Design of H_∞ Controller with Different Weighting Functions Using Convex Combination

Min-Chan Kim, Seung-Kyu Park and **Gun-Pyong Kwak**, *Member, KIMICS*

Abstract—In this paper, a combination problem of controllers which are the the same type of H_∞ controllers designed with different weighting functions. This approach can remove the difficulty in the selection of the weighting functions. As a sub-controller, the Youla type of H_∞ controller is used. In the H_∞ controller, Youla parameterization is used to minimize H_∞ norm of mixed sensitivity function by using polynomial approach. Computer simulation results show the robustness improvement and the performance improvement.

Index Terms—H_∞ controller, Convex combination, Robust control, Youla parameterization

I. INTRODUCTION

The H_∞ robust control system is very appropriate for the robustness improvement properties. It has been developed in two ways. One is state space approach [1][2][3] and the other is polynomial approach [4][5][6].

In state space approach, the relation between disturbance and its effect to output has to be expressed explicitly by using system matrices and H_∞ robust control design problem is to minimize the H_∞ norm of the transfer function between them. The algebraic Riccati equations are solved to obtain the control parameters in this approach.

The main idea in the polynomial approach is to minimize the H_∞ cost function which is usually the sum of weighted sensitivity function and complementary sensitivity cost function because only the transfer functions between inputs and outputs are known. Grimble's approach is also to minimize the mixed sensitivity cost function which is the sum of H_∞ norm of the weighted sensitivity function and weighted complementary sensitivity function. In this design, the minimization can be achieved functionally by using Kwakernaak's lemma. Most polynomial approach use weighting functions in its design procedure.

In H_∞ control theory, the frequency characteristic of the controlled system can be determined by the choice of weighting functions because the H_∞ norm of some designing value, which is multiplied by weighting function, is minimized. So the selection of weighting functions is very important and very difficult problem. Even if the weighting functions are determined, no one knows they are optimal weighting functions.

In this paper, a new type of H_∞ controller, which is based on the convex combination of H_∞ controllers designed with different weightings. This type of controller can remove the difficulty in the selection of the weighting functions.

II. PROBLEM FORMULATION

We consider a SISO discrete time system as in Fig. 1 For simplicity, the arguments of polynomial are often omitted.($A(z^{-1})$ is denoted by $\mathbf{A}$)

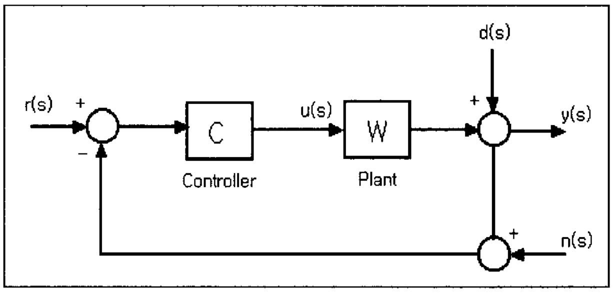

Fig. 1 SISO feedback control system

The transfer function of the system is as follows.

$$\mathbf{G} = \frac{\mathbf{B}}{\mathbf{A}} \tag{1}$$

The aim of controller design is to minimize the following mixed sensitivity cost function.

$$\mathbf{J} = \|\mathbf{W}_S \mathbf{S}\|_\infty^2 + \|\mathbf{W}_T \mathbf{T}\|_\infty^2 \tag{2}$$

where S is sensitivity function and T is complementary sensitivity function and $W_T = W_{Tn}/W_{Td}$, $W_S = W_{Sn}/W_{Sd}$ are the following weighting functions which have opposite shape to the desired sensitivity and complementary sensitivity function respectively.

Manuscript received August 9, 2004.

This work was supported in part by the Korea Science and Engineering Foundation (KOSEF) through the Machine Tool Research Center at Changwon National University.

Min-Chan Kim is with the Department of Electrical Engineering, Changwon National University, Kyong nam, Korea. (phone : 82-55-263-6885, Fax : 82-55-263-9956, E-mail : kmc710@kebi.com)

Seung-Kyu Park is with the Department of Electrical Engineering, Changwon National University, Kyong nam, Korea. (phone : 82-55-279-7514, Fax : 82-55-263-9956, E-mail : skpark@changwon.ac.kr)

Gun-Pyong Kwak is with the Department of Electrical Engineering, Changwon National University, Kyong nam, Korea. (phone : 82-55-279-7518, Fax : 82-55-263-9956, E-mail : kwak @sarim.changwon.ac.kr)

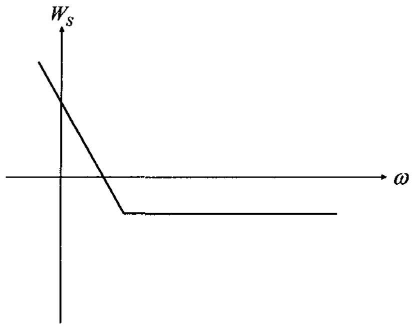

Fig. 2 Typical shape of weighting function W_S

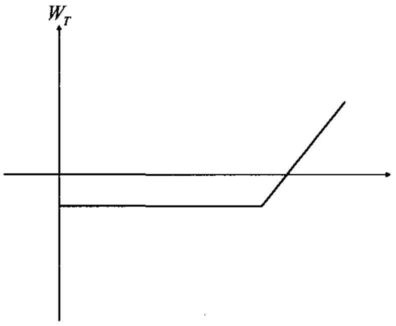

Fig. 3 Typical shape of weighting function W_T

However the selection of weighting functions is difficult because they must have desired shapes under the order limitation. The convex combination of the H∞ robust controller with low order weighting functions can solve this problem.

III. DESIGN OF H∞ CONTROLLER

To minimize the cost function, we use the following Youla controller.

$$C = \frac{N_0 + AK}{M_0 - BK} \tag{3}$$

With this controller, the sensitivity function and the complementary sensitivity function can be expressed respectively as follows.

$$S = \frac{1}{1 + CG} = (M_0 - BK)A \tag{4}$$

$$T = \frac{CG}{1 + CG} = (N_0 + AK)B \tag{5}$$

where $M_0 = M_{0n}/M_{0d}$, $N_0 = N_{0n}/N_{0d}$ have to satisfy the following equation and M_{0d} has the same denominator N_{0d}.

$$AM_{0n} + BN_{0n} = N_{0d} \tag{6}$$

The following lemma makes it possible to minimize H∞ -norm functionally.

Lemma 1

Consider the auxiliary problem of minimizing. [5]

$$J = \frac{1}{2\pi j}\oint_{|z=1|} (X_0(z^{-1})\Sigma(z^{-1}))\frac{dz}{z} \tag{7}$$

and suppose that for some real rational polynomial $\Sigma(z^{-1}) = \Sigma^*(z^{-1}) > 0$ the cost function J is minimized by a function $X_0(z^{-1})\lambda^2$(a real constant) on $|z|=1$.

Then $\sup_{|z|=1}\|X_0(z^{-1})\|$ is also minimized.

For the cost function is eq.(2), we have to minimize the eq.(7) with following $X_0$and Σ.

$$X_0 = (W_S S)^*(W_S S) + (W_T T)^*(W_T T) \tag{8}$$

$$\Sigma = \frac{B_\sigma {}^* B_\sigma}{A_\sigma {}^* A_\sigma} \tag{9}$$

To solve Youla Parameter K which can minimize the cost function, Eq.(7) is to satisfy the following equation.

$$\begin{aligned} X_0(z^{-1})\Sigma(z^{-1}) &= (\overline{W_S}S)^*(\overline{W_S}S) + (\overline{W_T}T)^*(\overline{W_T}T) \\ &= (\overline{W_S}(M_0 - BK)A)^*(\overline{W_S}(M_0 - BK)A) \\ &\quad + (\overline{W_T}(N_0 + AK)B)^*(\overline{W_T}(N_0 + AK)B) \end{aligned} \tag{10}$$

where $\overline{W_S} = W_S\frac{B_\sigma}{A_\sigma} = \frac{\overline{W_{Sn}}}{\overline{W_{Sd}}}$, $\overline{W_T} = W_T\frac{B_\sigma}{A_\sigma} = \frac{\overline{W_{Tn}}}{\overline{W_{Td}}}$, A_σ and B_σ are real rational function.

And the following equation is obtained by spectral factorization.

$$\begin{aligned} &(\overline{Y_f}Y_cK + \frac{(-\overline{W_S^*W_S}M_0A + \overline{W_T^*}\,\overline{W_T}BN_0)(AB)^*}{\overline{Y_c^*}Y_c^*})^* \cdot \\ &(\overline{Y_f}Y_cK + \frac{(-\overline{W_S^*W_S}M_0A + \overline{W_T^*}\,\overline{W_T}BN_0)(AB)^*}{\overline{Y_c^*}Y_c^*}) + \frac{\overline{W_S^*}\,\overline{W_S}\,\overline{W_T^*}\,\overline{W_T}}{Y_f^*Y_f} \end{aligned} \tag{11}$$

where $\overline{Y_f^*}\overline{Y_f} = \overline{W_S^*}\,\overline{W_S} + \overline{W_T^*}\,\overline{W_T}$, $Y_c^*Y_c = (BA)^*BA$ and $Y_f = Y_{fn}/Y_{fd}$

Because the youla parameter K must be stable, the following diophantine equation is satisfied.

$$\overline{W_{Sd}}M_{0d}F + \overline{W_{Sd}^*}\,\overline{Y_{fn}^*}Y_c^*Gz^{-g} = \overline{W_{Sn}^*}\,\overline{W_{Sn}}M_{0n}A(BA)^*Y_{fd}^*z^{-g} \tag{12}$$

$$\overline{W_{Td}}N_{0d}S + \overline{W_{Td}^*}\,\overline{Y_{fn}^*}Y_c^*Lz^{-gl} = \overline{W_{Tn}^*}\,\overline{W_{Tn}}N_{0n}B(BA)^*Y_{fd}^*z^{-gl} \tag{13}$$

And then the following equation is obtained.

$$\begin{aligned} &(\frac{\overline{Y_f}Y_cK\overline{W_{Sd}}M_{0d}\overline{W_{Td}}N_{0d} - (G\overline{W_{Td}}N_{0d} - L\overline{W_{Sd}}M_{0d})}{\overline{W_{Sd}}M_{0d}\overline{W_{Td}}N_{0d}} - \frac{F\overline{W_{Td}^*}z^g - S\overline{W_{Sd}^*}z^{gl}}{(\overline{W_{Sd}^*}\,\overline{W_{Td}^*}Y_{fn}^*Y_c^*)}) \\ &(\frac{\overline{Y_f}Y_cK\overline{W_{Sd}}M_{0d}\overline{W_{Td}}N_{0d} - (G\overline{W_{Td}}N_{0d} - L\overline{W_{Sd}}M_{0d})}{\overline{W_{Sd}}M_{0d}\overline{W_{Td}}N_{0d}} - \frac{F\overline{W_{Td}^*}z^g - S\overline{W_{Sd}^*}z^{gl}}{(\overline{W_{Sd}^*}\,\overline{W_{Td}^*}Y_{fn}^*Y_c^*)}) \\ &+ \frac{\overline{W_S^*}\,\overline{W_S}\,\overline{W_T^*}\,\overline{W_T}}{Y_f^*Y_f} \end{aligned} \tag{14}$$

where superscript g and gl are lowest degree of z^{-1}which multiply to have the solution of polynomial z^{-1} in diophantine equation.

Therefore Eq.(14) is simplified by the following equation.

$$(T_1 + T_2)^*(T_1 + T_2) + \Sigma\frac{W_S^* W_S W_T^* W_T}{Y_f^* Y_f} \quad (15)$$

where $G_a = G\overline{W_{Td}}N_{0d} - L\overline{W_{Sd}}M_{0d}$

$$T_1 = \frac{\overline{Y_f Y_c K W_{Sd} M_{0d} W_{Td} N_{0d}} - G_a}{\overline{W_{Sd} M_{0d}}\,\overline{W_{Td} N_{0d}}}$$

$$T_2 = -\frac{(F\overline{W_{Td}^*}z^g - S\overline{W_{Sd}^*}z^{gl})}{(\overline{W_{Sd}^*}\,\overline{W_{Td}^*}\,\overline{Y_{fn}^*}\,\overline{Y_c^*})}$$

$$\Sigma = (\frac{B_\sigma^* B_\sigma}{A_\sigma^* A_\sigma})$$

From the above equation, $(T_1 + T_2)^*(T_1 + T_2)$ is to satisfy the following equation.

$$T_1^*T_1 + T_2^*T_2 + T_1^*T_2 + T_1T_2^* \quad (16)$$

Because all poles of T_1 exist in unit circle, it is asymptotically stable. And all poles of T_2 is strictly unstable by existing outside unit circle.

In Eq.(16), $T_1^*T_2$ is analytic in the unit circle which can obtain the following equation by residue theorem.

$$\oint_{z=1}(T_1^*T_2)\frac{dz}{z} = 0 \quad (17)$$

where z, θ is defined as follows.

$$z = \exp(j\theta) \;,\; \theta_1 = -\theta \quad (18)$$

$$\begin{aligned}
&\oint_{z=1}(T_1(z)T_2(z^{-1}))\frac{dz}{z} \\
&= \oint_{-\pi}^{\pi}(T_1(\exp(j\theta))T_2(\exp(-j\theta)))jd\theta \\
&= \oint_{-\pi}^{\pi}(T_1(\exp(-j\theta_1))T_2(\exp(-j\theta_1)))jd\theta_1 \\
&= \oint_{z=1}(T_1(z_1)T_2(z_1))\frac{dz_1}{z_1} \\
&= \oint_{z=1}(T_1\,T_2^*)\frac{dz}{z} = 0 \quad (19)
\end{aligned}$$

From the relation of Eq.(17) and Eq.(19) , Eq.(20) is obtained.

$$T_1^*T_1 + T_2^*T_2 \quad (20)$$

Eq.(15) can be expressed by the following equation.

$$T_1^*T_1 + T_2^*T_2 + \Sigma\frac{W_S^* W_S W_T^* W_T}{Y_f^* Y_f} \quad (21)$$

The above equation is simplified as follows.

$$T_2^*T_2 + \Sigma\frac{W_S^* W_S W_T^* W_T}{Y_f^* Y_f} \quad (22)$$

where $K = \frac{G_a}{\overline{Y_f Y_c W_{sd} W_{TD} M_{0d} N_{0d}}}$

Eq.(22) have to satisfy the following relation in Eq.(7)

$$T_2^*T_2 + \Sigma\frac{W_S^* W_S W_T^* W_T}{Y_f^* Y_f} = \lambda^2.\frac{B_\sigma^* B_\sigma}{A_\sigma^* A_\sigma} \quad (23)$$

Because W_S and W_T are complement relation each other, one of two functions has small value at the same frequency.

Therefore $\frac{W_S^* W_S W_T^* W_T}{Y_f^* Y_f}$ can be omitted.

$$T_2^*T_2 = \Sigma\lambda^2$$

$$(\frac{F\overline{W_{Td}^*}z^g - S\overline{W_{Sd}^*}z^{gl}}{(\overline{W_{Sd}^*}\,\overline{W_{Td}^*}\,\overline{Y_{fn}^*}\,\overline{Y_c^*})})^*(\frac{F\overline{W_{Td}^*}z^g - S\overline{W_{Sd}^*}z^{gl}}{(\overline{W_{Sd}^*}\,\overline{W_{Td}^*}\,\overline{Y_{fn}^*}\,\overline{Y_c^*})}) = \Sigma\lambda^2 \quad (24)$$

$$(\frac{F_0 W_{Td}^* z^g - S W_{Sd}^* z^{gl}}{(W_{Sd}^* W_{Td}^* Y_{fn}^* Y_c^*)})^*(\frac{F_0 W_{Td}^* z^g - S W_{Sd}^* z^{gl}}{(W_{Sd}^* W_{Td}^* Y_{fn}^* Y_c^*)}) = \Sigma\lambda^2 \quad (25)$$

where $F_1 = F_0 W_{Td}^* z^g - S_0 W_{Sd}^* z^{gl}$

$D_{fc} = W_{SD}W_{TD}Y_{fn}Y_c$

$F = F_0 A_\sigma^* B_\sigma^*$

$S = S_0 A_\sigma^* B_\sigma^*$

In the above relation, $T_2^*T_2$ is calculated as follows.

$$T_2^*T_2 = \frac{F_1 F_1^*}{D_{fc} D_{fc}^*} \quad (26)$$

$$\frac{F_1 F_1^*}{D_{fc} D_{fc}^*} = \Sigma\lambda^2 \quad (27)$$

and A_σ, B_σ can be obtained.

$$A_\sigma = D_{fc}\lambda \quad (28)$$

$$B_\sigma = F_{ls} \quad (29)$$

where $F_{ls}F_{ls}^* = F_1 F_1^*$

Finally the Youla parameter K which minimize the H_∞ cost function is as follows.

$$K = \frac{G_1 Y_{fd}}{B_\sigma Y_{fn} Y_c W_{Sd} W_{Td} M_{0d} N_{0d}} \quad (30)$$

where $G_1 = GW_{Td}N_{0d} - LW_{Sd}m_{0d}$, G_1 and B_σ are the solution of the following diophantine equation.

$$\begin{aligned}
&A_\sigma W_{Sd} W_{Td} M_{0d} N_{0d} F_1 + W_{Sd}^* W_{Td}^* Y_{fn}^* Y_{cn}^* G_1 z^{-gl} \quad (31) \\
&= B_\sigma (BA)^* Y_{fd}^* (W_{Td} W_{Td}^* N_{0d} W_{Sn} W_{Sn}^* A M_{0n}^* + W_{Sd} W_{Sd}^* M_{0d} W_{Tn} W_{Tn}^* B N_{0n}^*) z^{-gl}
\end{aligned}$$

IV. THE CONVEX COMBINATION OF CONTROLLERS

The system with the controller which is the convex combination of sub-controllers is shown in the following figures.

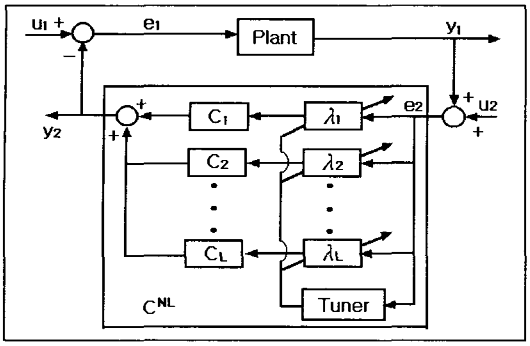

Fig. 4 Linked controller

The convex combination of controllers is as follows.

$$C = \sum_{i=1}^{L} C_i L_i \qquad (32)$$

where $\sum_{i=1}^{L} = 1$, C_i is i-th sub-controller.

The λ_i is time-varying. An adaptation algorithm is used to adjust the convex combination parameter λ_i. It is based on "pseudo-gradient" computation of the error.

algorithm 1.

For the two-controller case adaptation algorithm has the following form. [7]

$$\beta(k+1) = \beta(k) - \|e_2(k)\| \cdot \|e_2(k+1)\, e_2(k)\| \cdot \text{sgn}(\beta(k) - \beta(k-1))$$

$$\lambda_1(k) = \frac{0.5 \cdot \beta(k)}{1 + |\beta(k)|} + 0.5$$

$$\lambda_2(k) = 1 - \lambda_1(k)$$

Essentially, the algorithm defined in the above equation prescribes that the parameters continue being adjusted in the same direction if the "error" signal e_2 is decreasing in norm.

V. SIMULATION

Consider the transfer function of PMSM for discrete system with

$$G = \frac{0.0938 + 0.3615z^{-1} + 0.0907z^{-2}}{1 - 1.5599z^{-1} + 0.9354z^{-2}}$$

And disturbance of the system is given as follows.

$$d(t) = 0.01\sin(0.05t)$$

Weighting functions are considered by the following equations.

$$W_{1S} = \frac{0.5789 + 0.3727z^{-1}}{1 - 0.9048z^{-1}}, \quad W_{1T} = \frac{9.5211 - 9.5116z^{-1}}{1 - 0.9048z^{-1}}$$

$$W_{2S} = \frac{0.009 + 0.0727z^{-1}}{1 - 0.9999z^{-1}}, \quad W_{2T} = \frac{9.8902 - 9.9501z^{-1}}{1 - 0.9999z^{-1}}$$

And sub-controllers are calculated as follows.

$$C_1 = \frac{1 - 2.514z^{-1} + 2.5753z^{-2} - 1.1283z^{-3} + 0.1411z^{-4}}{1 - 0.8532z^{-1} - 0.1603z^{-2} + 0.0469z^{-3} + 0.0023z^{-4}}$$

$$C_2 = \frac{1 - 3.0393z^{-1} + 3.7328z^{-2} - 2.1477z^{-3} + 0.458z^{-4}}{1 - 1.2717z^{-1} + 0.1513z^{-2} + 0.1298z^{-3} - 0.0063z^{-4}}$$

The simulation results are shown in the following figures. In Fig. 5 and Fig. 6, plant output of H∞ controller with C_1 and C_2 converge to zero before 150 steps.

However, its robustness and performance in convex combination of H∞ controller are better than that of each controllers C_1 and C_2. And fig.8 plots that beta of convex combination. Fig. 9 and Fig. 10 show that the values of $\lambda_1(k)$ and $\lambda_2(k)$.

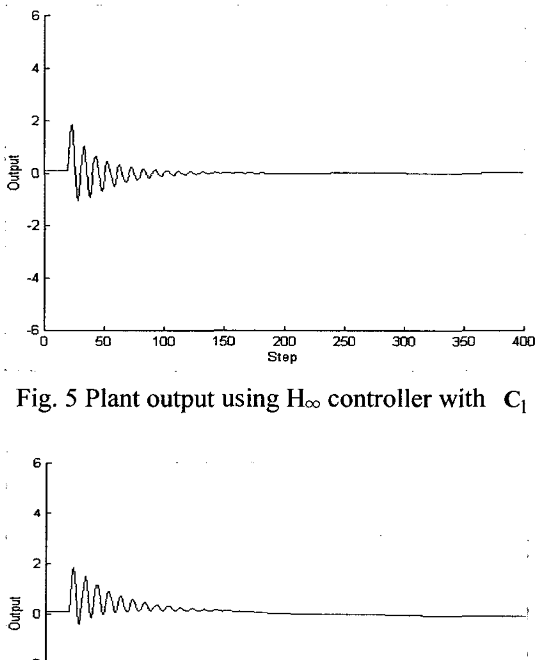

Fig. 5 Plant output using H∞ controller with C_1

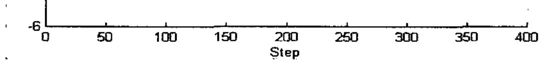

Fig. 6 Plant output using H∞ controller with C_2

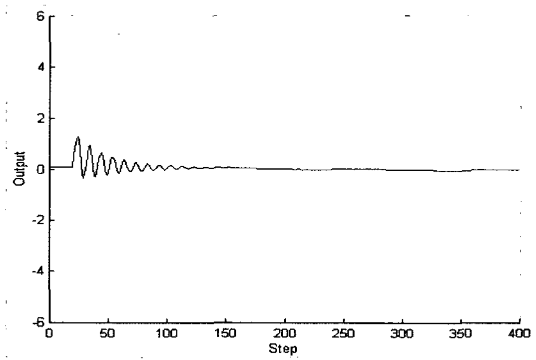

FIg. 7 plant output using convex combination of H∞ controllers

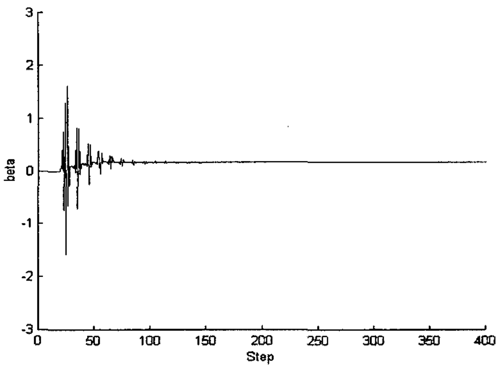

Fig. 8 beta of convex combination

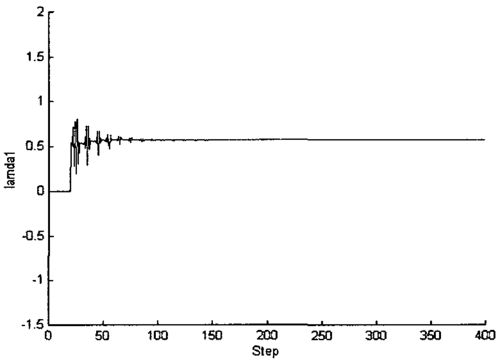

Fig. 9 $\lambda_1(k)$ of convex combination

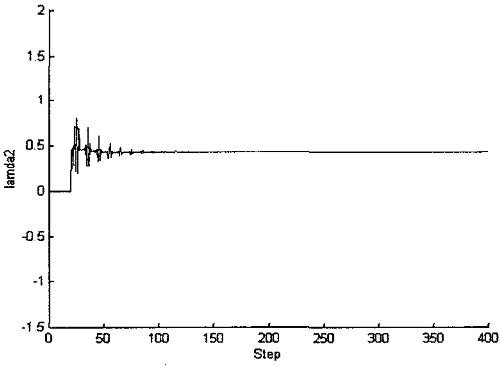

Fig. 10 $\lambda_2(k)$ of convex combination

VI. CONCLUSIONS

In this paper, convex combination of the two H∞ robust controller with the different weighting functions has been proposed. This removes the difficulty in the selection of the weighting functions. Its robustness and performance are better than that of each sub-controllers.

REFERENCES

[1] H. Kwakernaak, "Robustness optimization of linear feedback systems", *22nd CDC Conf.*, Texas, 1983.

[2] J. C. Doyle, K. Glover, P. Khargonekar, B. Francis, "State-space solutions to standard H∞ and H_2 control problems", *IEEE AC-34*, 841- 847, 1989.

[3] J. Doyle, K. Zhou, K. Glover, B. Bodenheimer, "Mixed H∞ and H_2 performance objective II : Optimal control", *IEEE, AC-39*, No. 8, pp. 1575-1587, 1994.

[4] M. J. Grimble, "H∞ robust controller for self-turning control application Part 2. Self-turning and Robustness", *International Journal of Control*, Vol. 46, No.5, pp. 1819-1940, 1987.

[5] M. J. Grimble, "H∞ Robust controller for self-turning control application Part 3. Self-turning and Robustness", *International Journal of Control*, Vol. 52, No. 1, pp. 15-36, 1990.

[6] M. J. Grimble, "H∞ / H_2 robust control design for a generalized control structure and flight control application", *International Journal of Systems Science*, Vol. 26, No. 11, pp. 2043-2068, 1995.

[7] S. J. Cusumanno, *Adaptive Robust Control*, University of Illinois at Urbana-Champaign, USA, 1988.

Min-Chan Kim
Received the B. S, M. S, Ph. D degrees in Electrical Engineering from Changwon National University in 1996, 1998 and 2003, respectively. Since 1998, he has been a part-time lecturer in the Department of Electrical Engineering at Changwon National University. His research interests include in the area of H∞robust control theory, sliding mode control theory, system identification and its applications. He is a member of KIEE of Korea and ICASE of Korea.

Seung-Kyu Park
Received the B. S, M. S, Ph. D degrees in Electrical Engineering from Korea University in 1984, 1986 and 1990, respectively. He was a visiting professor of Strathclyde University, England from 1995 to 1996. Since 1990, he is currently a professor in the Department of Electrical Engineering at Changwon National University, Changwon, Korea. His research interests include in the area of adaptive control theory, robust control theory and nonlinear control theory. He is a member of KIEE of Korea and ICASE of Korea.

Gun-Pyong Kwak
Received the B. S, M. S, Ph. D degrees in Electrical Engineering from Korea University in 1982, 1985 and 1990, respectively. He was a section chief of CNC team, LGIS from 1990 to 1997. Since 1998, he is currently a associate professor in the Department of Electrical Engineering at Changwon National University, Changwon, Korea. His research interests include in the area of control algorithm and motion controller. He is a member of KIEE of Korea and ICASE of Korea.