lambda,\mu)}$ and each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then $yx^{-1} \in A^{(\lambda,\mu)}$. Thus $\mu_A(yx^{-1}) \geq \lambda$ and $\nu_A(yx^{-1}) \leq \mu$. Since $x \in G \setminus A^{(\lambda,\mu)}$, $\mu_A(x) < \lambda$ and $\nu_A(x) > \mu$. On the other hand,

$$\mu_A(y) = \mu_A(yx^{-1}x) \geq \mu_A(yx^{-1}) \wedge \mu_A(x)$$

and

$$\nu_A(y) = \nu_A(yx^{-1}x) \leq \nu_A(yx^{-1}) \vee \nu_A(x).$$

Thus $\mu_A(y) \geq \mu_A(x)$ and $\nu_A(y) \leq \nu_A(x)$. By the similar arguments, we have that $\mu_A(y) \leq \mu_A(x)$ and $\nu_A(y) \geq \nu_A(x)$. Hence $A(x) = A(y)$.

Proposition 3.5. Let A be an IFNG of a group G and let $x \in G$. Then $Ax(xg) = Ax(gx) = A(g)$ for each $g \in G$.

Proof. Let $g \in G$, Then

$$\begin{aligned} Ax(xg) &= (\mu_{Ax}(xg),\ \nu_{Ax}(xg)) \\ &= (\mu_{Ax}(xgx^{-1}x),\ \nu_{Ax}(xgx^{-1}x)) \\ &= (\mu_A(xgx^{-1}xx^{-1}),\ \nu_A(xgx^{-1}xx^{-1}) \text{ (By the definition of } Ax) \\ &= (\mu_A(xgx^{-1}),\ \nu_A(xgx^{-1})) \\ &= (\mu_A(g),\ \nu_A(g)) \text{ (By Theorem2.11)} \\ &= A(g). \end{aligned}$$

Similarly, we have $Ax(gx) = A(g)$. This completes the proof.

Remark 3.6. Proposition 3.5 is analogous to the result in group theory that if $N \lhd G$, then $Nx = xN$ for each $x \in G$.

If N is a normal subgroup of a group G, then the cosets of G with respect to N form a group (called the *quotient group* G/N). For an IFNG, we have the analogous result :

Theorem 3.7. Let A be an IFNG of a group G and let G/A be the set of all the intuitionistic fuzzy cosets of A. We define an operation $*$ on G/A as follows : for any $x, y \in G$,
$Ax * Ay = Axy$.
Then $(G/A,\ *)$ is a group. In this case, G/A is called the *intuitionistic fuzzy quotient group induced* by A.

Proof. Let $x, y, x_0, y_0 \in G$ such that $Ax = Ax_0$ and $Ay = Ay_0$, and let $g \in G$. Then $Axy(g) = A(gy^{-1}x^{-1})$ and $Ax_0y_0(g) = A(gy_0^{-1}x_0^{-1})$. On the other hand,

$$\begin{aligned}\mu_A(gy^{-1}x^{-1}) &= \mu_A(gy_0^{-1}y_0y^{-1}x^{-1})\\ &= \mu_A(gy_0^{-1}x_0^{-1}x_0y_0y^{-1}x^{-1})\\ &\geq \mu_A(gy_0^{-1}x_0^{-1}) \wedge \mu_A(x_0y_0y^{-1}x^{-1}) \ (*)\\ &\quad \text{(Since } A \in \text{IFG}(G))\end{aligned}$$

and

$$\begin{aligned}\nu_A(gy^{-1}x^{-1}) &= \nu_A(gy_0^{-1}y_0y^{-1}x^{-1})\\ &= \nu_A(gy_0^{-1}x_0^{-1}x_0y_0y^{-1}x^{-1})\\ &\leq \nu_A(gy_0^{-1}x_0^{-1}) \vee \nu_A(x_0y_0y^{-1}x^{-1}) \ (**)\end{aligned}$$

Since $Ax = Ax_0$ and $Ay = Ay_0$, $A(gx^{-1}) = A(gx_0^{-1})$ and $A(gy^{-1}) = A(gy_0^{-1})$. In particular,

$$\begin{aligned}A(x_0y_0y^{-1}x^{-1}) &= A(x_0y_0y^{-1}x_0^{-1})\\ &= A(y_0y^{-1}) \text{ (Since } A \in \text{IFNG}(G))\\ &= A(e).\end{aligned}$$

So $(\mu_A(x_0y_0y^{-1}x^{-1}),\ \nu_A(x_0y_0y^{-1}x^{-1})) = (\mu_A(e),\ \nu_A(e))$. By Result 1.A, $\mu_A(e) \geq \mu_A(gy_0^{-1}x_0^{-1})$ and $\nu_A(e) \leq \nu_A(gy_0^{-1}x_0^{-1})$. Thus, by $(*)$ and $(**)$,

$$\mu_A(gy^{-1}x^{-1}) \geq \mu_A(gy_0^{-1}x_0^{-1}) \text{ and } \nu_A(gy^{-1}x^{-1}) \leq \nu_A(gy_0^{-1}x_0^{-1}).$$

By the similar arguments, we have that

$$\mu_A(gy_0^{-1}x_0^{-1}) \geq \mu_A(gy^{-1}x^{-1}) \text{ and } \nu_A(gy_0^{-1}x_0^{-1}) \leq \nu_A(gy^{-1}x^{-1}).$$

So $A(gy_0^{-1}x_0^{-1}) = A(gy^{-1}x^{-1})$, i.e., $Ax_0y_0(g) = Axy(g)$. Hence $*$ is well-defined. Furthermore, we can easily check that the followings are true:

(i) $*$ is associative.

(ii) Ax^{-1} is the inverse of Ax for each $x \in G$.

(iii) $Ae = A$ is the identity of G/A.

Therefore $(G/A, *)$ is a group. This completes the proof.

Proposition 3.8. Let A be an IFNG of a group G. We define a complex mapping $\overline{A} = (\mu_{\overline{A}}, \nu_{\overline{A}}) : G/A \to I \times I$ as follows : for each $x \in G$,

$$\overline{A}(Ax) = A(x), \text{ i.e., } \mu_{\overline{A}}(Ax) = \mu_A(x) \text{ and } \nu_{\overline{A}}(Ax) = \nu_A(x).$$

Then $\overline{A}$ is an *IFG* of G/A. In this case, $\overline{A}$ is called the *intuitionistic fuzzy subquotient group* determined by A.

Proof. From the definition of $\overline{A}$, it is clear that $\overline{A} \in \text{IFS}(G/A)$. Let $x, y \in G$. Then :

$$\begin{aligned} \mu_{\overline{A}}(Ax * Ay) &= \mu_{\overline{A}}(Axy) \\ &= \mu_A(xy) \\ &\geq \mu_A(x) \wedge \mu_A(y) \\ &= \mu_{\overline{A}}(Ax) \wedge \mu_{\overline{A}}(Ay) \end{aligned}$$

and

$$\begin{aligned} \nu_{\overline{A}}(Ax * Ay) &= \nu_{\overline{A}}(Axy) \\ &= \nu_A(xy) \\ &\leq \nu_A(x) \vee \nu_A(y) \\ &= \nu_{\overline{A}}(Ax) \vee \nu_{\overline{A}}(Ay). \end{aligned}$$

On the other hand,

$$\mu_{\overline{A}}((Ax)^{-1}) = \mu_{\overline{A}}(Ax^{-1}) = \mu_A(x^{-1}) \geq \mu_A(x) = \mu_{\overline{A}}(Ax)$$

and

$$\nu_{\overline{A}}((Ax)^{-1}) = \nu_{\overline{A}}(Ax^{-1}) = \nu_A(x^{-1}) \leq \nu_A(x) = \nu_{\overline{A}}(Ax).$$

Hence $\overline{A} \in \text{IFG}(G/A)$.

Proposition 3.9. Let A be an IFNG of a group G. We define a mapping $\pi : G \to G/A$ as follows : for each $x \in G$, $\pi(x) = Ax$. Then π is a homomorphism with $\text{Ker}(\pi) = G_A$. In this case π is called the *natural homomorphism.*

Proof. Let $x, y \in G$. Then $\pi(xy) = Axy = Ax * Ay = \pi(x) * \pi(y)$. So π is a homomorphism. Furthermore,

$$\begin{aligned}\text{Ker}(\pi) &= \{x \in G : \pi(x) = Ae\} \\ &= \{x \in G : Ax = Ae\} \\ &= \{x \in G : Ax(x) = Ae(x)\} \\ &= \{x \in G : A(e) = A(x)\} \\ &= G_A.\end{aligned}$$

This completes the proof.

Now we obtain for intuitionistic fuzzy subgroups an analogous result of the "Fundamental Theorem of Homomorphism of Groups".

Theorem 3.10. Let $A \in \text{IFNG}(G)$. Then each intuitionistic fuzzy (normal) subgroup of G/A corresponds in a natural way to an intuitionistic fuzzy (normal) subgroup of G.

Proof. Let A^* be an intuitionistic fuzzy subgroup of G/A. Define a complex mapping $B = (\mu_B, \nu_B) : G \to I \times I$ as follows : for each $x \in G$, $B(x) = A^*(Ax)$.
By the definition of B, it is clear that $B \in \text{IFS}(G)$. Let $x, y \in G$. Then

:

$$\begin{aligned}\mu_B(xy) &= \mu_{A^*}(Axy)\\ &= \mu_{A^*}(Ax * Ay)\\ &\geq \mu_{A^*}(Ax) \wedge \mu_{A^*}(Ay) \text{ (Since } A^* \in \text{IFG}(G/A))\\ &= \mu_B(x) \wedge \mu_B(y)\end{aligned}$$

and

$$\begin{aligned}\nu_B(xy) &= \nu_{A^*}(Axy)\\ &= \nu_{A^*}(Ax * Ay)\\ &\leq \nu_{A^*}(Ax) \vee \nu_{A^*}(Ay)\\ &= \nu_B(x) \vee \nu_B(y).\end{aligned}$$

Since $A^* \in \text{IFG}(G/A)$, $A^*(Ax^{-1}) = A^*(Ax)$. Thus :

$$\begin{aligned}B(x^{-1}) &= (\mu_B(x^{-1}),\ \nu_B(x^{-1})) = (\mu_{A^*}(Ax^{-1}),\ \nu_{A^*}(Ax^{-1}))\\ &= (\mu_{A^*}(Ax),\ \nu_{A^*}(Ax))\\ &= (\mu_B(x),\ \nu_B(x)) = B(x).\end{aligned}$$

Hence $B \in \text{IFG}(G)$. It is easy to see that if B is an IFNG of G/A, then B is an IFNG of G. This completes the proof.

Now we will obtain an intuitionistic fuzzy analog of the famous "Lagrange's Theorem" for finite groups which is a basic result in group theory . Let A be an IFG of a finite group G. Then it clear that G/A is finite.

Definition 3.11. Let A be an IFG of a finite group G. Then the cardinality $|G/A|$ of G/A is called the *index* of A.

Theorem 3.12(Intuitionistic Fuzzy Lagrange's Theorem). Let A be an IFG of a finite group G. Then the index of A divides the order of G.

Proof. By Proposition 3.9, there is the natural homomorphism $\pi : G \to G/A$. Let H be the subgroup of G defined by $H = \{h \in G : Ah = Ae\}$, where e is the identity of G. Let $h \in H$. Then $Ah(g) = Ae(g)$ or $A(gh^{-1}) = A(g)$ for each $g \in G$. In particular, $A(h^{-1}) = A(e)$. Since A is an IFG of G, by Result 1.A, $A(h) = A(e)$. Thus $h \in G_A$. So $H \subset G_A$. Now let $h \in G_A$. Then $A(h) = A(e)$. Thus, by Result 1.A, $A(h^{-1}) = A(e)$. By Lemma 2.2, $A(gh^{-1}) = A(g)$ or $Ah(g) = Ae(g)$ for each $g \in G$. Thus $Ah = Ae$, i.e., $h \in H$. So $G_A \subset H$. Hence $H = G_A$.

Now decompose G as a disjoint union of the cosets of G with respect to H :

$$G = Hx_1 \cup Hx_2 \cup \cdots \cup Hx_k \ (***)$$

where $Hx_1 = H$. We show that corresponding to each coset Hx_i given in $(***)$, there is an intuitionistic fuzzy coset belonging to G/A, and further that this correspondence is injective. Consider any coset Hx_i. Let $h \in H$. Then

$$\begin{aligned} \pi(hx_i) &= Ahx_i = Ah * Ax_i \\ &= Ae * Ax_i = Ax_i. \end{aligned}$$

Thus π maps each element of Hx_i into the intuitionistic fuzzy coset Ax_i. Now we define a mapping $\bar{\pi} : \{Hx_i : 1 \leq i \leq k\} \to G/A$ as follows : for each $i \in \{1, 2, \cdots, k\}$,

$$\bar{\pi}(Hx_i) = Ax_i.$$

Then clearly, $\bar{\pi}$ is well-defined. Suppose $Ax_i = Ax_j$. Then

$$Ax_i{x_j}^{-1} = Ae.$$

Thus $x_i{x_j}^{-1} \in H$. So $Hx_i = Hx_j$. Hence $\bar{\pi}$ is injective. From the above discussion, it is clear that $|G/A| = k$. Since k divides the order of G.

$|G/A|$ also divides the order of G. This completes the proof.

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[a] Division of Mathematics and Informational
Statistics,
and Institute of Basic Natural Science
Wonkwang University Iksan, Chonbuk, Korea 570-749
E-mail: kulhur@wonkwang.ac.kr
E-mail: soyoun12@yahoo.com

[b] Dept. of Mathematics Education
Woosuk University
Hujong-Ri Samrae-Eup, Wanju-kun Chonbuk, Korea 565-701
E-mail: khwon@woosuk.ac.kr