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AN APPLICATION OF CATEGORY THEORY TO THE NONLINEAR WAVE EQUATION WITH JUMPING NONLINEARITY

TACK SUN JUNG AND Q-HEUNG CHOI

Abstract. We investigate the multiplicity of the periodic solutions of the nonlinear wave equation with jumping nonlinearity. By category theory we prove that the jumping problem has at least $2k+1$ solutions for the positive source term.

1. INTRODUCTION

We investigate the multiplicity of the periodic solutions of the nonlinear wave equation with Dirichlet boundary condition

$$u_{tt} - u_{xx} + bu^+ - au^- = se_1^+ \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \tag{1.1}$$
$$u(\pm\frac{\pi}{2}, t) = 0,$$
$$u(x,t) = u(-x,t) = u(x,-t) = u(-x, t+\pi),$$

where $u^+ = \max\{u, 0\}$, $u^- = -\min\{u, 0\}$, $s \neq 0$, $s \in R$ and e_1^+ is the eigenfunction corresponding to the positive eigenvalue $\mu_1^+ = 1$ of the eigenvalue problem $u_{tt} - u_{xx} = \mu u$ with Dirichlet boundary condition. We look for π- periodic solutions of (1.1). Choi and Jung proved in [3] that if $-5 < a < -1, 3 < b < 7$ and $s > 0$, then (1.1) has at least four solutions. In this paper we improved the results of [2] and [3] when the

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jumping nonlinearity conditions are $-\mu_i^- < a < -\mu_{i+1}^-, \dots, -\mu_{i+k}^- < b < -\mu_{i+k+1}^-$ and $b \to -\mu_{i+k}^-$ or $-\mu_{i+k+1}^+ < b < -\mu_{i+k}^+, \dots, -\mu_{i+1}^+ < a < -\mu_i^+$, $b \to -\mu_{i+k}^+$, where μ_k^- and μ_k^+, $k \geq 1$, are the negative and positive eigenvalues of the problem $u_{tt} - u_{xx} = \lambda u$ with Dirichlet boundary condition. We prove the following result:

Theorem 1.1. Let μ_k^- be a negative eigenvalue such that $\mu_k^- < \mu_1^-$. Then there exists a number $\delta > 0$ such that for any a and b with $\mu_{i+k}^- - \delta < -b < \mu_{i+k}^- \leq \mu_{i+1}^- < -a < \mu_i^-$, $i \geq 1$, and $s > 0$, then problem (1.1) has at least $2k+1$ solutions.

2. VARIATIONAL APPROACH

The eigenvalue problem (1.2) has infinitely many eigenvalues $\lambda_{mn} = (2n+1)^2 - 4m^2$ $(m, n = 0, 1, 2, \dots)$ and corresponding normalized eigenfunctions $\phi_{mn}(x,t)$ given by

$$\phi_{0n} = \frac{\sqrt{2}}{\pi} \cos(2n+1)x \qquad \text{for } n \geq 0,$$

$$\phi_{mn} = \frac{2}{\pi} \cos 2mt \cdot \cos(2n+1)x \qquad \text{for } m > 0, n \geq 0.$$

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H' the Hilbert space defined by $H' = \{u \in L^2(Q) |\ u \text{ is even in } x \text{ and } t\}$. Then the set of functions $\{\phi_{mn}\}$ is an orthonormal basis in H'. Let us denote an element u, in H', as $u = \sum h_{mn}\phi_{mn}$ and we define a subspace H of H' as $H = \{u \in H' |\ \sum |\lambda_{mn}| h_{mn}^2 < \infty\}$. This is a complete normed space with a norm $\|u\| = [\sum |\lambda_{mn}| h_{mn}^2]^{\frac{1}{2}}$. Since the set $\{\lambda_{mn} |\ m, n = 0, 1, 2, \dots\}$ is unbounded from above and from below and has no finite accumulation point, it is convenient for the following to rearrange the eigenvalues λ_{mn} by increasing magnitude: From now on we denote by $(\mu_i^-)_{i \geq 1}$ the sequence of the negative eigenvalues of (1.2), by (μ_i^+) the sequence of the positive ones, so that

$$\ldots \leq \mu_i^- \leq \ldots \leq \mu_2^- \leq \mu_1^- < 0 < \mu_1^+ \leq \mu_2^+ \leq \ldots \leq \mu_i^+ \leq \ldots.$$

We note that each eigenvalue has a finite multiplicity and that $\mu_i^- \to -\infty$ and $\mu_i^+ \to +\infty$ as $i \to \infty$. Let $\{e_i^-, e_i^+, i \geq 1\}$ be an orthonormal system of eigenfunctions associated with the eigenvalues $\{\mu_i^-, \mu_i^+, i \geq 1\}$. If μ is any eigenvalue, we set

$$H^+(\mu) = \text{closure of span } \{\text{eigenfunctions with eigenvalue } \geq \mu\},$$

$$H^-(\mu) = \text{closure of span } \{\text{eigenfunctions with eigenvalue } \leq \mu\},$$

and set $H^+ = H^+(0)$, $H^- = H^-(0)$. We define the projections $P^- : H \to H^-$, $P^+ : H \to H^+$. We define two linear operators $R : H \to H^+$, $S : H \to H^-$ by

$$S(u) = \sum_{i=1}^{\infty} \frac{a_i^- e_i^-}{\sqrt{-\mu_i^-}}, \qquad R(u) = \sum_{i=1}^{\infty} \frac{a_i^+ e_i^+}{\sqrt{\mu_i^+}}$$

if $u = \sum_{i=1}^{\infty} a_i^- e_i^- + \sum_{i=1}^{\infty} a_i^+ e_i^+$. It is clear that S and R are compact and self adjoint on H. In this paper we study the nonlinear wave equation, $s \in R$,

$$u_{tt} - u_{xx} + bu^+ - au^- = se_1^+ \qquad \text{in } H. \tag{2.1}$$

Let us define the functional on H, corresponding to (2.1), $s \in R$,

$$I_{a,b}(u) = \frac{1}{2}\|P^+u\|^2 - \frac{1}{2}\|P^-u\|^2 + \frac{b}{2}\|(R+S)u^+\|^2 + \frac{a}{2}\|(R+S)\|^2 - s\langle e_1^+, u\rangle. \tag{2.2}$$

Then $I_{a,b}(u)$ is well defined, continuous and Frèchet differentiable in H. Moreover $I_{a,b} \in C^{1,1}(H, R)$ and the solutions of (2.1) coincide with the critical points of $I_{a,b}(u)$, that is, $\nabla I_{a,b}(u) = 0$ if and only if $(R+S)u$ is a weak solution of (1.1).

First we assume that $s > 0$, $\mu_{i+k+1}^- < -b < \mu_{i+k}^- \leq \mu_{i+1}^- < -a < \mu_i^-$ and $-b \to \mu_{i+k}^-$, $k, i \geq 1$. We will find the solutions of the form $u = \bar{u} + z$, so that z is a critical point for the functional $J_{a,b}(w) = I_{a,b}(\bar{u} + w) - I_{a,b}(\bar{u})$, where

$$J_{a,b}(z) = \frac{1}{2}\|P^+ z\|^2 - \frac{1}{2}\|P^- z\|^2 + \frac{b}{2}\|Az\|^2 - \frac{b}{2}\|[\bar{u}+Az]^-\|^2 + \frac{a}{2}\|[\bar{u}+Az]^-\|^2,$$

where we set $Az = (R+S)z$. We know that

$$\nabla J_{a,b}(z) = P^+ z - P^- z + bA^2 z + bA[\bar{u} + Az]^- - aA[\bar{u} + Az]^-,$$

and 0 is the trivial solution of $J_{a,b}$ with $J_{a,b}(0) = 0$.

3. CRITICAL POINT THEORY ON THE MANIFOLD

Let H be a Hilbert space and M be the closure of an open subset of H such that M can be endowed with the structure of C^2 manifold with boundary. Let $f : W \to R$ be a $C^{1,1}$ functional, where W is an open set containing M. For applying the usual topological methods of critical points theory we need a suitable notion of critical point for f on M. We recall the following notions: lower gradient of f on M, $(P.S.)_c^*$ condition and the limit relative category (see [4]).

Definition 3.1. If $u \in M$, the lower gradient of f on M at u is defined by

$$grad_{\bar{M}} f(u) = \begin{cases} \nabla f(u) & \text{if } u \in int(M), \\ \nabla f(u) + [< \nabla f(u), \nu(u) >]^- \nu(u) & \text{if } u \in \partial M, \end{cases} \tag{3.1}$$

where we denote by $\nu(u)$ the unit normal vector to ∂M at the point u, pointing outwards.

We say that u is a lower critical point for f on M, if $grad_{\bar{M}} f(u) = 0$.

Definition 3.2. Let $c \in R$. We say that f satisfies the $(P.S.)_c$ condition on M if for any sequence $(u_n)_n$ in M such that $f(u_n) \to c$ and $grad_{\bar{M}} f(u_n) \to 0$ there exists a subsequence $(u_{n_k})_k$ which converges to a point u in M such that $grad_{\bar{M}} f(u) = 0$.

Let Y be a closed subspace of M.

Definition 3.3. Let B be a closed subset of M with $Y \subset B$. We define the relative category $cat_{M,Y}(B)$ of B in (M,Y), as the least integer h such that there exist $h+1$ closed subsets U_0, $U_1, \ldots$, U_h with the following properties:
$B \subset U_0 \cup U_1 \cup \ldots \cup U_h$;
$U_1, \ldots, U_h$ are contractible in M;
$Y \subset U_0$ and there exists a continuous map $F : U_0 \times [0,1] \to M$ such that

$$\begin{aligned} F(x,0) &= x && \forall x \in U_0, \\ F(x,t) &\in Y && \forall x \in Y, \forall t \in [0,1], \\ F(x,1) &\in Y && \forall x \in U_0. \end{aligned}$$

If such an h does not exist, we say that $cat_{M,Y}(B) = +\infty$.

Now we recall a theorem which gives an estimate of the number of critical points of a functional, in terms of the relative category of its sublevels (see [5]).

Theorem 3.1. Let Y be a closed subset of M. For any integer i we set

$$c_i = \inf\{\sup f(B) \mid B \text{ is closed}, Y \subset B, cat_{M,Y}(B) \geq i\}.$$

Assume that $(P.S.)_c$ holds for $c = c_i$ and that $\sup f(Y) < c_i < +\infty$. Then c_i is a lower critical level for f, that is, there exists u in M such that $f(u) = c_i$ and $grad_{\bar{M}} f(u) = 0$. Moreover, if

$$c_i = c_{i+1} = \ldots = c_{i+k-1} = c,$$

then

$$cat_M(\{u \in M|\ f(u) = c,\ grad_{\tilde{M}} f(u) = 0\}) \geq k.$$

We need in the following a version of previous theorem suited to treat strongly indefinite functionals (see [1, 4]). In this case the notion of the $(P.S.)_c^*$ condition and limit relative category turn out to be a very useful tool.

Let $(H_n)_n$ be a sequence of closed finite dimensional subspace of H, defined by

$$H_n = \text{closure of span } < e_n^-, \ldots, e_1^-, e_1^+, \ldots, e_n^+ > .$$

Definition 3.4. Let $c \in R$. We say that f satisfy the $(P.S.)_c^*$ condition with respect to $(H_n)_n$, if for any sequence $(k_n)_n$, with $k_n \to +\infty$, and for any sequence $(u_n)_n$, with $u_n \in H_{k_n}$, $f(u_n) \to c$, $\nabla f_{k_n}(u_n) \to 0$, $f_{k_n} = f|_{H_{k_n}}$, there exists a subsequence of $(u_n)_n$ which converges in H to a critical point of f.

Lemma 3.1. The functional $J_{a,b} : H \to R$ satisfies the $(P.S.)_c^*$ condition with respect to $(H_n)_n$, for any $c \in R$.

Let $M_n = M \cap H_n$, for any n, be the closure of an open subset of H_n, which has the structure of a C^2 manifold with boundary in H_n. We assume that for any n there exists a retraction $r_n : M \to M_n$. For given $B \subset H$, we will write $B_n = B \cap H_n$.

Definition 3.5. Let $c \in R$. We say that f satisfies the $(P.S.)_c^*$ condition with respect to $(M_n)_n$, on the manifold with boundary M, if for any sequence $(k_n)_n$ in N and any sequence $(u_n)_n$ in M such that $k_n \to \infty$, $u_n \in M_{k_n}$, $\forall n$, $f(u_n) \to c$, $grad_{\tilde{M}_{k_n}} f(u_n) \to 0$, there exists a subsequence of $(u_n)_n$ which converges to a point $u \in M$ such that

$grad_{\bar{M}} f(u) = 0$.

Definition 3.6. Let (X, Y) be a topological pair and X_n be a sequence of subsets of X. For any subset B of X we define the limit relative category of B in (X, Y), with respect to $(X_n)_n$, by

$$cat^*_{(X,Y)}(B) = \lim_{n \to \infty} \sup cat_{(X_n, Y_n)}(B_n).$$

Let Y be a fixed subset of M. We set
$\mathcal{B}_i = \{B \subset M |\ cat^*_{(M,Y)}(B) \geq i\}$, $c_i = \inf_{B \in \mathcal{B}_i} \sup_{x \in B} f(x)$.
We have the following multiplicity theorems, which was proved in [5].

Theorem 3.2. Let $i \in N$. Assume that
(1) $c_i < +\infty$,
(2) $\sup_{x \in Y} f(x) < c_i$,
(3) The $(P.S.)^*_{c_i}$ condition with respect to $(M_n)_n$ holds.
Then there exists a lower critical point x such that $f(x) = c_i$. If

$$c_i = c_{i+1} = \ldots = c_{i+k-1} = c,$$

then

$$cat_M(\{x \in M | f(x) = c,\ grad_{\bar{M}} f(x) = 0\}) \geq k.$$

We recall the following multiplicity result in [5], which will be used in the proofs of our main theorems.

Theorem 3.3. Let H be a Hilbert space and let $H = X_1 \oplus X_2 \oplus X_3$, where X_1, X_2, X_3 are three closed subspaces of H with X_2 of finite dimension. Moreover let $(H_n)_n$ be a sequence of closed subspaces of H with finite dimension and such that for all n,

$$X_2 \subset H_n, \qquad P_{X_i} \circ P_{H_n} = P_{H_n} \circ P_{X_i} (= P_{X_i \cap H_n}), \ i = 1, 2, 3,$$

where, for a given subspace X of H, P_X is the orthogonal projection from H onto X. Set

$$C = \{x \in X | \ \|P_{X_2}x\| \geq 1\}$$

and let $f : W \to R$ be a $C^{1,1}$ function defined on a neighborhood W of C. Let $1 < \rho < R$, $R_1 > 0$, we define

$$\Delta_{12} = \{x_1 + x_2 | \ x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, 1 < \|x_2\| < R\},$$

$$\begin{aligned} \Sigma_{12} &= \{x_1 + x_2 | \ x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, \|x_2\| = 1\} \\ &\cup \{x_1 + x_2 | \ x_1 \in X_1, x_2 \in X_2, \|x_1\| \leq R_1, \|x_2\| = R\} \\ &\cup \{x_1 + x_2 | \ x_1 \in X_1, x_2 \in X_2, \|x_1\| = R_1, 1 \leq \|x_2\| \leq R\}, \\ S_{23} &= \{x \in X_2 \oplus X_3 | \ \|x\| = \rho\}. \end{aligned}$$

Let

$$\alpha = \inf f(S_{23}), \qquad \beta = \sup f(\Delta_{12}).$$

Assume that

$$\sup f(\Sigma_{12}) < \inf f(S_{23}).$$

Assume that the $(P.S.)^*_c$ condition holds for f on C, with respect to the sequence $(C_n)_n$, $C_n = C \cap H_n$, $\forall c \in [\alpha, \beta]$. Assume that $f|_{X_1 \oplus X_3}$ has no critical points with $\alpha \leq f(u) \leq \beta$. Moreover we assume $\beta < +\infty$. Then there exist two lower critical points u_1, u_2 for f on $Int\ C$ such that $\inf f(S_{23}) \leq f(u_i) \leq \sup f(\Delta_{12})$, $i = 1, 2$.

4. VARIATIONAL INEQUALITIES ON THE MANIFOLD

Let's take integers, $i, k \geq 1$ such that $\mu^-_{i+k} \leq \ldots \leq \mu^-_{i+1} < \mu^-_i \leq \mu^-_1 < 0$. First of all, we set

$$X^k_1 = H^-(\mu^-_{i+k+1}), \qquad X^k_2 = span\{e^-_{i+k}\}, \qquad X^k_3 = H^+(\mu^-_{i+k-1}).$$

In this case we will show that if $-b \longrightarrow \mu_{i+k}^{-}$, $\mu_{i+k+1}^{-} < -b < \mu_{i+k}^{-} \leq \mu_{i+1}^{-} < -a < \mu_{i}^{-} \leq \mu_{1}^{-}$, then we have a linking type inequality relative to the decomposition $H = X_1^k \oplus X_2^k \oplus X_3^k$, so we get the existence of nontrivial two critical points for $J_{a,b}$ in the subspace X_2^k. We need the following lemmas: Lemma 4.1, 4.2, 4.3, 4.4, 4.5 which can be proved by the same methods as in the results in [6].

Lemma 4.1. For any μ with $\mu_{i+k+1}^{-} < \mu < \mu_{i+k}^{-}$, $k \geq 1$, there exists a constant $\Gamma > 0$ such that for all a and b with $\mu \leq -b \leq \mu_{i+k}^{-} \cdots \leq \mu_{i+1}^{-} < -a < \mu_{i}^{-}$, if u is a critical point for $J_{a,b}|_{X_1^k \oplus X_3^k}$ with $0 \leq J_{a,b}(u) \leq \Gamma$, then $u = 0$.

Let X be a linear subspace of H and E be a subset of H such that $E \cap X = \emptyset$ and $R > 0$. Let

$$\Delta_R(E, X) = \{w + \sigma e |\ w \in X, \sigma \geq 0, e \in E, \|w + \sigma e\| \leq R\},$$

$$\Sigma_R(E, X) = \{w + \sigma e |\ w \in X, \sigma \geq 0, e \in E, \|w + \sigma e\| = R\} \cup \{w \in X |\ \|w\| \leq R\}.$$

Lemma 4.2. There exist $\delta_k > 0$, $R_k > 0$, $\rho_k > 0$, and $\bar{\rho} > 0$ with $0 < \rho_k < R_k$ such that if $\mu_{i+k}^{-} - \delta_k \leq -b < \mu_{i+k}^{-} \leq \ldots \leq \mu_{i+1}^{-} < -a < \mu_{i}^{-} \leq \mu_{1}^{-}$, then

$$\sup_{v \in \Sigma_{R_k}(S_k(\bar{\rho}), X_1^k)} J_{a,b}(v) < \inf_{\substack{w \in X_2^k \oplus X_3^k \\ \|w\| = \rho_k}} J_{a,b}(w),$$

where $S_k(\bar{\rho}) = \{v \in X_2^k |\ \|v\| = \bar{\rho}\}$.

Let us define a functional $\Psi : H \backslash (X_1^k \oplus X_3^k) \rightarrow H$ by

$$\Psi(u) = u - \frac{P_{X_2^k} u}{\|P_{X_2^k} u\|} = P_{X_1^k \oplus X_3^k} u + \left(1 - \frac{1}{\|P_{X_2^k} u\|}\right) P_{X_2^k} u.$$

We have

$$\Psi'(u)(v) = v - \frac{1}{\|P_{X_2^k}u\|}(P_{X_2^k}v - \langle \frac{P_{X_2^k}u}{\|P_{X_2^k}u\|}, v\rangle \frac{P_{X_2^k}u}{\|P_{X_2^k}u\|}).$$

Let

$$C = \{u \in H| \ \|P_{X_2^k}u\| \geq 1\}.$$

Then C is the smooth manifold with boundary. Let us define the function $\widetilde{J}_{a,b} : C \to H$ by

$$\widetilde{J}_{a,b} = J_{a,b} \circ \Psi.$$

Then $\widetilde{J}_{a,b} \in C^{1,1}_{loc}$. We note that if $\tilde{u}$ is the critical point of $\widetilde{J}_{a,b}$ and lies in the interior of C, then $u = \Psi(\tilde{u})$ is the critical point of $J_{a,b}$. We also note that

$$\|grad_{\bar{C}}\widetilde{J}_{a,b}(\tilde{u})\| \geq \|P_{X_1^k \oplus X_3^k}\nabla J_{a,b}(\Psi(\tilde{u}))\|, \qquad \forall \tilde{u} \in \partial C.$$

Lemma 4.3. $\widetilde{J}_{a,b}$ satisfies the $(P.S.)^*_\gamma$ condition with respect to $(C_n)_n$, $C_n = C \cap H_n$ for any γ such that

$$\inf_{\tilde{w} \in \widetilde{S}_{23}(\rho_k)} \widetilde{J}_{a,b}(\tilde{w}) \leq \gamma \leq \sup_{\tilde{v} \in \widetilde{\Delta}_{R_k}} \widetilde{J}_{a,b}(\tilde{v}),$$

where

$$S_{23}(\rho_k) = \{u \in X_2^k \oplus X_3^k| \ \|u\| = \rho_k\},$$

$$\widetilde{S}_{23}(\rho_k) = \Psi^{-1}(S_{23}(\rho_k)) = \{u_2 + u_3| \quad u_2 \in X_2^k, u_3 \in X_3^k, \|u_3\| \leq \rho_k, \\ \|u_2\| = 1 + \sqrt{\rho_k^2 - \|u_3\|^2}\},$$

$$\begin{aligned} \widetilde{\Delta}_{R_k} &= \Psi^{-1}(\Delta_{R_k}(S_k(\bar{\rho}), X_1^k)) \\ &= \{u = u_1 + \sigma(e) + \frac{1}{\sigma}e| \ u_1 \in X_1^k, e \in X_2^k, \|e\| = \bar{\rho}, \sigma \geq 0, \|u\| \leq R_k\}. \end{aligned}$$

Proposition 4.1. Let i, $k \geq 1$ be such that $\mu^-_{i+k+1} < \mu^-_{i+k} \leq \mu^-_{i+1}$. Let δ_k, ρ_k, $\bar{\rho}$, R_k be as in Lemma 4.2. Then for any a and b with

$\mu^-_{i+k} - \delta_k \leq -b < \mu^-_{i+k} \leq \cdots \leq \mu^-_{i+1} < -a < \mu^-_i \leq \mu^-_1$, the functional $J_{a,b}$ has two critical points $v^{(k)}_j$, $j = 1, 2$ such that

$$0 < \inf_{\substack{v \in H^+(\mu^-_{i+k}) \\ \|v\| = \rho_k}} J_{a,b}(v) \leq J_{a,b}(v^{(k)}_j) \leq \sup_{w \in \Delta_{R_k}(S_k(\bar{\rho}), H^-(\mu^-_{i+k+1}))} J_{a,b}(w).$$

Secondly, we set

$$X^{k-1}_1 = H^-(\mu^-_{i+k}), \qquad X^{k-1}_2 = span\{e^-_{i+k-1}\}, \qquad X^{k-1}_3 = H^+(\mu^-_{i+k-2}).$$

In this case we define a functional $\Psi : H \backslash (X^{k-1}_1 \oplus X^{k-1}_3) \to H$ and the manifold C on H with respect to the decomposition $H = X^{k-1}_1 \oplus X^{k-1}_2 \oplus X^{k-1}_3$. We also define $\widetilde{J}_{a,b} : C \to H$ by $\widetilde{J}_{a,b} = J_{a,b} \circ \Psi$. Then $\widetilde{J}_{a,b} \in C^{1,1}_{loc}$ and arguing as in the proof of Lemma 4.3 we can prove that for $-b \to \mu^-_{i+k}$, $-b < \mu^-_{i+k-1}$, $\widetilde{J}_{a,b}$ satisfies the $(P.S.)^*_c$ condition with respect to $(C_n)_n$, $C_n = C \cap H_n$, for any c such that

$$\inf_{\tilde{w} \in \widetilde{S}_{23}(\rho_{k-1})} \widetilde{J}_{a,b}(\tilde{w}) \leq c \leq \sup_{\tilde{v} \in \widetilde{\Delta}_{R_{k-1}}} \widetilde{J}_{a,b}(\tilde{v}),$$

where $S_{23}(\rho_{k-1}) = \{u \in X^{k-1}_2 \oplus X^{k-1}_3 |\ \|u\| = \rho_{k-1}\}$ and $\widetilde{S}_{23}(\rho_{k-1}) = \Psi^{-1}(S_{23}(\rho_{k-1}))$, $\widetilde{\Delta}_{R_{k-1}} = \Psi^{-1}(\Delta_{R_{k-1}}(S_{k-1}(\rho_{k-1}), X^{k-1}_1))$.

Lemma 4.4. Assume that $\mu^-_{i+k} \leq -b \leq \mu^-_{i+k-1} < -a$ and u be a critical point for $J_{a,b}|_{X^{k-1}_1 \oplus X^{k-1}_3} : X^{k-1}_1 \oplus X^{k-1}_3 \to R$. Then $J_{a,b}(u) = 0$.

Assume that $\mu^-_{i+k} \leq -b \leq \mu^-_{i+k-1} < -a$ and let u be a critical point of $J_{a,b}|_{X^{k-1}_1 \oplus X^{k-1}_3} : X^{k-1}_1 \oplus X^{k-1}_3 \to R$. Then, if $\mu^-_{i+k} < -b < -a$, we have $u = 0$, otherwise, if $\mu^-_{i+k} = -b$, u lies in the eigenspace associated with the eigenvalue μ^-_{i+k} and $\bar{u} + Su \geq 0$.

Lemma 4.5. Let α and β be such that $0 < \alpha < \beta$. Then there exists $\delta' > 0$ such that for any a and b with $\mu^-_{i+k} - \delta' \leq -b \leq \mu^-_{i+k-1} < -\mu^-_{i+1} < -a < \mu^-_i \leq \mu^-_1$, the functional $J_{a,b}|_{X^{k-1}_1 \oplus X^{k-1}_3}$ has no critical points $u \in X_1^{k-1} \oplus X_3^{k-1}$ with $\alpha \leq J_{a,b}(u) \leq \beta$.

Lemma 4.6. There exists a real number $\delta_{k-1} > 0$, R_{k-1}, ρ_{k-1}, $\bar{\rho} > 0$ with $0 < \rho_{k-1} < R_{k-1}$ such that if $\mu_{i+k}^- - \delta_{k-1} \leq -b \leq \mu_{i+k-1}^- \leq \mu_{i+1}^- < -a < \mu_i^- \leq \mu_1^-$, then we have

$$\sup_{v \in \Sigma_{R_{k-1}}(S_{k-1,k}(\bar{\rho}), X_1^k)} J_{a,b}(v) < \inf_{\substack{w \in H^+(\mu_{i+k-1}^-) \\ \|w\| = \rho_{k-1}}} J_{a,b}(w),$$

where $S_{k-1,k}(\bar{\rho}) = \{v \in X_2^{k-1} \oplus X_2^k | \ \|v\| = \bar{\rho}\}$.

Proof. We choose ρ' such that, if $e \in S_{k-1}(\rho')$, then $Se \leq 0$ in Ω. First we will prove that there exist ρ_{k-1} and δ_{k-1} such that if $\mu_{i+k}^- - \delta_{k-1} \leq -b \leq \mu_{i+k-1}^- \leq \mu_{i+1}^- < -a < \mu_i^- \leq \mu_1^-$,

$$\sup_{v \in X_1^{k-1}} J_{a,b}(v) < \inf_{\substack{w \in X_2^{k-1} \oplus X_3^{k-1} \\ \|w\| = \rho_{k-1}}} J_{a,b}(w).$$

Let $w \in H^+(\mu_{i+k-1}^-) = X_2^{k-1} \oplus X_3^{k-1}$. Then we can choose $\bar{\rho} > 0$ such that if $\|w\| \leq \bar{\rho}$, then $\bar{u} + Sw > 0$, where $Sw \in [e_{i+k-1}^-, \ldots, e_1^-]$. Choose $\rho_{k-1} > 0$ with $\rho_{k-1} < \bar{\rho}$. Then we have, for $u \in H^+(\mu_{i+k-1}^-)$ with $\|u\| \leq \bar{\rho}$,

$$J_{a,b}(u) \geq \frac{1}{2} min\{1, b, \frac{b}{|\mu_{i+k-1}^-|} - 1\}\|u\|^2 > 0.$$

Now we will show that for any a and b with $\mu_{i+k}^- \leq -b \leq \mu_{i+k-1}^- \leq \mu_{i+1}^- < -a < \mu_i^- \leq \mu_1^-$, $\sup_{u \in X_1^{k-1}} J_{a,b}(u) < 0$. Let $u \in H^-(\mu_{i+k}^-) = X_1^{k-1}$. Then $P^+u = Ru = 0$. Thus we have

$$J_{a,b}(u) \leq \frac{1}{2}(\frac{b}{|\mu_{i+k}^-|} - 1)\|P^-u\|^2 < 0.$$

Next, we will show that there exist $R_{k-1} > 0$, $\delta_{k-1} > 0$ and $\bar{\rho} > 0$ such that for any a and b with $\mu_{i+k}^- - \delta_{k-1} \leq -b \leq \mu_{i+k-1}^- \leq \mu_{i+1}^- < -a < \mu_i^- \leq \mu_1^-$,

$$\sup_{w \in \Sigma_{R_{k-1}}(S_{k-1,k}(\bar{\rho}), X_1^k)} J_{a,b}(w) < 0,$$

where $S_{k-1,k}(\bar{\rho}) = \{v \in X_2^{k-1} \oplus X_2^k | \; \|v\| = \bar{\rho}\}$.

We choose $\bar{\rho}$ such that, if $v + z \in S_{k-1,k}(\bar{\rho})$, then $Sv + Sz \leq 0$ in Q. Now, we can find $(b_n)_n$ and $(a_n)_n$ with $-b_n \to \mu_{i+k}^-$, $\mu_{i+k+1}^- < -b_n < \mu_{i+k-1}^- \leq \mu_{i+1}^- < -a < \mu_i^- \leq \mu_1^-$, $\|u_n\| \to +\infty$, $u_n = w_n + \sigma_n(v_n) + \sigma_n(z_n)$ with $w_n \in H^-(\mu_{i+k+1}^-)$, $\sigma_n > 0$, $v_n + z_n \in S_{k-1,k}(\bar{\rho})$, $v_n \in X_2^{k-1}$, $z_n \in X_2^k$, $\|v_n + z_n\| = \bar{\rho}$ and

$$\begin{aligned} J_{a_n,b_n}(u_n) &= -\frac{1}{2}\|w_n\|^2 - \frac{1}{2}\sigma_n^2\bar{\rho}^2 + \frac{b_n}{2}\|Sw_n + \sigma_n Sv_n + \sigma_n Sz_n\|^2 \\ &\quad -\frac{b_n - a_n}{2}\|[\bar{u} + Sw_n + \sigma_n Sv_n + \sigma_n Sz_n]^-\|^2. \end{aligned}$$

We can assume $\hat{w}_n = \frac{w_n}{\|u_n\|} \rightharpoonup \hat{w}$, $\hat{v}_n = \frac{v_n}{\|u\|} \to \hat{v}$, $\hat{z}_n = \frac{z_n}{\|u\|} \to \hat{z}$ and $\hat{\sigma}_n = \frac{\sigma_n}{\|u_n\|} \to \hat{\sigma} \geq 0$. Dividing by $\|u_n\|^2$ and passing to the limit, by the definition of $\bar{\rho}$, we have

$$\begin{aligned} \lim_{n\to\infty}\sup \frac{J_{a_n,b_n}(u_n)}{\|u_n\|^2} &\leq \frac{1}{2}\|\hat{w}\|^2\left(\frac{b}{|\mu_{i+k+1}^-|} - 1\right) + \frac{\hat{\sigma}^2}{2}\|\hat{z}\|^2\left(\frac{b}{|\mu_{i+k}|} - 1\right) \\ &\quad + \frac{\hat{\sigma}^2}{2}\|\hat{v}\|^2\left(\frac{b}{|\mu_{i+k-1}|} - 1\right) \leq 0. \end{aligned}$$

Thus we proved the lemma.

We have the following.

Proposition 4.2. Let δ_{k-1}, ρ_{k-1}, R_{k-1} with $\rho_{k-1} < R_{k-1}$ be as in Lemma 4.6. Then for any a and b with $\mu_{i+k}^- - \delta_{k-1} \leq -b < \mu_{i+k-1}^- \leq \mu_{i+1}^- < -a < \mu_i^- \leq \mu_1^-$, there exist two critical points $v_i^{(k-1)}$, $i = 1, 2$, for $J_{a,b}$ with

$$\begin{aligned} 0 < \inf_{\substack{v \in H^+(\mu_{i+k-1}^-) \\ \|v\| = \rho_{k-1}}} J_{a,b}(v) &\leq J_{a,b}(v_i^{(k-1)}) \\ &\leq \sup_{w \in \Delta_{R_{k-1}}(S_{k-1,k}(\bar{\rho}), X_1^k)} J_{a,b}(w), i = 1, 2, \end{aligned}$$

where $S_{k-1,k}(\bar{\rho}) = \{v \in X_2^{k-1} \oplus X_2^k | \; \|v\| = \bar{\rho}\}$.

Next we take integers i, k such that $\mu^-_{i+k} > \mu^-_{i+k-1} > \mu^-_{i+k-2} > \mu^-_{i+k-3}$. We set

$$X_1^{k-3} = H^-(\mu^-_{i+k-2}), X_2^{k-3} = span\{e^-_{i+k-3}\}, X_3^{k-3} = H^+(\mu^-_{i+k-4}).$$

As the previous case we define the functional Ψ and the manifold C on H with respect to the decomposition $H = X_1^{k-3} \oplus X_2^{k-3} \oplus X_3^{k-3}$. We also define $\widetilde{J} : C \to H$ by $\widetilde{J} = J \circ \Psi$. Then $\widetilde{J} \in C^{1,1}_{loc}$ and $\widetilde{J}_{a,b}$ satisfies the $(P.S.)^*_c$ condition with respect to $(C_n)_n$, $C_n = C \cap H_n$ for any c. We have the following lemma.

Lemma 4.7. Let α' and β' be such that $0 < \alpha' < \beta'$ and δ' be as in Lemma 4.5. Then there exists $\delta'' > 0$ such that for any a and b with

$$\mu^-_{i+k} - \delta' \leq -b \leq \mu^-_{i+k} \leq \dots \leq \mu^-_{i+1} < -a < \mu^-_i \leq \mu^-_1$$

or

$$\mu^-_{i+k-1} - \delta'' \leq -b \leq \mu^-_{i+k-2} \leq \dots \leq \mu^-_{i+1} < -a < \mu^-_i \leq \mu^-_1,$$

the functional $J_{a,b}|_{X_1^{k-2} \oplus X_3^{k-2}}$ has no critical points $u \in X_1^{k-2} \oplus X_3^{k-2}$ with $\alpha' \leq J_{a,b}(u) \leq \beta'$.

Lemma 4.8. Let α'' and β'' be such that $0 < \alpha'' < \beta''$ and δ' and δ'' be as in Lemma 4.5 and Lemma 4.7, respectively. Then there exist $\delta^{(3)} > 0$ such that for any a and b with

$$\mu^-_{i+k} - \delta' \leq -b \leq \mu^-_{i+k} \leq \dots \leq \mu^-_{i+1} < -a < \mu^-_i \leq \mu^-_1$$

or

$$\mu^-_{i+k-1} - \delta'' \leq -b \leq \mu^-_{i+k-1} \leq \dots \leq \mu^-_{i+1} < -a < \mu^-_i \leq \mu^-_1$$

or

$$\mu^-_{i+k-2} - \delta^{(3)} \leq -b \leq \mu^-_{i+k-3} \leq \dots \leq \mu^-_{i+1} < -a < \mu^-_i \leq \mu^-_1,$$

the functional $J_{a,b}|_{X_1^{k-3}\oplus X_3^{k-3}}$ has no critical points $u \in X_1^{k-3} \oplus X_3^{k-3}$ with $\alpha'' \le J_{a.b}(u) \le \beta''$.

Proof. By the same method as the proof of Lemma 4.5, there exists $\delta^{(3)} > 0$ such that for any a and b with $\mu_{i+k-2}^- - \delta^{(3)} \le -b \le \mu_{i+k-3}^- \le \dots \le \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-$, the functional $J_{a,b}|_{X_1^{k-3}\oplus X_3^{k-3}}$ has no critical points $u \in X_1^{k-3} \oplus X_3^{k-3}$ with $\alpha'' \le J_{a,b}(u) \le \beta''$. We claim that for δ'' as in Lemma 4.7 and any a and b with $\mu_{i+k-1}^- - \delta'' \le -b \le \mu_{i+k-1}^- \le \dots \le \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-$, $J_{a,b}|_{X_1^{k-3}\oplus X_3^{k-3}}$ has no critical points $u \in X_1^{k-3} \oplus X_3^{k-3}$ with $\alpha'' \le J_{a,b}(u) \le \beta''$. Now we have

$$\begin{aligned} J_{a,b}|_{X_1^{k-3}\oplus X_3^{k-3}} &= J_{a,b}|_{\{X_1^{k-2}\cup(X_1^{k-3}\setminus X_1^{k-2})\}\oplus X_3^{k-3}} \\ &= J_{a,b}|_{\{X_1^{k-2}\oplus X_3^{k-3}\}\cup\{(X_1^{k-3}\setminus X_1^{k-2})\oplus X_3^{k-3}\}}. \end{aligned}$$

Since $X_3^{k-3} \subset X_3^{k-2}$ and, by Lemma 4.7, for $\mu_{i+k-1}^- - \delta'' \le -b \le \mu_{i+k-2}^- \le \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-$, $J_{a,b}|_{X_1^{k-2}\oplus X_3^{k-2}}$ has no critical points $u \in X_1^{k-2} \oplus X_3^{k-2}$ with $\alpha'' \le J_{a,b}(u) \le \beta''$, it follows that $J_{a,b}|_{X_1^{k-2}\oplus X_3^{k-3}}$ has no critical points $u \in X_1^{k-2} \oplus X_3^{k-3}$ with $\alpha'' \le J_{a,b}(u) \le \beta''$. Next we consider $J_{a,b}|_{(X_1^{k-3}\setminus X_1^{k-2})\oplus X_3^{k-3}}$. Since $X_1^{k-3}\setminus X_1^{k-2} = X_2^{k-2} \subset X_3^{k-1}$, $X_3^{k-3} \subset X_3^{k-1}$, and by Lemma 4.5, for $\mu_{i+k}^- - \delta' \le -b \le \mu_{i+k-1}^- \le \dots \le \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-$, $J_{a,b}|_{X_1^{k-1}\oplus X_3^{k-1}}$ has no critical points $u \in X_1^{k-1} \oplus X_3^{k-1}$ with $\alpha'' \le J_{a,b}(u) \le \beta''$, it follows that $J_{a,b}|_{X_1^{k-3}\setminus X_1^{k-2}\oplus X_3^{k-3}}$ has no critical points in $(X_1^{k-3}\setminus X_1^{k-2}) \oplus X_3^{k-3}$. Thus for $\mu_{i+k-1}^- - \delta'' \le -b \le \mu_{i+k-1}^- \le \dots \le \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-$, $J_{a,b}|_{X_1^{k-3}\oplus X_3^{k-3}}$ has no critical points $u \in X_1^{k-3} \oplus X_3^{k-3}$ with $\alpha'' \le J_{a,b}(u) \le \beta''$. We also claim that for δ' as in Lemma 4.5 and any a and b with $\mu_{i+k}^- - \delta' \le -b \le \mu_{i+k}^- \le \dots \le \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-$, $J_{a.b}|_{X_1^{k-3}\oplus X_3^{k-3}}$ has no critical points $u \in X_1^{k-3} \oplus X_3^{k-3}$ with $\alpha'' \le J_{a.b}(u) \le \beta''$. We have

$$\begin{aligned} J_{a.b}|_{X_1^{k-3}\oplus X_3^{k-3}} &= J_{a.b}|_{\{X_1^{k-1}\cup(X_1^{k-3}\setminus X_1^{k-1})\}\oplus X_3^{k-3}} \\ &= J_{a.b}|_{\{X_1^{k-1}\oplus X_3^{k-3}\}\cup\{(X_1^{k-3}\setminus X_1^{k-1})\oplus X_3^{k-3}\}}. \end{aligned}$$

Since $X_3^{k-3} \subset X_3^{k-1}$ and by Lemma 4.5, for $\mu_{i+k}^- - \delta' \leq -b \leq \mu_{i+k-1}^- \leq \dots \leq \mu_{i+1}^- < -a < \mu_i^- \leq \mu_1^-$ (δ' is as in Lemma 4.5), $J_{a,b}|_{X_1^{k-1} \oplus X_3^{k-1}}$ has no critical points $u \in X_1^{k-1} \oplus X_3^{k-1}$ with $\alpha'' \leq J_{a,b}(u) \leq \beta''$, it follows that $J_{a,b}|_{X_1^{k-1} \oplus X_3^{k-3}}$ has no critical points $u \in X_1^{k-1} \oplus X_3^{k-3}$ with $\alpha'' \leq J_{a,b}(u) \leq \beta''$. Next we consider $J_{a,b}|_{(X_1^{k-3} \setminus X_1^{k-1}) \oplus X_3^{k-3}}$. Since $X_1^{k-3} \setminus X_1^{k-1} = X_2^{k-1} \cup X_2^{k-2} \subset X_3^k$, $X_3^{k-3} \subset X_3^k$, and by Lemma 4.1, for $\mu_{i+k+1}^- < -b \leq \mu_{i+k}^- \leq \dots \mu_{i+1}^- < -a < \mu_i^- \leq \mu_1^-$, $J_{a,b}|_{X_1^k \oplus X_3^k}$ has no critical points $u \in X_1^k \oplus X_3^k$ with $\alpha'' \leq J_{a,b}(u) \leq \beta''$, it follows that $J_{a,b}|_{(X_1^{k-3} \setminus X_1^{k-1}) \oplus X_3^{k-3}}$ has no critical points $u \in (X_1^{k-3} \setminus X_1^{k-1}) \oplus X_3^{k-3}$ with $\alpha'' \leq J_{a,b}(u) \leq \beta''$. Thus for $\mu_{i+k}^- - \delta' \leq -b \leq \mu_{i+k}^- \leq \dots \mu_{i+1}^- < -a < \mu_i^- \leq \mu_1^-$, $J_{a,b}|_{X_1^{k-3} \oplus X_3^{k-3}}$ has no critical points $u \in X_1^{k-3} \oplus X_3^{k-3}$ with $\alpha'' \leq J_{a,b}(u) \leq \beta''$. Thus we prove the lemma.

In general we take integers k, l with $0 \leq l \leq k$ such that $\mu_{i+k-l+1}^- > \mu_{i+k-l}^- > \mu_{i+k-l-1}^- > \dots$. We set

$$X_1^{k-l} = H^-(\mu_{i+k-l+1}^-),\ X_2^{k-l} = span\{e_{i+k-l}^-\},\ X_3^{k-l} = H^+(\mu_{i+k-l-1}^-).$$

We define the functional $\Psi : H \setminus (X_1^{k-l} \oplus X_3^{k-l}) \to H$ and the manifold C on H with respect to the decomposition $H = X_1^{k-l} \oplus X_2^{k-l} \oplus X_3^{k-l}$. We also define $\widetilde{J}_{a,b} : C \to H$ by $\widetilde{J}_{a,b} = J_{a,b} \circ \Psi$. Then $\widetilde{J}_{a,b} \in C_{loc}^{1,1}$ and for $-b \to \mu_{i+k}^-$, $-b < \mu_{i+k-l}^-$, $\widetilde{J}_{a,b}$ satisfies the $(P.S.)_c^*$ condition with respect to $(C_n)_n$, $C_n = C \cap H_n$, for any c We have the following lemma.

Lemma 4.9. Let α''' and β''' be such that $0 < \alpha''' < \beta'''$ and δ', δ'' and $\delta^{(3)}$ be as in Lemma 4.5, Lemma 4.7 and Lemma 4.8, respectively. Then there exists $\delta^{(l)} > 0$ such that for any a and b with

$$\mu_{i+k}^- - \delta' \leq -b \leq \mu_{i+k}^- \leq \dots \mu_{i+1}^- < -a < \mu_i^- \leq \mu_1^-,$$

$$\dots$$

or

$$\mu_{i+k-l+1}^- - \delta^{(l)} \leq -b \leq \mu_{i+k-l}^- \leq \dots \leq \mu_{i+1}^- < -a < \mu_i^- \leq \mu_1^-,$$

the functional $J_{a,b}|_{X_1^{k-l}\oplus X_3^{k-l}}$ has no critical points $u \in X_1^{k-l} \oplus X_3^{k-l}$ with $\alpha''' \le J_{a,b}(u) \le \beta'''$.

Lemma 4.10. There exist $\delta_{k-l} > 0$, $\rho_{k-l} > 0$, $\bar{\rho} > 0$ and $R_{k-l} > 0$ with $0 < \rho_{k-l} < R_{k-l}$ such that for any a and b with

$$\mu_{i+k}^- - \delta_k \le -b < \mu_{i+k}^- \le \dots \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-,$$

or

$$\mu_{i+k-1}^- - \delta_{k-2} \le -b < \mu_{i+k-1}^- \le \dots \le \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-,$$

$$\dots$$

or

$$\mu_{i+k-l+1}^- - \delta_{k-l} \le -b < \mu_{i+k-l}^- \le \dots \le \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-,$$

$$\sup_{v \in \Sigma_{R_{k-l}}(S_{k-l,k-l+1,\dots,k}(\bar{\rho}), X_1^k)} J_{a,b}(v) < \inf_{\substack{w \in H^+(\mu_{i+k-l}^-) = X_2^{k-l} \oplus X_3^{k-l} \\ \|w\| = \rho_{k-l}}} J_{a,b}(w),$$

where $S_{k-l,k-l+1,\dots,k}(\bar{\rho}) = \{z \in X_2^{k-l} \oplus X_2^{k-l+1} \oplus \dots \oplus X_2^k \mid \|z\| = \bar{\rho}\}$.

Proof. The proof has the same processure as the proof of Lemma 4.6.

Proposition 4.3. Let δ_k be as in Lemma 4.2. Then there exist $\delta_{k-1}, \dots,$ δ_{k-l} such that for any a and b with

$\mu_{i+k}^- - \delta_k \le -b < \mu_{i+k}^- \le \dots \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-$, or

$\mu_{i+k-1}^- - \delta_{k-2} \le -b < \mu_{i+k-1}^- \le \dots \le \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-$,

$\dots$

or

$\mu_{i+k-l+1}^- - \delta_{k-l} \le -b < \mu_{i+k-l}^- \le \dots \le \mu_{i+1}^- < -a < \mu_i^- \le \mu_1^-$, there exist two critical points $v_i^{(k-l)}$, $i = 1, 2$, in X_2^{k-l} for $J_{a,b}$ with

$$0 < \inf_{\substack{w \in H^+(\mu_{i+k-l}^-) \\ \|w\| = \rho_{k-l}}} J_{a,b}(w) \le J_{a,b}(v_i^{(k-l)}) \le \sup_{v \in \Delta_{R_{k-l}}(S_{k-l,k-l+1,\dots,k}(\bar{\rho}), X_1^k)} J_{a,b}(v).$$

Proof. Let δ_k, δ' and $\delta^{(l)}$ be as in Lemma 4.2 and Lemma 4.9 respectively. By the same processure of the proof of Proposition 4.1, there exist $\delta_{k-1},\dots,\ \delta_{k-l}$ with $\delta_{k-1}\ \le\ \delta'\dots,\ \delta_{k-l}\ \le\ \delta^{(l)}$ such that for any a and b with the regions in the assumption of Lemma 4.10, there exist two critical points $v_i^{(k-l)}$, $i=1,2$, in X_2^{k-l} for $J_{a,b}$ such that $0<\inf_{\substack{w\in H^+(\mu^-_{i+k-l})=X_2^{k-l}\oplus X_3^{k-l}\\ \|w\|=\rho_{k-l}}} J_{a,b}(w)\le J_{a,b}(v_i^{(k-l)})$ $\le \sup_{v\in\Delta_{R_{k-l}}(S_{k-l,k-l+1,\dots,k}(\bar\rho),X_1^k)} J_{a,b}(v)$. Since X_2^{k-l+1}, $X_2^{k-l+2},\dots,X_2^k\subset X_1^{k-l}$, by Lemma 4.9, the critical points $v_i^{(k-l)}$, $i=1,2$, exist only in the subspace X_2^{k-l}.

PROOF OF THEOREM 1.1. Let

$$\delta=\min\{\delta_k,\delta_{k-1},\delta'\}.$$

By Proposition 4.1, 4.2 and 4.3, for fixed k and any a and b such that $\mu^-_{i+k}-\delta\le -b<\mu^-_{i+k}\le\dots\le\mu^-_{i+1}<-a<\mu^-_i\ \le\mu^-_1$, there exist at least two nontrivial solutions $v_i^{(j)}$, $i=1,2$, in X_2^j for each j, $1\le j\le k$, for $J_{a,b}$, which satisfy the followings:

$$0<\inf_{\substack{w\in H^+(\mu^-_{i+k})\\ \|w\|=\rho_k}} J_{a,b}(w)\le J_{a,b}(v_i^{(k)})\le \sup_{v\in\Delta_{R_k}(S_k(\bar\rho),X_1^k)} J_{a,b}(v),$$

$$0<\inf_{\substack{w\in H^+(\mu^-_{i+k-1})\\ \|w\|=\rho_{k-1}}} J_{a,b}(w)\le J_{a,b}(v_i^{(k-1)})\le \sup_{v\in\Delta_{R_{k-1}}(S_{k-1,k}(\bar\rho),X_1^k)} J_{a,b}(v),$$

$$0<\inf_{\substack{w\in H^+(\mu^-_{i+k-2})\\ \|w\|=\rho_{k-2}}} J_{a,b}(w)\le J_{a,b}(v_i^{(k-2)})\le \sup_{v\in\Delta_{R_{k-2}}(S_{k-2,k-1,k}(\bar\rho),X_1^k)} J_{a,b}(v),$$

$$\dots$$

$$0<\inf_{\substack{w\in H^+(\mu^-_{i+k-l})\\ \|w\|=\rho_{k-l}}} J_{a,b}(w)\le J_{a,b}(v_i^{(k-l)})\le \sup_{v\in\Delta_{R_{k-l}}(S_{k-l,k-l+1,\dots,k}(\bar\rho),X_1^k)} J_{a,b}(v).$$

We recall that we can suppose $R_{k-l}>\dots>R_{k-1}>R_k$, $2\le l\le k-1$, so $\Delta_{R_k}(S_k(\bar\rho),X_1^k)\subset\Sigma_{R_{k-1}}(S_{k-1,k}(\bar\rho),X_1^k)$ and $\Delta_{R_{k-l+1}}(S_{k-l+1,k-l+2,\dots,k}(\bar\rho),$

$X_1^k) \subset \Sigma_{R_{k-l}}(S_{k-l,k-l+1,\ldots,k}(\bar{\rho}), X_1^k)$. Thus we have

$$\begin{aligned}
0 < \inf_{\substack{w \in X_2^k \oplus X_3^k \\ \|w\| = \rho_k}} J_{a,b}(w) \le J_{a,b}(v_i^{(k)}) &\le \sup_{v \in \Delta_{R_k}(S_k(\bar{\rho}), X_1^k)} J_{a,b}(v) \\
&\le \sup_{v \in \Sigma_{R_{k-1}}(S_{k-1,k}(\bar{\rho}), X_1^k)} J_{a,b}(v) \\
< \inf_{\substack{w \in X_2^{k-1} \oplus X_3^{k-1} \\ \|w\| = \rho_{k-1}}} J_{a,b}(w) \le J_{a,b}(v_i^{(k-1)}) &\le \sup_{v \in \Delta_{R_{k-1}}(S_{k-1,k}(\bar{\rho}), X_1^k)} J_{a,b}(v) \le \ldots \\
\le \sup_{v \in \Delta_{R_{k-l+1}}(S_{k-l+1,k-l+2,\ldots,k}(\bar{\rho}), X_1^k)} J_{a,b}(v) &\le \sup_{v \in \Sigma_{R_{k-l}}(S_{k-l,k-l+1,\ldots,k}(\bar{\rho}), X_1^k)} J_{a,b}(v) \\
< \inf_{\substack{w \in X_2^{k-l} \oplus X_3^{k-l} \\ \|w\| = \rho_{k-l}}} J_{a,b}(w) \le J_{a,b}(v_i^{(k-l)}) &\le \sup_{v \in \Delta_{R_{k-l}}(S_{k-l,k-l+1,\ldots,k}(\bar{\rho}), X_1^k)} J_{a,b}(v).
\end{aligned}$$

Thus $J_{a,b}(u)$ has at least $2k$ nontrivial critical points, which proves the theorem.

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Tack Sun Jung
Department of Mathematics,
Kunsan National University,
Kunsan 573-701, Korea
E-mail :tsjung@kunsan.ac.kr

Q-Heung Choi
Department of Mathematics Education,
Inha University,
Incheon 402-751, Korea
E-mail :qheung@inha.ac.kr